MA22S3, sample for my four questions on the schol exams: outline solutions

Eight questions do six in a three hour exam.

1. Prove that for two periodic functions $f(t)$ and $g(t)$ with the same period L then
$$\frac{1}{L}\int_{-L/2}^{L/2} f(t)g^*(t)dt = \int_{n=-\infty}^{\infty} c_n d_n^*$$
where c_n and d_n are the coefficients in the complex Fourier series for $f(t)$ and $g(t)$ respectively. Deduce Parseval's theorem from this.

   Solution: Just substitute in
$$\frac{1}{l}\int_c^{c+l} dt f(t)g(t) = \frac{1}{l}\int_c^{c+l} dt \sum c_n e^{intl/2\pi} \sum d_m e^{imtl/2\pi} \tag{1}$$
and then do a change of index to send m to $-m$ and use $d_{-m} = d_m^*$ to get
$$\frac{1}{l}\int_c^{c+l} dt f(t)g(t) = \sum_n \sum_m c_n d_m^* \frac{1}{l}\int_c^{c+l} dt e^{intl/2\pi} e^{-imtl/2\pi} \tag{2}$$
and then the integral gives the required Kronecker δ-function. Setting $f = g$ gives Parseval's theorem.

2. Write the triangular pulse
$$f(t) = \begin{cases} At/T + A & -T < t < 0 \\ -At/T + A & 0 < t < T \\ 0 & |t| > T \end{cases}$$
as a Fourier integral.

   Solution: So, substituting in
$$\begin{aligned} f(\tilde{k}) &= \frac{1}{2\pi}\int_{\infty}^{\infty} f(t)e^{-ikt}dt \\ &= \frac{1}{2\pi}\int_{-T}^{0}\left(\frac{At}{T} + A\right)e^{-ikt}dt \\ &\quad + \frac{1}{2\pi}\int_{0}^{T}\left(-\frac{At}{T} + A\right)e^{-ikt}dt \end{aligned} \tag{3}$$

Now, doing a change of variable from t to $-t$ gives

$$\int_{-T}^{0} \frac{At}{T} e^{-ikt} dt = -\int_{0}^{T} \frac{At}{T} e^{ikt} dt \tag{4}$$

so we can put everything together

$$f(\tilde{k}) = \frac{A}{2\pi} \int_{-T}^{T} e^{-ikt} dt - \frac{A}{\pi T} \int_{0}^{T} t \cos kt dt \tag{5}$$

The second integral has to be done by parts

$$\begin{aligned} \int_{0}^{T} t \cos kt dt &= \left. \frac{1}{k} t \sin kt \right]_{0}^{T} - \frac{1}{k} \int_{0}^{T} \sin kt dt \\ &= \frac{T}{k} \sin kT + \frac{1}{k^2} \cos kT - \frac{1}{k^2} \end{aligned} \tag{6}$$

The first integral is

$$\int_{-T}^{T} e^{-ikt} = \left. \frac{1}{-ik} e^{-ikt} \right]_{-T}^{T} = \frac{2}{k} \sin kT \tag{7}$$

Thus, we have

$$\begin{aligned} f(\tilde{k}) &= \frac{A}{\pi k} \sin kT - \frac{A}{\pi T} \left(\frac{T}{k} \sin kT + \frac{1}{k^2} \cos kT - \frac{1}{k^2} \right) \\ &= \frac{A}{\pi T k^2} (1 - \cos kT) \end{aligned} \tag{8}$$

3. Solve the Euler-Cauchy equation

$$t^2 \frac{d^2 y}{dt^2} + 3t \frac{dy}{dt} + y = 0.$$

Solution: By the usual change of variable, see the notes,

$$t = e^z \tag{9}$$

this become

$$\frac{d^2 y}{dz^2} + 2 \frac{dy}{dz} + z = 0 \tag{10}$$

This equation has auxiliary

$$\lambda^2 + 2\lambda + 1 = 0 \tag{11}$$

which has a repeated root $\lambda = -1$; hence the solution is

$$y = C_1 e^{-z} + C_2 z e^{-z} \tag{12}$$

or

$$y = \frac{C_1 + C_2 \ln t}{t} \tag{13}$$

4. Using the method of Fröbenius, or otherwise, find the general solution to

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - \alpha t y = 0$$

where $\alpha > 0$.

Solution: So, to use the method of Froebenius, we assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} a_n t^{n+r} \tag{14}$$

This gives

$$\begin{aligned} t\ddot{y} &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-1} \\ 2\dot{y} &= \sum_{n=0}^{\infty} 2a_n (n+r) t^{n+r-1} \\ t\dot{y} &= \sum_{n=0}^{\infty} a_n t^{n+r+1} \end{aligned} \tag{15}$$

which means we need to shift the indices for the first two terms forward two steps by setting $n = m+2$, after taking terms out of the sum to get the range of the index correct, and renaming m back to n, this gives

$$\begin{aligned} t\ddot{y} &= r(r-1)a_0 t^{r-1} + (r+1)r a_1 t^r \\ &\quad + \sum_{n=0}^{\infty} a_{n+2}(n+r+2)(n+r+1) t^{n+r+1} \\ 2\dot{y} &= 2r a_0 t^{r-1} + 2(r+1) a_1 t^r + \sum_{n=0}^{\infty} 2a_{n+2}(n+r+2) t^{n+r+1} \end{aligned} \tag{16}$$

Thus, the equation becomes

$$r(r+1)a_0 t^{r-1} \quad + \quad (r+1)(r+2)a_1 t^r$$

$$
\begin{aligned}
&+ \sum_{n=0}^{\infty} [a_{n+2}(n+r+2)(n+r+3) - \alpha a_n] t^{n+r+1} \\
&= 0
\end{aligned} \tag{17}
$$

The indicial equation, which is a consequence of the assumption that $a_0 \neq 0$ is, therefore

$$
r(r+1) = 0 \tag{18}
$$

so $r = 0$ and $r = -1$ are the possilities, hence $r = 0$ or $r = -1$, giving general solution

$$
y = C_1 y_1 + C_2 y_2 \tag{19}
$$

where

$$
y_1 = \sum_{n=0}^{\infty} a_n t^n \tag{20}
$$

where

$$
a_{n+2} = \frac{\alpha}{(n+2)(n+1)} \tag{21}
$$

with $a_0 = 1$ and $a_1 = 0$ and, where where

$$
y_2 = \frac{1}{t} \sum_{n=0}^{\infty} a_n t^n \tag{22}
$$

where

$$
a_{n+2} = \frac{\alpha}{(n+1)n} \tag{23}
$$

with $a_0 = 1$ and $a_1 = 0$.