MA22S3, sample for my questions on the summer exams

Six questions do four in a two hour exam.

1. (a) The Fourier series relies on the orthogonality of the sine and cosine functions: as an example, show

$$\int_{-\pi}^{\pi} \sin nt \cos mt dt = 0$$

for integers n and m.

(b) In a similar way it can be shown that

$$\int_{-\pi}^{\pi} \sin nt \sin mt dt = \pi\delta_{nm}, \qquad \int_{-\pi}^{\pi} \cos nt \cos mt dt = \pi\delta_{nm}$$

How are these identities used to derive formulas for the a_0, a_n and b_n in the Fourier series.

(c) Find the Fourier series of the function $f(t)$ with period 2π defined by

$$f(t) = t$$

2. Determine the Fourier transform of the one-off pulse:

$$f(t) = \begin{cases} 0 & |t| > T \\ -A & -T < t < 0 \\ A & 0 < t < T \end{cases}$$

3. What are

$$\int_{-\infty}^{\infty} \delta(t)\frac{1+t^2}{1-t^2}dt, \qquad \int_{-\infty}^{\infty} \delta(t-1)e^{t-1}dt,$$
$$\int_{-5}^{5} \delta(t-3)t^2dt, \qquad \int_{-1}^{1} \delta(t-3)t^2dt$$

Calculate the Fourier transform of $\delta(t)$.

4. Find the general solution of

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 0, \qquad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = e^{5t}.$$

5. What is the solution of

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t$$

with $y(0) = \dot{y}(0) = 1$?

6. Find a series solution for

$$(1+t)\frac{dy}{dt} + y = 0.$$

Some useful formula

- A function with period L has the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right)dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right)dt \end{aligned}$$

- A function with period L has the Fourier series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\frac{2i\pi nt}{L}.$$

where

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\exp\left(\frac{-2i\pi nt}{L}\right)dt$$

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikt} \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{-ikt} \end{aligned}$$