MA22S3, sample for my four questions on the schol exams

Eight questions do six in a three hour exam.

1. Prove that for two periodic functions $f(t)$ and $g(t)$ with the same period L then
$$\frac{1}{L}\int_{-L/2}^{L/2} f(t)g^*(t)dt = \int_{n=-\infty}^{\infty} c_n d_n^*$$
where c_n and d_n are the coefficients in the complex Fourier series for $f(t)$ and $g(t)$ respectively. Deduce Parseval's theorem from this.

2. Write the triangular pulse
$$f(t) = \begin{cases} At/T + A & -T < t < 0 \\ -At/T + A & 0 < t < T \\ 0 & |t| > T \end{cases}$$
as a Fourier integral.

3. Solve the Euler-Cauchy equation
$$t^2\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y = 0.$$

4. Using the method of Fröbenius, or otherwise, find the general solution to
$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - \alpha t y = 0$$
where $\alpha > 0$.

Some useful formula

- A function with period L has the Fourier series expansion
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right).$$
where
$$a_0 = \frac{2}{L}\int_{-L/2}^{L/2} f(t)dt$$

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right)dt \\
b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right)dt
\end{aligned}
$$

- A function with period L has the Fourier series expansion

$$
f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\frac{2i\pi nt}{L}.
$$

where

$$
c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\exp\left(\frac{-2i\pi nt}{L}\right)dt
$$

- The Fourier integral or Fourier transform:

$$
\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt} \\
\widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{-ikt}
\end{aligned}
$$