Credit will be given for the best three questions answered.

Each question is worth 20 marks.

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.
1. (a) Let \((M, g)\) be a manifold endowed with a metric. Let \(\nabla\) be the covariant derivative:
\[
\nabla_j \xi^i = \partial_j \xi^i + \xi^k \Gamma^i_{kj}, \quad \nabla_j \xi_i = \partial_j \xi_i - \xi_k \Gamma^k_{ij}.
\]
Show that a torsion-free connection that is compatible with the metric \((\nabla_k g_{ij} = 0)\) defines a unique connection, known as the Levi-Civita (or metric) connection. Obtain an expression for the Levi-Civita connection coefficients \(\Gamma^k_{ij}\). [5 marks]

**Solution:** We have
\[
\nabla_k g_{ij} = \partial_k g_{ij} - g_{pj} \Gamma^p_{ik} - g_{ip} \Gamma^p_{jk} = \partial_k g_{ij} - \Gamma_{i,kj} - \Gamma_{k,ij} = 0,
\]
where \(\Gamma_{j,ik} \equiv g_{pj} \Gamma^p_{ik} = g_{jp} \Gamma^p_{ik}\). Taking the sum of the last two equations and subtracting from the sum the first equation, one gets
\[
\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij} - 2\Gamma_{k,ij} = 0,
\]
where we took into account that the connection is torsion-free, and therefore \(\Gamma_{k,ij} = \Gamma_{k,ji}\). Thus, one finds
\[
\Gamma_{k,ij} = \frac{1}{2} \left( \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} \right) \quad \Rightarrow \quad \Gamma^k_{ij} = \frac{1}{2} g^{kp} \left( \partial_i g_{pj} + \partial_j g_{ip} - \partial_p g_{ij} \right),
\]
(b) Consider the 2-dimensional de Sitter space with the metric
\[
ds^2 = dt^2 - \cosh^2 t d\theta^2.
\]
i. Show that nontrivial components of the Levi-Civita connection are given by
\[
\Gamma^t_{\theta\theta} = \frac{1}{2} \sinh 2t, \quad \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \tanh t.
\]
while all the other connection components vanish. [3 marks]

**Solution:** We have \(g_{tt} = 1, \ g_{t\theta} = g_{\theta t} = 0, \ g_{\theta\theta} = -\cosh^2 t\). Since the metric depends only on \(t\), the nonvanishing components of the connection must have at least one index equal to \(t\). Thus one gets
\[
\Gamma^t_{\theta\theta} = -\frac{1}{2} g^{tt} \partial_t g_{\theta\theta} = \sinh t \cosh t = \frac{1}{2} \sinh 2t, \quad \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \frac{1}{2} g^{\theta\theta} \partial_t g_{\theta\theta} = \tanh t,
\]
and all the other connection components vanish.
ii. Show that the geodesics $x^m(\tau)$ satisfy

$$
\dot{\theta} \cosh^2 t = K, \quad t^2 \cosh^2 t = \cosh^2 t, 
$$

where a dot $'$ denotes differentiation with respect to $\tau$, and $K$ and $L$ are constants that should be identified.

**Solution:** The geodesic equations are

$$
\ddot{t} + \sinh t \cosh t \dot{\theta}^2 = 0 \\
\frac{d}{d\tau}(\dot{\theta} \cosh^2 t) = 0. 
$$

(4)

Thus

$$
\dot{\theta} \cosh^2 t = K, 
$$

(5)

where $K$ is the velocity in the $\theta$ direction at $t = 0$, and the equation for $t$ becomes

$$
\ddot{t} + \frac{\sinh t}{\cosh^3 t} K^2 = 0.
$$

(6)

Multiplying the equation by $2t$ one finds

$$
\frac{d}{dt}(t^2 - \frac{K^2}{\cosh^2 t}) = 0 \quad \Rightarrow \quad t^2 - \frac{K^2}{\cosh^2 t} = L,
$$

(7)

Note that the equation can be written in the form

$$
\dot{t}^2 - \cosh^2 t \dot{\theta}^2 = \frac{ds^2}{d\tau^2} = L.
$$

(8)

Thus the constant $L$ determines whether the geodesic is timelike ($L > 0$), null ($L = 0$) or spacelike ($L < 0$). Note that (7) implies that $L \geq -K^2$.

iii. The first equation above allows one to choose $\theta$ as a parameter of the geodesics. Then the second equation takes the form

$$
t^2 \frac{K^2}{\cosh^2 t} = K^2 + L \cosh^2 t,
$$

where $t$ denotes differentiation with respect to $\theta$. Find the change of the variable $t$: $t = f(v)$ which brings the equation to the form

$$
v'^2 = M^2 - v^2,
$$
where the constant $M$ should be identified. [5 marks]

**Solution:** The change is

$$v = \tanh t, \quad dt = dv \cosh^2 t, \quad \frac{1}{\cosh^2 t} = 1 - v^2. \quad (9)$$

By using the formulae one gets

$$v'^2 = M^2 - v^2, \quad M^2 = 1 + \frac{L}{K^2} \geq 0. \quad (10)$$

iv. Determine and sketch the geodesics (timelike, null and spacelike) passing through $(t, \theta) = (0, 0)$. [2 marks]

**Solution:** The geodesics are found by computing the integral

$$\theta = \pm \int \frac{dv}{\sqrt{M^2 - v^2}} = \pm \sin^{-1}\left(\frac{v}{M}\right) \quad \Rightarrow \quad v = \pm M \sin(\theta) \quad \Rightarrow \quad t = \pm \tanh^{-1}(M \sin(\theta)), \quad (11)$$

where we fixed a constant from the initial condition which also rules out the $M = 0$ case. Since $\tanh^{-1}(\pm 1) = \pm \infty$, the null geodesics ($M = 1$) diverge at $\theta = \pm \pi/2$, while the timelike geodesics $M > 1$ diverge at $\theta = \pm \sin^{-1} \frac{1}{M}$. The spacelike geodesics do not diverge.

The geodesics are sketched below.
2. The Riemann curvature tensor $R^i_{jkl}$ is defined by means of the formula
\[
[\nabla_k, \nabla_l] \xi^i = R^i_{jkl} \xi^j + T^i_{kl} \nabla_j \xi^i.
\]

(a) Use the formula to find explicit formulae for $R^i_{jkl}$ and $T^i_{kl}$ in terms of the connection coefficients $\Gamma^i_{jk}$. What is $T^j_{kl}$? [5 marks]

Solution: We have by definition
\[
\nabla_l \xi^i = \partial_l \xi^i + \xi^p \Gamma^i_{pl}, \quad \nabla_k \xi_l = \partial_k \xi_l - \xi^p \Gamma^p_{lk},
\]
and therefore
\[
[\nabla_k, \nabla_l] \xi^i = \partial_k (\partial_l \xi^i + \xi^p \Gamma^i_{pl}) + \nabla_l \xi^p \Gamma^i_{pl} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l)
\]
\[
= \partial_k \xi^p \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} + \nabla_l \xi^p \Gamma^i_{pl} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l)
\]
\[
= (\nabla_k \xi^p - \xi^q \Gamma^p_{qk}) \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} + \nabla_l \xi^p \Gamma^i_{pl} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l)
\]
\[
= -\xi^q \Gamma^p_{qk} \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} - \nabla_q \xi^i \Gamma^q_{lk} - (k \leftrightarrow l)
\]
\[
= (\partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{pk} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{jk}) \xi^j + (\Gamma^j_{kl} - \Gamma^i_{lk}) \nabla_j \xi^i.
\]
Thus
\[
R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{pk} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{jk},
\]
\[
T^j_{kl} = \Gamma^j_{kl} - \Gamma^i_{lk},
\]
and $T^j_{kl}$ is the torsion tensor.

(b) Let the manifold be endowed with a metric, and let the connection be torsion-free and compatible with the metric. List the algebraic symmetries the components of the Riemann tensor satisfy. How many linearly independent components does $R_{ijkl}$ have in arbitrary dimensions? Prove the formula. [8 marks]

Solution: We define $R_{ijkl} = g_{ip} R^p_{jkl}$. Then $R_{ijkl}$ satisfies the following algebraic relations
\[
R_{ijkl} + R_{ijlk} = 0,
\]
\[
R_{ijkl} + R_{jikl} = 0,
\]
\[
R_{ijkl} - R_{klij} = 0,
\]
\[
R_{ijkl} + R_{klij} + R_{ijkl} = 0.
\]

(15)
Thanks to $R_{ijkl} = R_{klij}$ one can think about $R_{ijkl}$ as a symmetric matrix $R_{AB} = R_{BA}$ where $A = ij$ and $B = kl$ are multi-indices. Since $R_{ijkl} = -R_{ijlk}$, $R_{ijkl} = -R_{jikl}$ the indices $A$ and $B$ take $n(n - 1)/2$ values where $n$ is the dimension of the manifold. Thus, a generic symmetric matrix $R_{AB}$ would have

$$\frac{1}{2} \frac{n(n - 1)}{2} \left( \frac{n(n - 1)}{2} + 1 \right) = \frac{n(n - 1)(n^2 - n + 2)}{8}$$

independent components. One has however to subtract from this number the number of independent relations due to $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. To this end we note that

$$R_{ijkl} \equiv R_{ijkl} - R_{ikjl} - R_{ilkj} = R_{ijkl} + R_{iklj} + R_{iljk} = 0,$$  

and it is skew-symmetric under permutations of $j, k, l$. Moreover, one gets

$$R_{[ijkl]} \equiv R_{ijkl} - R_{ikjl} - R_{ilkj} = R_{jilk} - R_{jlkk} - R_{jkli} = -R_{[jkl]},$$

and similarly

$$R_{[ijkl]} = R_{[klij]} = -R_{[ijkl]},$$

Now consider the skew-symmetric part of $R_{ijkl}$

$$R_{[ijkl]} \equiv R_{[ijkl]} - R_{[ikjl]} - R_{[ilkj]} - R_{[jilk]} = 4R_{[ijkl]} = 0.$$

Thus, the relation $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ together with the first three relations is equivalent to vanishing of the skew-symmetric part of $R_{ijkl}$. Since a fourth-rank skew-symmetric tensor has $n(n - 1)(n - 2)(n - 3)/4!$ independent components we finally get

$$\frac{n(n - 1)(n^2 - n + 2)}{8} - \frac{n(n - 1)(n^2 - 5n + 6)}{24} = \frac{n(n - 1)(3n^2 - 3n - n^2 + 5n)}{24} = \frac{n^2(n^2 - 1)}{12}.$$

(c) Prove Bianchi’s identities for the curvature tensor of a symmetric connection

$$\nabla_m R_{ijkl}^n + \nabla_i R_{mkl}^n + \nabla_k R_{iml}^n = 0.$$
Solution: We use normal coordinate, so that $\Gamma^k_{li} = 0$ at a given point. Then we have at this point

$$R^n_{ikl} = \partial_k \Gamma^n_{il} - \partial_l \Gamma^n_{ik} + \Gamma^n_{pk} \Gamma^p_{il} - \Gamma^n_{pl} \Gamma^p_{ik}$$

$$\partial_m R^n_{ikl} = \partial_m \partial_k \Gamma^n_{il} - \partial_m \partial_l \Gamma^n_{ik}$$

$$\partial_l R^n_{imk} = -\partial_l \partial_k \Gamma^n_{im} + \partial_l \partial_m \Gamma^n_{ik}.$$  \hfill (22)

$$\partial_k R^n_{ilm} = -\partial_k \partial_m \Gamma^n_{il} + \partial_k \partial_l \Gamma^n_{im}.$$  \hfill (23)

Thus

$$\nabla_m R^n_{ikl} + \nabla_l R^n_{imk} + \nabla_k R^n_{ilm} = \partial_m R^n_{ikl} + \partial_l R^n_{imk} + \partial_k R^n_{ilm} = 0.$$

Since Bianchi’s identities are tensor they hold in any coordinate system.

3. A hyperbolic $n$-space, $H^n$, is one sheet of the hyperboloid of two sheets realised as a surface

$$-(x^0)^2 + \sum_{i=1}^n (x^i)^2 = -1, \quad x^0 > 0,$$

in the Minkowski space $\mathbb{R}^{1,n}$ with the standard metric

$$ds^2 = -(dx^0)^2 + \sum_{i=1}^n (dx^i)^2.$$

(a) Plot $H^1$ in $\mathbb{R}^2$ (where it is a curve) by taking the $x^0$-axis as a vertical one. Plot a straight line through the point $(x^0, x^1) = (-1, 0)$ and a point of $H^1$.  \[1 \text{ mark}\]
(b) Prove that \( H^n \) is topologically an \( n \)-dimensional disc by considering the “stereographic” projection from the point \((x^0, x^1, \ldots, x^n) = (-1, 0, \ldots, 0)\) onto the plane \( x^0 = 0 \), i.e. by drawing a straight line through the point \((-1, 0, \ldots, 0)\) and a point of \( H^n \). Let \( u^i \) be the coordinates of the point of intersection of the line with the plane \( x^0 = 0 \). Express \( x^0, x^1, \ldots, x^n \) of \( H^n \) in terms of \( u^i \). The “stereographic” coordinates \( u^i \) of the \( n \)-dimensional disc provide the Poincaré disc model of \( H^n \).

What is the radius of the disc? [4 marks]

Solution: The stereographic projection maps \( H^1 \) onto the interval \((-1, 1)\). Then, it is clear that due to the rotational symmetry of \( H^n \) it is mapped to the \( n \)-dimensional disc of radius 1.

A line through the point \((-1, 0, \ldots, 0)\) and a point of \( H^n \) can be parametrised as

\[
x^0 = -1 + t, \quad x^i = u^i t, \quad i = 1, \ldots, n,
\]

where \( t = 1 \) corresponds to the point of intersection of the line with the plane \( x^0 = 0 \), and \( u^i \) are the coordinates of the point. Then, the value of \( t \) corresponding to the point of intersection of the line with the hyperboloid is

\[
t = \frac{2}{1 - \bar{u}^2}, \quad \bar{u}^2 = \sum_{j=1}^{n} (u^j)^2,
\]

and therefore

\[
x^0 = \frac{1 + \bar{u}^2}{1 - \bar{u}^2}, \quad x^i = \frac{2u^i}{1 - \bar{u}^2}.
\]

Since \( x^0 > 0 \) we get again that \( \bar{u}^2 < 1 \), and therefore \( H^n \) is topologically the \( n \)-dimensional disc of radius 1.

(c) The metric on the hyperbolic space \( H^n \) is the metric induced from the ambient Minkowski metric. Find the metric, \( ds_{H^n}^2 \), on \( H^n \) and the volume form, \( \Omega_{H^n} \), of \( H^n \) in terms of the coordinates \( u^i \). [3 marks]

Solution:
The induced metric on $H^n$ is equal to

$$ds^2_{H^n} = (dx^i)^2 - (dx^0)^2 = \left( \frac{2du^i}{1-u^2} + \frac{4u^i u^j du^j}{(1-u^2)^2} \right)^2 - \left( \frac{4u^i du^i}{1-u^2} \right)^2$$

$$= \frac{4(du^i)^2}{(1-u^2)^2} + \frac{16u^i u^j du^i du^j}{(1-u^2)^3} + \frac{16\bar{u}^2 u^i u^j du^i du^j}{(1-u^2)^4} - \frac{16u^i u^j du^i du^j}{(1-u^2)^4}$$

$$= \frac{4(du^i)^2}{(1-u^2)^2} = e^{2\phi}(du^i)^2,$$

(27)

where $\phi = \log 2 - \log(1-u^2)$. Thus, the induced metric is conformally flat.

The volume form is

$$\Omega_{S^n} = \frac{2^n}{(1-u^2)^n} du^1 \wedge \cdots \wedge du^n. \quad (28)$$

(d) Use the coordinates $u^i$ to calculate the components of the Levi-Civita connection of $H^n$. [4 marks]

Solution:

The connection is given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kp}(\partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij}) = \partial_i \phi \delta^k_j + \partial_j \phi \delta^k_i - \partial_k \phi \delta_{ij}. \quad (29)$$

(e) Calculate the components of the Riemann tensor, Ricci tensor, and scalar curvature of $H^n$. [8 marks]

Solution:

The equation of the hyperboloid $-(x^0)^2 + \sum_{i=1}^n (x^i)^2 = -1$ is invariant under the action of the Lorentz group $SO(1, n)$ in the Minkowski space $\mathbb{R}^{1,n}$. Let $p$ be an arbitrary point of $H^n$ with coordinates $x^i_p$. One can find a Lorentz transformation $A$ such that in the transformed coordinates $y^i = A^i_j x^j$ the point $p$ has coordinates $(y^i_p) = (1, 0, \ldots, 0)$. We then use the stereographic projection from the point $(y^0_p) = (-1, \ldots, 0)$ onto the $y^0 = 0$ plane. The corresponding coordinates $u^i$ are normal in the vicinity of the point $p$.

The derivatives of the connection at $p$ are

$$\partial_m \Gamma^k_{ij} = \partial_m \partial_i \phi \delta^k_j + \partial_m \partial_j \phi \delta^k_i - \partial_m \partial_k \phi \delta_{ij} = 2(\delta_m \delta^k_j + \delta_m \delta^k_i - \delta^k_m \delta_{ij}). \quad (30)$$
The Riemann tensor at \( p \) is therefore equal to

\[
R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} = -4\delta_{jk}\delta^i_l + 4\delta_{jl}\delta^i_k \quad \Rightarrow \quad R_{ijkl} = -16(\delta_{jl}\delta^i_k - \delta_{jk}\delta^i_l) = -g_{ij}g_{ik} + g_{jk}g_{il}.
\]

(31)

Since the last equality is tensor it holds in any coordinate system. So,

\[
R_{ijkl} = -g_{ij}g_{ik} + g_{jk}g_{il}, \quad R_{ij} = -(n - 1)g_{ij}, \quad R = -n(n - 1).
\]

(32)

Thus, \( H^n \) is an \( n \)-dimensional Riemann manifold of constant negative curvature.

4. Consider the set \( \widetilde{SL}(2, \mathbb{R}) \) of all transformations of the real line of the form

\[
x \rightarrow x + 2\pi a + \frac{1}{i} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}},
\]

where \( x \in \mathbb{R}, \, a \in \mathbb{R}, \, z \in \mathbb{C}, \, |z| < 1 \) and \( \ln \) is the main branch of the natural logarithmic function, i.e. the continuous branch determined by \( \ln 1 = 0 \).

(a) Show that \( \widetilde{SL}(2, \mathbb{R}) \) is a connected 3-dimensional Lie group. \[10\text{ marks}\]

**Solution:** It is obvious that \( \widetilde{SL}(2, \mathbb{R}) \) is a connected 3-dimensional manifold because it is \( \mathbb{R} \times D_2 \) where \( D_2 \) is the disk \( |z| < 1 \). To show that it is a Lie group we consider the composition of two transformations with parameters \( a_i, \, z_i, \, i = 1, 2 \). We get

\[
T_{g_2}T_{g_1}(x) = T_{g_2}(x + 2\pi a_1 + \frac{1}{i} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}})
\]

\[
= x + 2\pi a_1 + \frac{1}{i} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}} + 2\pi a_2 + \frac{1}{i} \ln \frac{1 - z_2 e^{-i(x+2\pi a_1)} + \frac{1}{i} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}}}{1 - \bar{z}_2 e^{ix}}
\]

\[
= x + 2\pi(a_1 + a_2) + \frac{1}{i} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}} - \frac{z_2 e^{-i(x+2\pi a_1)} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}}}{1 - \bar{z}_2 e^{ix}}
\]

\[
= x + 2\pi(a_1 + a_2) + \frac{1}{i} \ln \frac{1 - z_1 e^{-ix} - z_2 (1 - \bar{z}_1 e^{ix}) e^{-i(x+2\pi a_1)}}{1 - \bar{z}_1 e^{ix} - \bar{z}_2 (1 - z_1 e^{-ix}) e^{i(x+2\pi a_1)}}
\]

\[
= x + 2\pi(a_1 + a_2) + \frac{1}{i} \ln \frac{1 + z_2 \bar{z}_1 e^{-2\pi i a_1} - (\bar{z}_1 + z_2 e^{-2\pi i a_1}) e^{-ix}}{1 + \bar{z}_2 z_1 e^{2\pi i a_1} - (\bar{z}_1 + z_2 e^{2\pi i a_1}) e^{ix}}
\]

\[
= x + 2\pi(a_1 + a_2) + \frac{1}{i} \ln \frac{1 + z_2 \bar{z}_1 e^{-2\pi i a_1} - (\bar{z}_1 + z_2 e^{-2\pi i a_1}) e^{-ix}}{1 + \bar{z}_2 z_1 e^{2\pi i a_1} - (\bar{z}_1 + z_2 e^{2\pi i a_1}) e^{ix}}
\]

\[
= x + 2\pi(a_1 + a_2) + \frac{1}{i} \ln \frac{1 - z_2 \bar{z}_1 e^{-2\pi i a_1}}{1 - \bar{z}_2 z_1 e^{2\pi i a_1}} - \frac{1}{i} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}}
\]

\[
= x + 2\pi a_2 + \frac{1}{i} \ln \frac{1 - z_2 \bar{z}_1 e^{-2\pi i a_1}}{1 - \bar{z}_2 z_1 e^{2\pi i a_1}} - \frac{1}{i} \ln \frac{1 - z_1 e^{-ix}}{1 - \bar{z}_1 e^{ix}} = T_{g_2 g_1}(x),
\]

(33)
where the parameters $a_{21}$ and $z_{21}$ of $T_{g_{2g_1}}$ are given by
\[ a_{21} = a_1 + a_2 + \frac{1}{2\pi i} \ln \frac{1 + z_2 \bar{z}_1 e^{-2\pi i a_1}}{1 + \bar{z}_2 z_1 e^{2\pi i a_1}} , \quad z_{21} = \frac{z_1 + z_2 e^{-2\pi i a_1}}{1 + z_2 \bar{z}_1 e^{-2\pi i a_1}}. \] (34)

Note that $|z_{21}| < 1$ if $|z_1| < 1$ and $|z_2| < 1$.

(b) Calculate the Lie algebra $sl(2, \mathbb{R})$ of the Lie group $SL(2, \mathbb{R})$. [4 marks]

Solution: The $sl(2, \mathbb{R})$ algebra is the space of real traceless matrices. We choose as its basis the matrices
\[ L_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (35)

They satisfy the $sl(2, R)$ Lie algebra commutation relations
\[ [L_3, L_1] = 2L_2, \quad [L_3, L_2] = 2L_1, \quad [L_1, L_2] = -2L_3. \] (36)

(c) Calculate the Lie algebra of $\tilde{SL}(2, \mathbb{R})$ and show that it is isomorphic to $sl(2, \mathbb{R})$. [6 marks]

Solution: Since the identity element of $\tilde{SL}(2, \mathbb{R})$ corresponds to $a = z = 0$, one gets expanding (34)
\[ a_{21} = a_1 + a_2 + \frac{1}{2\pi i} (z_2 \bar{z}_1 - \bar{z}_2 z_1) + \ldots , \quad z_{21} = z_1 + z_2 - 2\pi i a_1 z_2 + \ldots \] (37)

Introducing $z_k = u_k + iv_k, k = 1, 2$, one gets
\[ a_{21} = a_1 + a_2 + \frac{1}{\pi} (v_2 u_1 - u_2 v_1) + \ldots , \quad u_{21} = u_1 + u_2 + 2\pi a_1 v_2 + \ldots , \quad v_{21} = v_1 + v_2 - 2\pi a_1 u_2 + \ldots \] (38)

Now, choosing $x^1 = u, \ x^2 = \pi a, \ x^3 = v$, as the local coordinates, we get the nonzero structure constants of the Lie algebra
\[ c^1_{32} = 2, \quad c^2_{31} = 2, \quad c^3_{12} = -2. \] (39)

Thus, the Lie algebra of $\tilde{SL}(2, \mathbb{R})$ is
\[ [J_3, J_2] = 2J_1, \quad [J_3, J_1] = 2J_2, \quad [J_1, J_2] = -2J_3, \] (40)

which the same as (36).