MA22S3, sample for my questions on the summer exams, solutions to some questions

Six questions do four in a two hour exam.

1. (a) The Fourier series relies on the orthogonality of the sine and cosine functions: as an example, show

$$\int_{-\pi}^{\pi} \sin nt \cos mt dt = 0$$

for integers n and m.

(b) In a similar way it can be shown that

$$\int_{-\pi}^{\pi} \sin nt \sin mt dt = \pi\delta_{nm}, \qquad \int_{-\pi}^{\pi} \cos nt \cos mt dt = \pi\delta_{nm}$$

How are these identities used to derive formulas for the a_0, a_n and b_n in the Fourier series.

(c) Find the Fourier series of the function $f(t)$ with period 2π defined by

$$f(t) = t$$

*Solution:*Done in problem sheets.

2. Determine the Fourier transform of the one-off pulse:

$$f(t) = \begin{cases} 0 & |t| > T \\ -A & -T < t < 0 \\ A & 0 < t < T \end{cases}$$

*Solution:*Done in problem sheets

3. What are

$$\int_{-\infty}^{\infty} \delta(t)\frac{1+t^2}{1-t^2}dt, \qquad \int_{-\infty}^{\infty} \delta(t-1)e^{t-1}dt,$$
$$\int_{-5}^{5} \delta(t-3)t^2dt, \qquad \int_{-1}^{1} \delta(t-3)t^2dt$$

Calculate the Fourier transform of $\delta(t)$.

*Solution:*In the first case the integration region includes $t = 0$ where the delta function has a zero argument, so just set $t = 0$ in the integrand

$$\int_{-\infty}^{\infty} \delta(t) \frac{1+t^2}{1-t^2} dt = 1 \tag{1}$$

In the next one the delta function has zero argument when $t = 1$ and again that's included in the integration region

$$\int_{-\infty}^{\infty} \delta(t-1) e^{t-1} dt = e^0 = 1 \tag{2}$$

Next, $t = 3$ is the zero of the delta and so

$$\int_{-5}^{5} \delta(t-3) t^2 dt = 9 \tag{3}$$

Finally, $t = 3$ lies ourside the integration region so the answer is zero

$$\int_{-1}^{1} \delta(t-3) t^2 dt = 0 \tag{4}$$

As for the Fourier transform

$$\widetilde{\delta(k)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(t) e^{-ikt} = \frac{1}{2\pi} \tag{5}$$

4. Find the general solution of

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0, \qquad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = e^{5t}.$$

*Solution:*The first differential equation is homogeneous so we just substitute $y = \exp \lambda t$ giving

$$\lambda^2 - \lambda - 6 = 0 \tag{6}$$

Now that factorizes handily enough $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$ giving $\lambda = 3$, $\lambda = -2$ and solution

$$y = C_1 e^{3t} + C_2 e^{-2t} \tag{7}$$

Neither of these two lambda values is five, the exponent in the inhomogeneous equation, so use $y = C \exp 5t$ as an ansatz for the particular solution

$$(25 - 5 - 6)C = 1 \tag{8}$$

so $C = 1/14$ and

$$y = C_1 e^{3t} + C_2 e^{-2t} + \frac{1}{14} e^{5t} \tag{9}$$

5. What is the solution of

$$\frac{d^2 y}{dt^2} - 2\frac{dy}{dt} + y = e^t$$

with $y(0) = \dot{y}(0) = 1$?

*Solution:*First solve the homogeneous equation, the auxiliary equation is

$$\lambda^2 - 2\lambda + 1 = 0 \tag{10}$$

which factorizes to give $(\lambda - 1)^2 = 0$ and hence the solution to the homogeneous equation is

$$y = (C_1 + C_2 t)e^t \tag{11}$$

Unfortunately the exponent matches the exponent of the inhomogeneity, so we need to use $y = Ct^2 \exp t$ as the ansatz for the inhomogeneous problem

$$\begin{aligned} \frac{dy}{dt} &= 2Cte^t + Ct^2 e^t \\ \frac{d^2 y}{dt^2} &= 2Ce^t + 4Cte^t + Ct^2 e^t \end{aligned} \tag{12}$$

and hence, cancelling the $\exp t$

$$2C + 4Ct + Ct^2 - 4Ct - 2Ct^2 + Ct^2 = 1 \tag{13}$$

or $C = 1/2$ and

$$y = \left(C_1 + C_2 t + \frac{1}{2}t^2\right) e^t \tag{14}$$

6. Find a series solution for

$$(1+t)\frac{dy}{dt} + y = 0.$$

*Solution:*So we assume there is a solution of series form

$$y = \sum_{n=0}^{\infty} a_n t^n \tag{15}$$

and then calculate the terms

$$\begin{aligned} \frac{dy}{dt} &= \sum_{n=0}^{\infty} a_n n t^{n-1} \\ t\frac{dy}{dt} &= \sum_{n=0}^{\infty} a_n n t^n \end{aligned} \tag{16}$$

So there is only one term where the power of t is lower and we fix that with the usual change of variables

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} a_{n+1}(n+1)t^n \tag{17}$$

so the equation becomes

$$\sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + (n+1)a_n\right] t^n = 0 \tag{18}$$

and hence

$$a_{n+1} = -a_n \tag{19}$$

and a_0 is arbitrary.