2009/10 annual paper, all questions, draft version.

1. (a) The Fourier series relies on the orthogonality of the sine and cosine functions: as an example, show

$$\int_{-\pi}^{\pi} \sin nt \cos mt dt = 0$$

for integers n and m.

(b) In a similar way it can be shown that

$$\int_{-\pi}^{\pi} \sin nt \sin mt dt = \pi\delta_{nm}, \qquad \int_{-\pi}^{\pi} \cos nt \cos mt dt = \pi\delta_{nm}$$

How are these identities used to derive formulas for the a_0, a_n and b_n in the Fourier series.

(c) Find the Fourier series of the function $f(t)$ with period 2π defined by

$$f(t) = t$$

2. Write the on-off pulse

$$f(t) = \begin{cases} 1 & -\pi < t < 0 \\ -1 & 0 < t < \pi \\ 0 & \text{otherwise} \end{cases}$$

as a Fourier integral.

3. What are

$$\int_{-\infty}^{\infty} (t^4+t^2-1)\delta(t)dt, \int_{-\infty}^{\infty} (t^4+t^2-1)\delta(t+1)dt, \int_{0}^{\infty} (t^4+t^2-1)\delta(t+1)dt$$

where $\delta(t)$ is the Dirac delta function. Show that the Fourier intergral representation of $\delta(t)$ is

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt} dk$$

Starting with

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)f^*(t')\delta(t-t')dtdt'$$

substitute the Fourier integral representation of $\delta(t-t')$ and derive the Plancherel formula.

4. Show that $y(t) = 1$ is a particular solution of
$$\ddot{y} + 4\dot{y} + 4y = 4$$
and find the general solution. What is the solution if $y(0) = \dot{y}(0) = 0$.

5. Using the substitution $t = e^z$ or otherwise, find the general solution of Euler equation
$$t^2\ddot{y} + 3t\dot{y} + y = t^2$$

6. Calculate the first four non-zero terms in the power series solution to
$$(1 - t^2)\dot{y} - 2ty = 0$$
with $y(0) = 2$.

Some useful formula

- A function with period L has the Fourier series expansion
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right).$$
where
$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right)dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right)dt \end{aligned}$$

- A function with period L has the Fourier series expansion
$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\frac{2i\pi nt}{L}.$$
where
$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\exp\left(\frac{-2i\pi nt}{L}\right)dt$$

- The Fourier integral or Fourier transform:
$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt} \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)e^{-ikt} \end{aligned}$$