Covariant Differentiation and the Metric, and the Curvature Tensor

1 Covariant Differentiation

The differential $d$ transforms a skew-symmetric tensor $T = (T_{i_1...i_k})$ to another skew-symmetric tensor $dT$

$$(dT)_{i_1...i_{k+1}} = \sum_{q=1}^{k+1} (-1)^{q-1} \frac{\partial T_{i_1...\hat{i}_q...i_{k+1}}}{\partial x^q} = \sum_{q=1}^{k+1} (-1)^{q-1} \partial_q T_{i_1...i_q...i_{k+1}}, \quad (1.1)$$

where $\hat{i}_q$ means that the index $i_q$ is omitted, and we introduce the notation

$$
\partial_k T_{j_1...j_q}^{i_1...i_p} = \frac{\partial T_{j_1...j_q}^{i_1...i_p}}{\partial x^k}.
$$

(1.2)

In particular for $k = 1$

$$(dT)^i_j = \frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j} = \partial_j T_i - \partial_i T_j. \quad (1.3)$$

It is important that we have the difference here because $\partial_j T_i$ is not a tensor.

Let us find how $\partial_k T_i$ and $\partial_k T^i$ transform under arbitrary coordinate changes

$$x^i = x^i(z^1, ..., z^n), \quad i = 1, ..., n. \quad (1.4)$$

We have

$$
\partial_q \tilde{T}_j = \frac{\partial \tilde{T}_j}{\partial z^q} = \partial \left( T_i \frac{\partial x^i}{\partial z^j} \right) = \frac{\partial T_i}{\partial z^q} \frac{\partial x^i}{\partial z^j} + T_i \frac{\partial^2 x^i}{\partial z^q \partial z^j} = \frac{\partial T_i}{\partial x^p} \frac{\partial x^i}{\partial z^j} + T_i \frac{\partial^2 x^i}{\partial x^p \partial z^j}. \quad (1.5)
$$

Thus,

$$
\partial_q \tilde{T}_j = \partial_p T_i \frac{\partial x^p}{\partial z^q} \frac{\partial x^i}{\partial z^j} + T_i \frac{\partial^2 x^i}{\partial z^q \partial z^j}, \quad (1.6)
$$

and the appearance of $T_i \frac{\partial^2 x^i}{\partial z^q \partial z^j}$ shows that it is not a tensor but its skew-symmetric part is.

Similarly for $\partial_k T^i$ we get

$$
\partial_q \tilde{T}^j = \frac{\partial \tilde{T}^j}{\partial z^q} = \partial \left( T^i \frac{\partial z^j}{\partial x^i} \right) = \frac{\partial T^i}{\partial z^q} \frac{\partial z^j}{\partial x^i} + T^i \frac{\partial \partial z^j}{\partial z^q \partial x^i} = \partial_p T^i \frac{\partial x^p}{\partial z^q} \frac{\partial z^j}{\partial x^i} + T^i \frac{\partial^2 z^j}{\partial x^p \partial x^i \partial z^q}. \quad (1.7)
$$

In particular the divergence $\partial_j T^i$ transforms as

$$
\partial_j \tilde{T}^j = \partial_j T^i + T^i \frac{\partial^2 z^j}{\partial x^p \partial x^i \partial z^q} \frac{\partial x^p}{\partial z^j}, \quad (1.8)
$$

and it is not a scalar.

On the other hand in Euclidean space with Euclidean coordinates $x^1, ..., x^n$ we need the quantities like $\partial_k T_{j_1...j_q}^{i_1...i_p}$, and we want them to transform like tensors under general coordinate transformations. So, in $\mathbb{R}^n$ we can introduce the tensor denoted by $\nabla_k T_{j_1...j_q}^{i_1...i_p}$ which is equal to
\[
\frac{\partial T_{j_1 \ldots j_q}^{i_1 \ldots i_p}}{\partial x^k} \text{ in Euclidean coordinates. Since it is a tensor we can find its components in any system with coordinates } z^1, \ldots, z^n \text{.}
\]

\[
\nabla_r \tilde{T}^{(k)}_{(l)} = \nabla_s T^{(i)}_{(j)} \frac{\partial x^s}{\partial z^r} \frac{\partial x^{(i)}}{\partial x^{(j)}} \frac{\partial z^{(k)}}{\partial x^{(l)}},
\]

where \((k) = k_1 \ldots k_p\), \((l) = l_1 \ldots l_q\), and so on, and

\[
\frac{\partial x^{(j)}}{\partial z^{(l)}} = \frac{\partial x^{j_1}}{\partial z^{l_1}} \cdots \frac{\partial x^{j_p}}{\partial z^{l_q}}, \quad \frac{\partial z^{(k)}}{\partial x^{(l)}} = \frac{\partial z^{k_1}}{\partial x^{l_1}} \cdots \frac{\partial z^{k_p}}{\partial x^{l_p}}, \quad \nabla_s T^{(i)}_{(j)} = \frac{\partial T_{j_1 \ldots j_q}^{i_1 \ldots i_p}}{\partial x^s} = \partial_s T^{(i)}_{(j)}.
\]

Notation

\[
T_{j_1 \ldots j_q}^{i_1 \ldots i_p} \equiv \nabla_k T_{j_1 \ldots j_q}^{i_1 \ldots i_p}.
\]

For simplicity consider \((T^i)\) and \((T_i)\). We have (note that \(\tilde{T}^k = T^i \frac{\partial x^k}{\partial z^i}\) and \(T^i = \tilde{T}^k \frac{\partial x^i}{\partial z^k}\))

\[
\nabla_r \tilde{T}^k = \partial_s T^i \frac{\partial x^s}{\partial z^r} \frac{\partial z^k}{\partial x^i} = \frac{\partial T^i}{\partial z^r} \frac{\partial z^k}{\partial x^i} = \frac{\partial}{\partial z^r} \left( T^i \frac{\partial z^k}{\partial x^i} \right) - T^i \frac{\partial}{\partial z^r} \frac{\partial z^k}{\partial x^i}
\]

\[
= \frac{\partial \tilde{T}^k}{\partial z^r} - \tilde{T}_s \frac{\partial x^s}{\partial z^r} \frac{\partial^2 z^k}{\partial x^m \partial x^i} \frac{\partial x^m}{\partial z^r}.
\]

Introducing

\[
\Gamma_{sr}^k = - \frac{\partial x^i}{\partial z^s} \frac{\partial x^m}{\partial z^r} \frac{\partial^2 z^k}{\partial x^m \partial x^i},
\]

we get

\[
\nabla_r \tilde{T}^k = \frac{\partial \tilde{T}^k}{\partial z^r} + \Gamma_{sr}^k \tilde{T}^s.
\]

Thus, we have proven

**Theorem 28.1.2.** Let \((T^i)\) be a vector field, and let \(\nabla_k T^i\) be a tensor given in terms of Euclidean coordinates \(x^1, \ldots, x^n\) by the formula \(\nabla_k T^i = \frac{\partial T^i}{\partial x^k}\). Then, in arbitrary coordinates \(z^1, \ldots, z^n\) the transformed components \(\nabla_k \tilde{T}^i\) are given by the formula

\[
\nabla_r \tilde{T}^k = \tilde{T}_r^k = \frac{\partial \tilde{T}^k}{\partial z^r} + \Gamma_{sr}^k \tilde{T}^s,
\]

where the coefficients \(\Gamma_{sr}^k\) are defined in (1.13).

Similarly, we have

**Theorem 28.1.3.** Let \((T_i)\) be a covector field, and let \(\nabla_k T_i\) be a tensor given in terms of Euclidean coordinates \(x^1, \ldots, x^n\) by the formula \(\nabla_k T_i = \frac{\partial T_i}{\partial x^k}\). Then, in arbitrary coordinates \(z^1, \ldots, z^n\) the transformed components \(\nabla_k \tilde{T}_i\) are given by the formula

\[
\nabla_r \tilde{T}_i = \tilde{T}_{i,r} = \frac{\partial \tilde{T}_i}{\partial z^r} - \Gamma_{kr}^i \tilde{T}_s.
\]

**Proof.** We have (note that \(\tilde{T}_k = T_i \frac{\partial x^i}{\partial z^k}\) and \(T_i = \tilde{T}^k \frac{\partial x^i}{\partial z^k}\))

\[
\nabla_r \tilde{T}_k = \partial_s T_i \frac{\partial x^s}{\partial z^r} \frac{\partial x^i}{\partial z^k} = \frac{\partial T_i}{\partial z^r} \frac{\partial x^i}{\partial z^k} = \frac{\partial}{\partial z^r} \left( T_i \frac{\partial x^i}{\partial z^k} \right) - T_i \frac{\partial^2 x^i}{\partial z^r \partial z^k}
\]

\[
= \frac{\partial \tilde{T}_k}{\partial z^r} - \tilde{T}_s \frac{\partial x^s}{\partial z^r} \frac{\partial^2 x^i}{\partial x^m \partial x^i} \frac{\partial x^m}{\partial z^r} = \frac{\partial \tilde{T}_k}{\partial z^r} - \Gamma_{kr}^i \tilde{T}_s.
\]
Comparing the two expressions, one gets
\[ \frac{\partial z^s}{\partial x^i} \frac{\partial^2 x^i}{\partial z^r \partial z^k} = \frac{\partial}{\partial z^r} \left( \frac{\partial z^s}{\partial x^i} \frac{\partial x^i}{\partial z^k} \right) - \left( \frac{\partial}{\partial z^r} \frac{\partial z^s}{\partial x^i} \right) \frac{\partial x^i}{\partial z^k} = \frac{\partial}{\partial z^r} \delta^s_k \frac{\partial^2 x^i}{\partial x^m \partial x^i} \frac{\partial x^i}{\partial z^k} = \Gamma^r_k. \] (1.18)

Generalising the computations above, one gets

**Theorem 28.1.4.** Let \( T^{(i)}_{(j)} \) be the components of a tensor of type \((p, q)\), and let \( \nabla_k T^{(i)}_{(j)} \) be a tensor given in terms of Euclidean coordinates \(x^1, \ldots, x^n\) by the formula \( \nabla_k T^{(i)}_{(j)} = \frac{\partial T^{(i)}_{(j)}}{\partial x^k} \). Then, in arbitrary coordinates \(z^1, \ldots, z^n\) the transformed components \( \nabla_r \bar{T}^{(k)}_{(l)} \) are given by the formula

\[ \nabla_r \bar{T}^{(k)}_{(l)} \equiv \bar{T}^{(k)}_{(l);r} = \frac{\partial \bar{T}^{(k)}_{(l)}}{\partial z^r} + \sum_{a=1}^{p} \Gamma_{sr}^{ka} \bar{T}^{k_{1\ldots a \rightarrow s}...k_p}_{l_{1\ldots l_q}} - \sum_{a=1}^{q} \Gamma_{l_{a \rightarrow s}r}^{k} \bar{T}^{k_{1\ldots a \rightarrow s}...k_p}_{l_{1\ldots l_q}}. \] (1.19)

For second-rank tensors one gets
\[ \nabla_r \bar{T}^{k}_{l} \equiv \bar{T}^{k}_{l;r} = \frac{\partial \bar{T}^{k}_{l}}{\partial z^r} + \Gamma_{sr}^{kl} \bar{T}^{s}_{r} - \Gamma_{lr}^{sr} \bar{T}^{k}_{s}, \]
\[ \nabla_r \bar{T}^{lm}_{r} \equiv \bar{T}^{lm}_{r;r} = \frac{\partial \bar{T}^{lm}_{r}}{\partial z^r} - \Gamma_{r}^{sr \rightarrow l} \bar{T}^{sm}_{r} - \Gamma_{mr}^{sr} \bar{T}^{l}_{r s}, \] (1.20)
\[ \nabla_r \bar{T}^{km}_{r} \equiv \bar{T}^{km}_{r;r} = \frac{\partial \bar{T}^{km}_{r}}{\partial z^r} + \Gamma_{sr}^{km} \bar{T}^{sm}_{r} + \Gamma_{mr}^{sm} \bar{T}^{l}_{r s}. \]

Let us now determine how \( \Gamma^k_i \) transform under an arbitrary coordinate change \(z^i = z^i(z'), i = 1, \ldots, n\). In what follows to simplify the notations we use the convention
\[ z^i \equiv z^i, \quad i = 1, \ldots, n, \quad \text{e.g. } z^{i'} = z^2. \] (1.21)

Then, we have
\[ \nabla_k \bar{T}^{i}_{k} = \frac{\partial \bar{T}^{i}_{k}}{\partial z^k} - \Gamma_{ik}^{r} \bar{T}^{r}_{r}, \quad \text{in } z \text{ coordinates}, \]
\[ \nabla_k \bar{T}^{i}_{k} = \frac{\partial \bar{T}^{i}_{k}}{\partial z^{i'}} - \Gamma_{ik}^{i'} \bar{T}^{i'}_{r'}, \quad \text{in } z' \text{ coordinates}. \] (1.22)

Since they are tensors
\[ \nabla_k \bar{T}^{i}_{k} = \nabla_k \bar{T}^{i}_{k} \frac{\partial z^i}{\partial z^{i'}} \frac{\partial z^k}{\partial z^{k'}} = \delta^i_k \bar{T}^i_{k} \frac{\partial z^i}{\partial z^{i'}} \frac{\partial z^k}{\partial z^{k'}} - \Gamma_{ik}^{r} \bar{T}^{i}_{r} \frac{\partial z^i}{\partial z^{i'}} \frac{\partial z^k}{\partial z^{k'}}. \] (1.23)

On the other hand eq.(1.6) shows how to express \( \partial_k \bar{T}^{i}_{k} \) in terms of \( \partial_k \bar{T}^{i}_{k} \)
\[ \nabla_k \bar{T}^{i}_{k} = \frac{\partial \bar{T}^{i}_{k}}{\partial z^{k'}} - \Gamma_{ik}^{i'} \bar{T}^{i'}_{r'} = \delta^i_k \bar{T}^i_{k} \frac{\partial z^i}{\partial z^{i'}} \frac{\partial z^k}{\partial z^{k'}} + \bar{T}^{i}_{k} \frac{\partial^2 z^r}{\partial z^r \partial z^{k'}} - \Gamma_{ik}^{i'} \bar{T}^{i'}_{r'} \frac{\partial z^r}{\partial z^{k'}}. \] (1.24)

Comparing the two expressions, one gets
\[ \Gamma_{ik}^{r} \frac{\partial z^r}{\partial z^{k'}} = \frac{\partial}{\partial z^{k'}} \frac{\partial z^i}{\partial z^{i'}} \frac{\partial z^k}{\partial z^{k'}} + \frac{\partial^2 z^r}{\partial z^{i'} \partial z^{k'}}. \] (1.25)

\[ ^1\text{Obviously for zero-rank tensors (functions or scalars) } T, \text{ one gets } \nabla_r T = \partial_r T. \]
Multiplying by $\frac{\partial s'}{\partial z^r}$, and summing over $r$ one gets
\[
\Gamma^s'_{ikr} = \Gamma^r_{ik} \frac{\partial z^i}{\partial z'^{k'}} \frac{\partial z'^{k'}}{\partial z^r} + \frac{\partial^2 z^r}{\partial z^i \partial z^{k'}} \frac{\partial z'^{k'}}{\partial z^r} .
\] (1.26)

This formula motivates the general definition

**Def 28.1.5.** An operation of **covariant differentiation** of tensors is said to be defined if we are given, in terms of any system of coordinates $z^1, \ldots, z^n$ a family of functions $\Gamma^k_{pq}(z)$ which transform under arbitrary coordinate changes $z = z(z')$ according to the formula (1.26). The quantities $\Gamma^k_{pq}(z)$ are called **Christoffel symbols**. The covariant derivatives of tensors are given by eq. (1.19) which for zero-rank tensors (functions or scalars) $T$ coincide with partial derivatives $\nabla_r T = \partial_r T$, and for vectors and covectors take the form
\[
\nabla_k T^i = \frac{\partial T^i}{\partial z^k} + \Gamma^i_{rk} T^r , \quad \nabla_k T_i = \frac{\partial T_i}{\partial z^k} - \Gamma^r_{ik} T^r .
\] (1.27)

**Def.** An operation of covariant differentiation is called a **connection**.

**Def.** A connection is said to be **Euclidean or affine** if there exist coordinates $x^1, \ldots, x^n$ in terms of which $\Gamma^i_{jk} = 0$. Such coordinates are often called Euclidean or affine.

It is clear from (1.26) that Christoffel symbols are not components of a tensor. On the other hand
\[
T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i = \Gamma_{[jk]}^i ,
\] (1.28)
is a tensor because the second term in (1.26) cancels out. It is called the **torsion tensor**.

**Def 28.1.5.** A connection $\Gamma_{jk}^i$ is said to be **symmetric or torsion-free** if the torsion tensor is identically zero, i.e. if $\Gamma_{jk}^i = \Gamma_{kj}^i$.

For example, if the connection is affine then in Euclidean coordinates $\Gamma_{jk}^i = 0$, and therefore $T_{jk}^i = 0$. Since it is a tensor, it is equal to zero in any coordinate system.

The two main properties of the operation of covariant differentiation are

1. The operation is linear, and commutes with the operation of contraction. The linearity is obvious, and to show the commutativity it is sufficient to consider tensors of rank $(1,1)$. Since $T = T_k^k$ is a scalar $\nabla_r T = \partial_r T$. Then we get
\[
\nabla_r T_k^k = \frac{\partial T_k^k}{\partial z^r} + \Gamma^k_{sr} T^s_k - \Gamma^s_{kr} T^s_k = \partial_r T .
\] (1.29)

2. The covariant derivative of a product $T^{(i)(j)}_{(p)(q)} = R^{(i)}_{(p)s} S^{(j)}_{(q)}$ of tensors is calculated using the usual Leibniz product rule
\[
\nabla_k T^{(i)(j)}_{(p)(q)} = (\nabla_k R^{(i)}_{(p)s}) S^{(j)}_{(q)} + R^{(i)}_{(p)} (\nabla_k S^{(j)}_{(q)}) .
\] (1.30)

It follows from (1.19), and the product rule for usual derivatives.

In fact these properties together with the formulae for the covariant derivatives of scalars, vectors and covectors uniquely determine the tensor operation of covariant differentiation corresponding to a given connection $\Gamma_{jk}^i$.

**Theorem 28.2.6.** Let $\Gamma_{jk}^i$ be a connection. If a tensor operation $\nabla_k$ satisfies the following four conditions
1. The operation is linear, and commutes with the operation of contraction,

2. On zero-rank tensors (functions) $T$ the operation coincides with the partial derivative $\nabla_k T = \partial_k T$,

3. On vectors and covectors the operation is given by (1.27),

4. On a product $T^{(i)(j)} = R^{(i)}_{(p)} S^{(j)}_{(q)}$ of tensors the operation acts according to the Leibniz rule

\[
\nabla_k T^{(i)(j)} = (\nabla_k R^{(i)}_{(p)}) S^{(j)}_{(q)} + R^{(i)}_{(p)} (\nabla_k S^{(j)}_{(q)}) ,
\]

then it is the operation of covariant differentiation determined by the $\Gamma^i_{jk}$, and on general tensors it is given by (1.19).

**Proof.** Let $e_1, \ldots, e_n$ be the standard basis of vector fields, and let $e^1, \ldots, e^n$ be the dual basis of covector fields. The components of these tensors are $(e_i)^j = \delta^j_i = (e^j)_i$. For a vector $\xi = \xi^i e_i$ we have

\[
\nabla_k \xi = \left( \frac{\partial \xi^a}{\partial x^k} + \Gamma^a_{jk} \xi^j \right) e_a ,
\]

and therefore if $\xi = e_i$ we get

\[
\nabla_k e_i = \Gamma^a_{ik} e_a .
\]

Analogously,

\[
\nabla_k e^i = -\Gamma^i_{ak} e^a .
\]

These equations can be viewed as defining $\Gamma^i_{jk}$. Every tensor $T$ with components $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ has the form

\[
T = T^{i_1 \ldots i_p}_{j_1 \ldots j_q} e_i \otimes \cdots \otimes e_{i_p} \otimes e^j \otimes \cdots \otimes e^j ,
\]

where the components $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ should be considered as scalars i.e. usual functions because the bases $(e_i)$ and $(e^j)$ transform under a change of coordinates. Consider for simplicity tensors of type $(0,2)$ only. Then, $T = T_{ij} e^i \otimes e^j$, and by using the conditions (1-4), we get

\[
\nabla_k (T) = \nabla_k (T_{ij}) e^i \otimes e^j + T_{ij} (\nabla_k e^i) \otimes e^j + T_{ij} e^i \otimes (\nabla_k e^j) \\
= \partial_k T_{ij} e^i \otimes e^j - T_{ij} \Gamma^a_{ak} e^a \otimes e^j - T_{ij} e^i \otimes \Gamma^j_{ak} e^a \\
= (\partial_k T_{ij} - T_{ij} \Gamma^k_{ik} - T_{il} \Gamma^l_{jk}) e^i \otimes e^j .
\]

Thus, the components of $\nabla_k T$ have the form

\[
\partial_k T_{ij} - T_{ij} \Gamma^k_{ik} - T_{il} \Gamma^l_{jk} ,
\]

which proves the theorem for tensors of type $(0,2)$. 

5
2 Parallel Transport of Tensor Fields

Let $\xi \in T_PM$, $P \in M$, $\dim M = n$, and let $T = (T^{(i)}_{(j)})$, $(i) \equiv i_1, \ldots, i_p$, $(j) \equiv j_1, \ldots, j_q$ be a tensor of type $(p,q)$.

**Def.** The directional derivative of $T$ at $P$ along (or relative to) the vector $\xi$

$$\nabla_\xi T^{(i)}_{(j)} = \xi^k \nabla_k T^{(i)}_{(j)}, \quad (2.38)$$

is a tensor of the same type $(p,q)$.

For a scalar $f$ (rank-zero tensor)

$$\nabla_\xi f = \xi^k \partial_k f = \partial_\xi f, \quad (2.39)$$

it coincides with the directional derivative of a function. Let $\xi(t)$ be a velocity vector of some curve $C$: $x^i = x^i(t)$, $\xi^i(t) = dx^i/dt$, $i = 1, \ldots, n$. If $\partial_\xi f = 0$ for all points of $C$ then

$$f(x^1(t), \ldots, x^n(t)) = \text{const}. \quad (2.40)$$

If we have a vector or a tensor field then the question whether it is constant along $C$ is meaningless because a tensor of rank $> 0$ has different components in different coordinates. Thus, even if the tensor components are constant in one system of coordinates they are not in another (unless it is the zero tensor). A given connection (covariant differentiation) provides us with a way to compare two tensors attached to different points of $M$.

**Def 29.1.1.** Let $\Gamma^i_{kj}$ be a connection defined on a manifold $M$ with local coordinates $x^1, \ldots, x^n$ (in some chart $U$), and let $x^i(t), a \leq t \leq b$, be a segment $C$ of an arbitrary curve. We say that a tensor field $T$ is covariantly constant or parallel along $C$ if

$$\nabla_\xi T = \xi^k \nabla_k T = 0, \quad a \leq t \leq b, \quad \xi^k = \frac{dx^k}{dt}. \quad (2.41)$$

For vector fields one gets

$$\nabla_\xi T^i = \xi^k \nabla_k T^i = \xi^k \left( \frac{\partial T^i}{\partial x^k} + \Gamma^i_{jk} T^j \right) = 0. \quad (2.42)$$

**Remarks**

1. The concept of parallelism is coordinate independent because covariant differentiation is a tensor operation.

2. The concept of parallelism depends on the connection given.

3. In general the concept of parallelism depends on $C$.

4. However if the connection is Euclidean (or affine) and if $\nabla_\xi T^{(i)}_{(j)} = 0$ along $C$ then $T^{(i)}_{(j)}$ is parallel along any other curve in chart $U$. This is because in affine coordinates $\Gamma^i_{jk} = 0$, and $\nabla_\xi T^{(i)}_{(j)} = \xi^k \partial_\xi T^{(i)}_{(j)} = 0$ implies that $T^{(i)}_{(j)} = \text{const}$ along $C$, and it is parallel along any other curve in chart $U$. 


5. This provides a relation to the fifth postulate of Euclid:

“Given a line through a point $P$, and a point $Q$ not on the line, there is exactly one line
through $Q$ parallel to the given line”.

This can be rephrased as

“In Euclidean geometry for each nonzero vector $(T^i)_P$ at the point $P$ there one (up to
scalar multiples) and only one parallel vector at any point $Q$”.

In affine coordinates a parallel vector has the same components (up to scalar multiples)
as the vector $(T^i)_P$.

6. A product of two covariantly constant tensors $T = (T^{(i)})_{(j)}$ and $S = (S^{(k)})_{(l)}$ is covariantly
constant

$$\nabla_t (T^{(i)}_{(j)} S^{(k)}_{(l)}) = (\nabla_t T^{(i)}_{(j)} S^{(k)}_{(l)}) + T^{(i)}_{(j)} (\nabla_t S^{(k)}_{(l)}) = 0.
$$

If the connection is not Euclidean then

**What do we mean when we say that two vectors (or tensors), one at each of two distinct
points, are parallel?**

To compare the vectors we need to move one vector to the second one. This is done by
means of “parallel transport”.

**Def 29.1.2.** Let $(T^i)_P$ be a vector at a point $P(x_0^1, \ldots, x_0^n)$ and let $x^i(t), 0 \leq t \leq 1$, be a
curve segment $C$ joining $P$ to a point $Q(x_1^1, \ldots, x_1^n)$. The unique vector field $(T^i)$ defined at all
points of $C$, taking value $(T^i)_P$ at $P(t = 0)$, and parallel along $C$ is said to result from *parallel
transport* of the vector $(T^i)_P$ along $C$ to $Q(t = 1)$. The value of the field $(T^i)_P$ at $Q$ is denoted
by $(T^i)_Q$, and is called the *result of parallel transport* of $(T^i)_P$ along $C$ from $P$ to $Q$, relative to
the given connection.

The vector field $(T^i)$ is determined by the equation of parallel transport

$$\frac{dx^k}{dt} \nabla_k T^i = \frac{dx^k}{dt} \left( \frac{\partial T^i}{\partial x^k} + \Gamma^i_{jk} T^j \right) = \frac{dT^i}{dt} + \left( \frac{dx^k}{dt} \Gamma^i_{jk} \right) T^j = 0,
$$

and the initial conditions

$$T^i(0) = T^i, \quad i = 1, \ldots, n.
$$

As we saw in Euclidean geometry the result of parallel transport of a vector from one point
to another is independent of the path.
3 Geodesics

Given an arbitrary connection, which curves play the role of “straight lines”?

Def 29.2.1. A curve $x^i = x^i(t)$ is called geodesics if the vector field defined by its tangent vector $T^i = dx^i/dt$ is parallel along the curve itself, i.e. if the curve parallel transports its own tangent vector:

$$\nabla_T(T) = \nabla_\dot{x}(\dot{x}) = 0.$$  

(3.46)

In components one gets

$$\nabla_T(T)^i = \frac{dx^k}{dt} \nabla_k \left( \frac{dx^i}{dt} \right) = \frac{d}{dt} \frac{dx^i}{dt} + \left( \frac{dx^k}{dt} \Gamma^i_{jk} \right) \frac{dx^j}{dt} = 0,$$

(3.47)

and the equations for the geodesics take the form

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad i = 1, \ldots, n.$$  

(3.48)

Remarks

1. If $\Gamma^i_{jk} = 0$ then the solutions are just straight lines.

2. The geodesics do not depend on the torsion. They depend only on the symmetric part $\Gamma^i_{(jk)} = \Gamma^i_{jk} + \Gamma^i_{kj}$ of the connection.

3. The equations are second-order ordinary differential equations. They can be interpreted as equations of motion of a freely moving particle on the manifold $M$ (to be discuss later).

4. The system has a unique solution satisfying the initial conditions

$$x^i|_{t=0} = x^i_0, \quad \frac{dx^i}{dt}|_{t=0} = v^i_0, \quad i = 1, \ldots, n,$$

(3.49)

for each choice of $x^i_0, v^i_0$.

To be precise

Theorem 29.2.2. Let $\Gamma^i_{jk}$ be a connection defined on a manifold $M$. Then for any point $P$ in some chart $U$ and any vector $(T^i)_P$ attached to the point there exists a unique geodesic starting from $P$ and with the initial tangent vector $(T^i)_P$.

4 Connections Compatible with the Metric

In general a (pseudo-)Riemann metric and a connection defined on a manifold $M$ are completely independent.

It would however be better if they were related because

1. A given connection can be used to parallel transport a metric defined at a point $P$ to any point $Q$ of $M$. We want the results to be independent of the curves used.

2. A metric can be used to lower indices of a tensor. We want covariant differentiation to commute with lowering indices.
3. We want the set of geodesics connecting two nearby points to contain the shortest curve (the distance is determined by the metric).

**Def 29.2.1.** A connection $\Gamma^i_{kj}$ is said to be **compatible with a metric** $g_{ij}$ if the covariant derivative of the metric tensor $(g_{ij})$ is identically zero:

$$\nabla_k g_{ij} \equiv 0, \quad i, j, k = 1, \ldots, n.$$  \hspace{1cm} (4.50)

**Remarks.** For a given connection is compatible with the metric on a manifold

1. The corresponding operation of covariant differentiation commutes with the operation of lowering any index of a tensor

$$\nabla_k (g_{ij}) = g_{lj} \nabla_k^j (g_{ij}) + g_{il} \nabla_k^j (g_{ij}) - g_{ij} \nabla_k^j (g_{lj}).$$

(4.51)

2. If vector fields $T^i(t)$ and $S^i(t)$ are both parallel along a curve $x^i = x^i(t)$, then their scalar product is constant along the curve

$$\frac{d}{dt} \langle T, S \rangle = \frac{d}{dt} (g_{ij} T^i S^j) = g_{ij} \frac{dx^k}{dt} \nabla_k (g_{ij} T^i S^j) = 0.$$  \hspace{1cm} (4.52)

In other words

Parallel transport of vectors from a point $P$ to a point $Q$ along a given curve defines an orthogonal transformation from the tangent space at $P$ to the tangent space at $Q$.

### 5 Symmetric $\Gamma^i_{kj}$ Compatible with the Metric

**Theorem 29.3.2.** If the metric $g_{ij}$ is non-singular (i.e. if $g = \det(g_{ij}) \neq 0$) on a chart $U$ of the manifold $M$ under consideration, then there is a unique torsion-free (symmetric) connection which is compatible with the metric. It is given in any system of coordinates $x^1, \ldots, x^n$ by Christoffel’s formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$  \hspace{1cm} (5.53)

**Proof.** We have

$$\nabla_k g_{ij} = \partial_k g_{ij} - g_{pj} \Gamma^p_{ik} - g_{ip} \Gamma^p_{jk} = \partial_k g_{ij} - \Gamma^p_{j,ik} - \Gamma^p_{i,jk} = 0,$$

$$\implies \partial_j g_{ki} - \Gamma^p_{i,jk} - \Gamma^p_{k,ij} = 0,$$

$$\implies \partial_l g_{jk} - \Gamma^p_{k,ji} - \Gamma^p_{j,ki} = 0,$$  \hspace{1cm} (5.54)

where $\Gamma^p_{j,ik} \equiv g_{pj} \Gamma^p_{ik} = g_{ip} \Gamma^p_{ik}$. Taking the sum of the last two equations and subtracting from the sum the first equation, one gets

$$\partial_j g_{ki} + \partial_l g_{jk} - \partial_k g_{ij} - 2\Gamma^p_{k,ij} = 0,$$

(5.55)

where we took into account that the connection is torsion-free, and therefore $\Gamma^p_{k,ij} = \Gamma^p_{k,ji}$. Thus, one finds

$$\Gamma^p_{k,ij} = \frac{1}{2} (\partial_l g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}) \implies \Gamma^k_{ij} = \frac{1}{2} g^{kp} (\partial_l g_{pj} + \partial_j g_{lp} - \partial_p g_{ij}).$$  \hspace{1cm} (5.56)

**Remarks.** We consider symmetric and compatible connections
1. If $\partial_i g_{jk} = 0$ at a given point for any $i, j, k = 1, \ldots, n$, then at that point all $\Gamma_{ij}^k = 0$.

**Example.** Consider a surface $\sigma$

$$x^1 = x^1(z^1, z^2), \quad x^2 = x^2(z^1, z^2), \quad x^3 = x^3(z^1, z^2), \quad (5.57)$$
in $\mathbb{R}^3$ with Euclidean coordinates $x^i$. At a given point $P$ we introduce new Euclidean coordinates $x, y, z$ so that the $z$-axis is perpendicular to the surface at $P$ while the $x$-axis and the $y$-axis are tangent to the surface at $P$. Then in some neighbourhood of $P$ the surface $\sigma$ is given by

$$z = f(x, y), \quad \frac{\partial f}{\partial x}|(0,0) = \frac{\partial f}{\partial y}|(0,0) = 0. \quad (5.58)$$
The induced metric is given by

$$ds^2_{\sigma} = ds^2_{\mathbb{R}^3}|_{z=f(x,y)} = dx^2 + dy^2 + df(x,y)^2 = g_{ij}dz^idz^j, \quad z^1 = x, \quad z^2 = y, \quad (5.59)$$
where

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial z^j}. \quad (5.60)$$
Thus at $P$ where $\frac{\partial f}{\partial z^i} = 0$ we have $g_{ij} = \delta_{ij}$, and

$$\frac{\partial g_{ij}}{\partial z^k} = \frac{\partial}{\partial z^k} \left( \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial z^j} \right) = \left( \frac{\partial}{\partial z^k} \frac{\partial f}{\partial z^i} \right) \frac{\partial f}{\partial z^j} + \frac{\partial f}{\partial z^i} \left( \frac{\partial}{\partial z^k} \frac{\partial f}{\partial z^j} \right) = 0. \quad (5.61)$$
Thus at $P$ the symmetric and compatible connection is zero.

2. The divergence of a vector field

$$\text{div} T = \nabla_i T^i = \frac{\partial T^i}{\partial x^i} + \Gamma_{ki}^i T^k, \quad (5.62)$$
can be expressed as

$$\nabla_i T^i = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} T^i \right), \quad (5.63)$$
by using

$$\Gamma_{ki}^i = \frac{1}{2} g^{ip} \left( \partial_k g_{pj} + \partial_l g_{kp} - \partial_p g_{kj} \right) = \frac{1}{2} g^{ip} \partial_k g_{pj} = \frac{1}{2} g \partial_k g. \quad (5.64)$$
Thus

$$\sqrt{|g|} \nabla_i T^i = \frac{\partial}{\partial x^i} \left( \sqrt{|g|} T^i \right), \quad (5.65)$$
and therefore if $M$ is an oriented closed manifold then

$$\int_M \nabla_i T^i \Omega_M = \int_M \nabla_i T^i \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n = 0. \quad (5.66)$$
6 Geodesics and the Metric

The length of a curve \( C: x^i = x^i(t), \ a \leq t \leq b \) between the points \( P \) and \( Q \) is given by

\[
L = \int_C ds = \int_a^b \sqrt{g_{ij} \dot{x}^i \dot{x}^j} \, dt = \int_a^b \mathcal{L}(x, \dot{x}) \, dt. \tag{6.67}
\]

The shortest curve satisfies equations which are obtained by considering the variation of \( L \) and setting it to 0

\[
\delta L = \int_a^b \delta \mathcal{L} \, dt = \int_a^b \left( \frac{\partial \mathcal{L}}{\partial x^k} \delta x^k + \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \delta \dot{x}^k \right) \, dt = 0,
\]

where we integrated by parts, and used \( \delta x^k(a) = \delta x^k(b) = 0 \). Thus we get the Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^k} - \frac{\partial \mathcal{L}}{\partial x^k} = 0. \tag{6.69}
\]

We then have

\[
\frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \frac{1}{\mathcal{L}} g_{ik} \dot{x}^i, \quad \frac{\partial \mathcal{L}}{\partial x^k} = \frac{1}{2\mathcal{L}} \partial_k g_{ij} \dot{x}^i \dot{x}^j, \tag{6.70}
\]

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \frac{1}{\mathcal{L}} g_{ik} \ddot{x}^i + \frac{1}{\mathcal{L}} \partial_j g_{ik} \dot{x}^i \dot{x}^j - \frac{\dot{\mathcal{L}}}{\mathcal{L}^2} g_{ik} \dot{x}^i. \tag{6.71}
\]

Thus, the Euler-Lagrange equations are

\[
g_{ik} \ddot{x}^i + \partial_j g_{ik} \dot{x}^i \dot{x}^j - \frac{1}{2} \partial_k g_{ij} \dot{x}^i \dot{x}^j = \frac{\dot{\mathcal{L}}}{\mathcal{L}} g_{ik} \dot{x}^i. \tag{6.72}
\]

Multiplying by \( g^{kl} \), and then replacing \( l \leftrightarrow i \), one gets

\[
\dddot{x}^i + g^{ki} \partial_j g_{lk} \dot{x}^l \dot{x}^j - \frac{1}{2} g^{ki} \partial_k g_{lj} \dot{x}^l \dot{x}^j = \frac{\ddot{\mathcal{L}}}{\mathcal{L}} \dot{x}^i, \tag{6.73}
\]

and

\[
\dddot{x}^i + \frac{1}{2} g^{ki} (\partial_j g_{lk} + \partial_l g_{jk} - \partial_k g_{lj}) \dot{x}^l \dot{x}^j = \frac{\ddot{\mathcal{L}}}{\mathcal{L}} \dot{x}^i \implies \dddot{x}^i + \Gamma^i_{ij} \dot{x}^j \dot{x}^j = \frac{1}{2} \dot{x}^i \frac{d}{dt} \ln(g_{ij} \dot{x}^i \dot{x}^j). \tag{6.74}
\]

We see that the geodesic equation is retrieved only when the right-hand side is zero, which will occur in general if

\[
g_{ij} \dot{x}^i \dot{x}^j = K = \text{const}. \tag{6.76}
\]

The quantity \( K \) is the length of the tangent vector. For any given parametrisation of the curve \( C \) with parameter \( t \) one can find a new one with parameter \( s = s(t) \) such that \( K \) is a constant. To this end one solves the equation

\[
K = g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \left( \frac{dt}{ds} \right)^2 \implies \frac{ds}{dt} = \sqrt{\frac{g_{ij} \dot{x}^i \dot{x}^j}{K}} \implies s(t) = \int_a^t \sqrt{\frac{g_{ij} \dot{x}^i \dot{x}^j}{K}} \, d\tau, \tag{6.77}
\]
This is a generalisation of an arc length parametrisation, which is called affine parametrisation. For such a parametrisation the length of the tangent vectors remain constant as it is parallel transported along the geodesic.

**Remarks.**

1. If the manifold is Riemannian then \( K > 0 \), and one can rescale \( s \) so that the new \( K \) is equal to 1. Then, \( s \) is equal to the distance from the point with coordinates \( x^i(0) \) to the point with coordinates \( x^i(s) \). Tangent vectors to the curve have length 1.

2. If the manifold is Minkowski then \( K \) can take any real value, and if \( K \neq 0 \) one can normalise it to \( \pm 1 \). The formula (6.67) then is the action of a free relativistic particle moving in the Minkowski manifold. If the signature of the Minkowski manifold is \((- , +, \ldots, +)\), then the curve is called (i) time-like if \( K = -1 \), (ii) space-like if \( K = +1 \), (iii) light-like or null if \( K = 0 \). The corresponding parameters are proper time and proper distance.

3. If the manifold is Riemannian then the formula

\[
A = \frac{m}{2} \int_a^b g_{ij} \dot{x}^i \dot{x}^j \, dt,
\]

(6.78)

the action of a free non-relativistic particle of mass \( m \) moving in the manifold. The Euler-Lagrange equations which follow from (6.78) are the geodesic equations for any choice of \( t \) (check it!). In fact the simplest way to derive the geodesic equations is to use (6.78).

### 7 The General Curvature Tensor

The result of parallel transport of a vector \( \xi \) is determined by the equation of parallel transport

\[
\frac{dx^k}{dt} \nabla_k \xi^i = \frac{dx^k}{dt} \left( \frac{\partial \xi^i}{\partial x^k} + \Gamma^i_{jk} \xi^j \right) = \frac{dx^k}{dt} \Gamma^i_{jk} \xi^j = 0.
\]

(7.79)

If \( \Gamma^i_{jk} = 0 \) the solution to this equation is trivial. It would useful to have a criterion for the existence of coordinates in terms of which \( \Gamma^i_{jk} \) vanish. The solution to this problem turns out to be the familiar “equality of mixed partial derivatives”:

\[
\frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial^2 f}{\partial x^i \partial x^j}.
\]

(7.80)

To show this, let us assume that there are Euclidean coordinates such that \( \Gamma^i_{jk} = 0 \). Then in those coordinates

\[
\nabla_k T^{(i)}_{(j)} = \partial_k T^{(i)}_{(j)},
\]

(7.81)

and by “equality of mixed partials”

\[
(\nabla_k \nabla_l - \nabla_l \nabla_k) T^{(i)}_{(j)} = 0.
\]

(7.82)

Since the lhs of this equation is a tensor the equality will hold in all coordinate systems.

If our connection is arbitrary then for a vector field \( (\xi^i) \) we have

\[
\nabla_l \xi^i = \partial_l \xi^i + \xi^p \Gamma^{i}_{pl}, \quad \nabla_k \xi_l = \partial_k \xi_l - \xi_p \Gamma_{lk}^p,
\]

(7.83)
and therefore
\[ [\nabla_k, \nabla_l] \xi^i = \partial_k (\partial_l \xi^i + \xi^p \Gamma^i_{pl}) + \nabla_i \xi^p \Gamma^i_{pk} - \nabla_q \xi^q \Gamma^i_{lk} - (k \leftrightarrow l) \]
\[ = \partial_k \xi^p \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} + \nabla_i \xi^p \Gamma^i_{pk} - \nabla_q \xi^q \Gamma^i_{lk} - (k \leftrightarrow l) \]
\[ = (\nabla_k \xi^p - \xi^q \Gamma^p_{qk}) \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} + \nabla_i \xi^p \Gamma^i_{pk} - \nabla_q \xi^q \Gamma^i_{lk} - (k \leftrightarrow l) \] (7.84)
\[ = -\xi^q \Gamma^p_{qk} \Gamma^i_{pl} + \xi^p \partial_k \Gamma^i_{pl} - \nabla_q \xi^q \Gamma^i_{lk} - (k \leftrightarrow l) \]
\[ = (\partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^p_{pl} \Gamma^i_{jl} - \Gamma^p_{pl} \Gamma^i_{jk}) \xi^j + (\Gamma^i_{kl} - \Gamma^i_{lk}) \nabla_j \xi^i . \]

Thus, introducing the torsion tensor
\[ T^j_{kl} = \Gamma^j_{kl} - \Gamma^j_{lk} , \] (7.85)
and the Riemann curvature tensor
\[ R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^p_{pl} \Gamma^i_{jl} - \Gamma^p_{pl} \Gamma^i_{jk} , \] (7.86)
one gets
\[ [\nabla_k, \nabla_l] \xi^i = R^i_{jkl} \xi^j + T^j_{kl} \nabla_j \xi^i . \] (7.87)
The equation (7.87) is called the Ricci identity.

For any symmetric connection we have
\[ [\nabla_k, \nabla_l] \xi^i = R^i_{jkl} \xi^j . \] (7.88)
Thus, if the Riemann curvature tensor is not identically zero then the corresponding connection is not Euclidean.

Let’s derive coordinate-free formulae for the curvature and torsion tensors. For arbitrary vector fields \( \xi, \eta, \zeta \) we set
\[ [T(\xi, \eta)]^i = T^i_{jkl} \xi^j \eta^l , \] [R(\xi, \eta)\zeta] = R^i_{jkl} \xi^j \eta^l \zeta^i . \] (7.89)

**Lemma 30.1.3.** For arbitrary vector fields \( \xi, \eta, \zeta \), the following equations hold
\[ T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta] , \] (7.90)
\[ R(\xi, \eta)\zeta = [\nabla_\xi, \nabla_\eta] \zeta - \nabla_{[\xi, \eta]} \zeta , \] (7.91)
where \([\xi, \eta]\) is the commutator of the vector fields \( \xi, \eta \).

**Proof.** We first show that the rhs of (7.90) and (7.91) are linear in \( \xi, \eta, \zeta \). We replace \( \xi \to f \xi \) where \( f \) is a function, and use
\[ \nabla_{f \xi} \eta = f \nabla_\xi \eta , \quad [f \xi, \eta] = f [\xi, \eta] - (\partial_\eta f) \xi , \]
to get
\[ T(f \xi, \eta) = \nabla_{f \xi} \eta - \nabla_\eta (f \xi) - [f \xi, \eta] \]
\[ = f (\nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]) - (\nabla_\eta f) \xi + (\partial_\eta f) \xi = f T(\xi, \eta) , \] (7.92)
\[ R(f \xi, \eta)\zeta = (\nabla_{f \xi} \nabla_\eta - \nabla_\eta \nabla_{f \xi}) \zeta - \nabla_{[f \xi, \eta]} \zeta \]
\[ = f ([\nabla_\xi, \nabla_\eta] \zeta - \nabla_{[\xi, \eta]} \zeta) - \nabla_\eta f \nabla_\xi \zeta + \partial_\eta f \nabla_\xi \zeta = f R(\xi, \eta)\zeta . \] (7.93)
The linearity in \( \eta \) follows similarly.
Finally, $\zeta \to f\zeta$

$$R(\xi, \eta)(f\zeta) = \left(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi \right)(f\zeta) - \nabla_{[\xi, \eta]}(f\zeta)$$

$$= f(\nabla_\xi \nabla_\eta \zeta - \nabla_{\xi, \eta} \zeta)$$

$$+ \left(\nabla_\eta f\right) \left(\nabla_\xi \zeta\right) + \left(\nabla_\xi f\right) \left(\nabla_\eta \zeta\right) - \left(\nabla_\xi f\right) \left(\nabla_\eta \zeta\right) - \left(\nabla_\eta f\right) \left(\nabla_\xi \zeta\right)$$

$$+ \left(\nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta\right) - \left(\nabla_{[\xi, \eta]} f\right) \zeta$$

$$= f(\nabla_\xi \nabla_\eta \zeta - \nabla_{\xi, \eta} \zeta)$$

$$+ \left(\partial_\xi \partial_\eta - \partial_\eta \partial_\xi\right) f \zeta - \left(\partial_\xi \partial_\eta - \partial_\eta \partial_\xi\right) f = f \, R(\xi, \eta) \zeta .$$

(7.94)

where we used $\nabla_{[\xi, \eta]} f = \partial_{[\xi, \eta]} f = \left(\partial_\xi \partial_\eta - \partial_\eta \partial_\xi\right) f$. 

Since the rhs of (7.90) and (7.91) are linear in $\xi, \eta, \zeta$ it is suffices to check that (7.90) and (7.91) hold for the basis vector fields $\xi = e_k$, $\eta = e_l$, $\zeta = e_j$ in which case $\xi^i = \delta^i_k$, $\eta^i = \delta^i_l$, $\zeta^i = \delta^i_j$. But for these vector fields (7.90) and (7.91) follow immediately from the definitions of $T_{i kl}$ and $R_{ijkl}$, e.g.

$$\left[T(e_k, e_l)\right]^i = T_{i kl} = \Gamma^i_{kl} - \Gamma^i_{lk} = \left(\nabla e_k e_l - \nabla e_l e_k\right)^i = \left(\nabla e_k e_l - \nabla e_l e_k - [e_k, e_l]\right)^i$$

(7.95)

and similarly for $\left[R(e_k, e_l)e_j\right]^i$.

8 The symmetries of the Curvature Tensor

Theorem 30.2.1.

1. The tensor $R_{ijkl}$ is always skew-symmetric in the indices $k$ and $l$

$$R_{ijkl} + R_{jikl} = 0 .$$

(8.96)

2. If the connection is symmetric, then

$$R_{ijkl} + R_{klij} + R_{ijlk} = 0 .$$

(8.97)

3. If the connection is compatible with the metric $g_{ij}$, and we define

$$R_{ijkl} \equiv g_{ij} R_{jikl}^p ,$$

then the tensor $R_{ijkl}$ is skew-symmetric in the indices $i$ and $j$

$$R_{ijkl} + R_{jikl} = 0 .$$

(8.99)

4. If the connection is both symmetric and compatible with the metric $g_{ij}$, then the tensor $R_{ijkl}$ is symmetric under the exchange of the pairs of the indices $ij$ and $kl$

$$R_{ijkl} - R_{klij} = 0 .$$

(8.100)

Proof.

1. is obvious
2. Let $e_i$, $i = 1, \ldots, n$ be the standard basis of vectors at every point. Since the connection is symmetric one has

$$[\nabla_k, \nabla_l]^i = R^i_{jkl} \xi^j \implies [\nabla_k, \nabla_l]^i = R^i_{jkl} \xi^j e_i. \quad (8.101)$$

Choosing $\xi^i e_i = e_j$, i.e. $\xi^i = \delta^i_j$, one gets

$$R^i_{jkl} e_i = [\nabla_k, \nabla_l] e_j. \quad (8.102)$$

Hence, $(8.97) R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$ will follow if we show that

$$I_{jkl} \equiv [\nabla_k, \nabla_l] e_j + [\nabla_l, \nabla_j] e_k + [\nabla_j, \nabla_k] e_l \quad (8.103)$$

vanish. We have

$$I_{jkl} = \nabla_k (\nabla_l e_j - \nabla_j e_l) + \nabla_l (\nabla_j e_k - \nabla_k e_j) + \nabla_j (\nabla_k e_l - \nabla_l e_k). \quad (8.104)$$

Now,

$$(\nabla_l e_j)^i = \partial_l e^j_i + \Gamma^i_{kl} e^k_j = \Gamma^i_{jl}. \quad (8.105)$$

Since the connection is symmetric $\nabla_l e_j - \nabla_j e_l = 0$, and therefore $I_{jkl} = 0$.

3. Since the connection is compatible with the metric $g_{ij}$, we have $\nabla_k g_{ij} = 0$. From

$$[\nabla_k, \nabla_l]^i = R^i_{jkl} \xi^j + T^i_{kl} \nabla_j \xi^j, \quad (8.106)$$

we get (note that the proof in the book is wrong!)

$$\langle [\nabla_k, \nabla_l] \xi, \xi \rangle = g_{ip} R^i_{jkl} \xi^j \xi^p + g_{ip} T^i_{kl} \nabla_j \xi^j \xi^p$$

$$= R_{pjkl} \xi^j \xi^p + T^i_{kl} \langle \nabla_j \xi, \xi \rangle = R_{pjkl} \xi^j \xi^p + \frac{1}{2} T^i_{kl} \partial_j \langle \xi, \xi \rangle. \quad (8.107)$$

Now, if $f$ is a scalar, e.g. $\langle \xi, \xi \rangle$, one has

$$[\nabla_k, \nabla_l] f = \nabla_k \nabla_l f - (k \leftrightarrow l) = -\Gamma^i_{lk} \partial_j f + \Gamma^i_{kl} \partial_j f = T^i_{kl} \partial_j f. \quad (8.108)$$

Hence,

$$\frac{1}{2} T^i_{kl} \partial_j \langle \xi, \xi \rangle = \frac{1}{2} \nabla_k \nabla_l \langle \xi, \xi \rangle = \frac{1}{2} (2 \nabla_k \langle \nabla_l \xi, \xi \rangle - 2 \nabla_l \langle \nabla_k \xi, \xi \rangle)$$

$$= \langle [\nabla_k, \nabla_l] \xi, \xi \rangle. \quad (8.109)$$

Thus,

$$R_{pjkl} \xi^j \xi^p = 0 \implies R_{ijkl} + R_{jikl} = 0. \quad (8.110)$$

4. If the connection is both symmetric and compatible with the metric $g_{ij}$, then we can use the previous three symmetries. We have from $(8.97)$

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0, \quad R_{ijkl} + R_{jikl} + R_{jlki} = 0 \implies \quad (8.111)$$

$$2R_{ijkl} + R_{iklj} + R_{iljk} - R_{jlki} - R_{ijkl} = 0, \quad (8.111)$$

$$R_{kijl} + R_{klij} + R_{klji} = 0, \quad R_{ijkl} + R_{lijk} + R_{lijk} = 0 \implies \quad (8.112)$$

$$2R_{klij} - R_{klij} - R_{lkij} - R_{ljki} - R_{lijk} = 0. \quad (8.112)$$

Subtracting $(8.112)$ from $(8.111)$, and using $(8.96)$ and $(8.99)$, one gets $(8.100)$

$$R_{ijkl} - R_{klji} = 0. \quad (8.113)$$
9 Ricci Tensor, Scalar Curvature, Einstein Tensor and Weyl Tensor

Def 30.3.1. The trace (or contraction)

\[ R_{jl} = R^{i}_{jl} \]  \hspace{1cm} (9.114)

of the Riemann curvature tensor is called the Ricci tensor.

Explicitly it is given by

\[ R_{jl} = R^{i}_{jl} = \partial_{l} \Gamma^{i}_{jl} - \partial_{l} \Gamma^{i}_{ji} + \Gamma^{i}_{pu} \Gamma^{p}_{jl} - \Gamma^{i}_{pl} \Gamma^{p}_{ji} , \]  \hspace{1cm} (9.115)

If the connection is symmetric and compatible with the metric, then

\[ R_{jl} = R_{lj} , \]  \hspace{1cm} (9.116)

because

\[ \partial_{l} \Gamma^{i}_{ji} = \partial_{l} \left( \frac{1}{2g} \partial_{j} g \right) g_{ij} = \frac{1}{2} \partial_{l} \partial_{j} \ln |g| , \]  \hspace{1cm} (9.117)

and all the other terms are obviously symmetric too.

Def 30.3.2. The scalar

\[ R = g^{jl} R_{jl} = g^{jl} R^{i}_{jl} \]  \hspace{1cm} (9.118)

is called the scalar curvature of the underlying Riemann manifold with the metric tensor \( g_{ij} \).

Def 3. If the connection is symmetric and compatible with the metric, then the tensor

\[ G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} \]  \hspace{1cm} (9.119)

is called the Einstein tensor. Note that (\( \dim M = n \))

\[ g^{ij} G_{ij} = R - \frac{1}{2} R n = \frac{2 - n}{2} R . \]  \hspace{1cm} (9.120)

It appears in Einstein’s equations

\[ G_{ij} + \Lambda g_{ij} = \frac{8 \pi G}{c^{4}} T_{ij} , \]  \hspace{1cm} (9.121)

where \( \Lambda \) is the cosmological constant, \( G \) is Newton’s gravitational constant, \( c \) is the speed of light in vacuum, and \( T_{ij} \) is the stress-energy (or energy-momentum) tensor of matter.

In the vacuum \( T_{ij} = 0 \), and one gets

\[ G_{ij} + \Lambda g_{ij} = 0 \implies \frac{2 - n}{2} R + \Lambda n = 0 \implies R = \frac{2n}{n - 2} \Lambda = \text{const} , \]  \hspace{1cm} (9.122)

and

\[ R_{ij} = \frac{2}{n - 2} \Lambda g_{ij} . \]  \hspace{1cm} (9.123)

If the metric satisfies this equation, then the manifold is called the Einstein manifold.
Def 4. The tensor
\[ C_{ijkl} = R_{ijkl} - \frac{2}{n-2} (g_{ik}R_{lj} - g_{jk}R_{li}) + \frac{2}{(n-2)(n-1)} g_{ik}g_{lj}R \] (9.124)
is called the Weyl tensor. Here \([kl]\) means anti-symmetrisation with respect to \(k\) and \(l\), e.g. \(g_{ik}R_{lj} = \frac{1}{2}(g_{ik}R_{lj} - g_{jl}R_{ki})\).

All contractions over the Weyl tensor vanish (check it!).

The Weyl tensor is sometimes called the conformal tensor because it is invariant under conformal transformations \[ g_{ij} \rightarrow \lambda g_{ij}, \quad C_{ijkl} \rightarrow C_{ijkl}. \] (9.125)

If \(C_{ijkl} = 0\) and \(\dim M \geq 4\), then the manifold is locally conformally flat.

10 The Curvature Tensor in 2 dimension

In 2 dimensions the curvature tensor is given by
\[ R_{abcd} = \frac{1}{2} R(g_{ac}g_{bd} - g_{ad}g_{bc}), \]
and evaluate the Einstein tensor in two dimensions.

It follows from the skew-symmetry properties of the curvature tensor that in 2D all non-vanishing components of \(R_{ijkl}\) are expressed through \(R_{1212}\). Then the components \(g^{ij}\) are expressed through \(g_{ij}\) as
\[ g^{11} = \frac{g_{22}}{g}, \quad g^{22} = \frac{g_{11}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}, \]
\[ g = g_{11}g_{22} - g_{12}g_{21}, \quad \frac{1}{g} = g^{11}g^{22} - g^{12}g^{21}. \] (10.126)

Using these formulae, one finds
\[ R_{11} = g^{pq}R_{q1p1} = g^{22}R_{2121} = g^{22}R_{1212}, \]
\[ R_{12} = g^{pq}R_{q1p2} = g^{12}R_{2112} = -g^{12}R_{1212}, \]
\[ R_{22} = g^{pq}R_{q2p2} = g^{11}R_{1212}, \]
\[ R = g^{pq}R_{qp} = g^{11}R_{11} + 2g^{12}R_{12} + g^{22}R_{22} = 2(g^{11}g^{22} - g^{12}g^{21})R_{1212} = \frac{2}{g}R_{1212}. \] (10.127)

So,
\[ R_{1212} = \frac{g}{2}R = \frac{1}{2}(g_{11}g_{22} - g_{12}g_{21})R \quad \Rightarrow \quad R_{abcd} = \frac{1}{2} R(g_{ac}g_{bd} - g_{ad}g_{bc}) \quad \Rightarrow \] (10.128)
\[ R_{ab} = \frac{1}{2} R g_{ab} \quad \Rightarrow \quad G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 0. \]

Another (more elegant) way to derive the formula is to notice that the combination \(g_{ac}g_{bd} - g_{ad}g_{bc}\) is a tensor which has the same symmetries as \(R_{abcd}\) (Check this!). Thus
\[ R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}), \]
where \(K\) is a scalar. Computing \(R_{ab}\) one gets
\[ R_{ab} = Kg_{ab} \quad \Rightarrow \quad R = 2K. \] (10.130)
11 The Curvature Tensor in 3 dimensions

In 3D, the curvature tensor is given by

\[ R_{abcd} = R_{ac}g_{bd} - R_{ad}g_{bc} + g_{ac}R_{bd} - g_{ad}R_{bc} - \frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}). \]  

(11.131)

It follows from the skew-symmetry properties of the curvature tensor that in 3D all non-vanishing components of \( R_{ijkl} \) are expressed through the following six components

\[ R_{1212}, \ R_{1213}, \ R_{1223}, \ R_{1313}, \ R_{1323}, \ R_{2323}. \]  

(11.132)

The Ricci tensor being symmetric also has six independent components. On the other hand the definition \( R_{bd} = g^{ac}R_{abcd} \) can be considered as a set of linear equations on the Riemann tensor components. Since the number of independent components of \( R_{ab} \) and \( R_{acbd} \) is the same, the linear equations must have a unique solution, and the Riemann tensor components \( R_{abcd} \) have to be linear combinations of \( R_{ab} \). Thus, all one has to do is to check that the formula (11.131) is consistent with \( R_{bd} = g^{ac}R_{abcd} \). This is indeed so

\[ g^{ac}R_{abcd} = R_{gbd} - R_{bd} + \frac{1}{2}R(3g_{bd} - g_{bd}) = R_{bd}. \]  

(11.133)

12 The Curvature Tensor in \( n \geq 4 \) dimensions

Let the connection be torsion-free and compatible with the metric. How many linearly independent components does \( R_{ijkl} \) have in arbitrary dimensions?

Recall that \( R_{ijkl} \) satisfies the following algebraic relations

\[ R_{ijkl} + R_{ijlk} = 0, \]
\[ R_{ijkl} + R_{jikl} = 0, \]
\[ R_{ijkl} - R_{klij} = 0, \]
\[ R_{ijkl} + R_{iklj} + R_{iljk} = 0. \]  

(12.134)

Thanks to \( R_{ijkl} = R_{klij} \) one can think about \( R_{ijkl} \) as a symmetric matrix \( R_{AB} = R_{BA} \) where \( A = ij \) and \( B = kl \) are multi-indices. Since \( R_{ijkl} = -R_{ijlk} \), \( R_{ijkl} = -R_{jikl} \) the indices \( A \) and \( B \) take \( n(n-1)/2 \) values. Thus, a generic symmetric matrix \( R_{AB} \) would have

\[ \frac{1}{2} \frac{n(n-1)}{2} \left( \frac{n(n-1)}{2} + 1 \right) = \frac{n(n-1)(n^2 - n + 2)}{8} \]  

(12.135)

independent components. One has however to subtract from this number the number of independent relations due to \( R_{ijkl} + R_{iklj} + R_{ijlk} = 0 \). To this end we note that

\[ R_{i[jkl]} = R_{ijkl} - R_{ikjl} - R_{ilkj} = R_{ijkl} + R_{iklj} + R_{ijlk} = 0, \]  

(12.136)

and it is skew-symmetric under permutations of \( j, k, l \). Moreover, one gets

\[ R_{i[jkl]} = R_{ijkl} - R_{ikjl} - R_{ilkj} = R_{jilk} - R_{jikl} - R_{jkli} = -R_{j[kli]}. \]  

(12.137)
and similarly
\[ R_{ijkl} = R_{klij} = -R_{iljk}. \] (12.138)

Now consider the skew-symmetric part of \( R_{ijkl} \)
\[ R_{ijkl} \equiv R_{ijkl} - R_{i[jkl]} - R_{k[jil]} - R_{l[ijk]} = 4R_{ijkl} = 0. \] (12.139)

Thus, the relation \( R_{ijkl} + R_{ilkj} + R_{dijk} = 0 \) together with the first three relations is equivalent to vanishing of the skew-symmetric part of \( R_{ijkl} \). Since a fourth-rank skew-symmetric tensor has \( n(n-1)(n-2)(n-3)/4! \) independent components we finally get
\[ \frac{n(n-1)(n^2-n+2)}{8} - \frac{n(n-1)(n^2-5n+6)}{24} = \frac{n(n-1)(3n^2-3n-n^2+5n)}{24} = \frac{n^2(n^2-1)}{12}. \] (12.140)

### 13 Normal Coordinates

Given \( p \in M \) and a chart \((x^i)\) we can find a new chart \((\hat{x}^i)\) such that
\[ \hat{\Gamma}^i_{(jk)}(p) = \frac{1}{2}(\hat{\Gamma}^i_j(p) + \hat{\Gamma}^i_k(p)) = 0, \] (13.141)
or equivalently \( \hat{\Gamma}^i_{jk}(p) = \frac{1}{2}\hat{\Gamma}^i_{jk}(p) \).

Let \( p \) have coordinates \( x^i = 0, \hat{x}^i = 0, i = 1, \ldots, n \). Let us look for \((\hat{x}^i)\) of the form
\[ \hat{x}^i = x^i + \frac{1}{2}Q^i_{jk}x^jx^k, \] (13.142)
where \( Q^i_{jk} = Q^i_{kj} \) are some constants to be determined from the condition \( \hat{\Gamma}^i_{(jk)}(p) = 0 \). For small \( |x| = \max_i |x^i| \) we can invert (13.142)
\[ x^i = \hat{x}^i - \frac{1}{2}Q^i_{jk}\hat{x}^j\hat{x}^k + \mathcal{O}(|x|^3), \] (13.143)

Then
\[ \frac{\partial \hat{x}^i}{\partial x^j} = \delta_j^i + Q^i_{jk}x^k, \quad \frac{\partial \hat{x}^i}{\partial \hat{x}^j} = \delta_j^i - Q^i_{jk}x^k + \mathcal{O}(|x|^2), \]
\[ \frac{\partial^2 x^i}{\partial \hat{x}^j \partial \hat{x}^k} = -Q^i_{jk} + \mathcal{O}(|x|), \] (13.144)

The connection transforms as
\[ \hat{\Gamma}^i_{jk} = \frac{\partial \hat{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \hat{x}^j} \frac{\partial x^c}{\partial \hat{x}^k} \Gamma^a_{bc} + \frac{\partial \hat{x}^i}{\partial x^a} \frac{\partial^2 x^a}{\partial \hat{x}^j \partial \hat{x}^k}. \] (13.145)

Thus, at \( p \) we have
\[ \hat{\Gamma}^i_{jk}(p) = \Gamma^i_{jk}(p) - Q^i_{jk}. \] (13.146)

So, if we choose \( Q^i_{jk} = \Gamma^i_{(jk)}(p) \), then \( \hat{\Gamma}^i_{(jk)}(p) = 0 \).
If the connection is symmetric, then at $p$ we have

$$ R^i_{jkl}(p) = \partial_k \Gamma^i_{jl}(p) - \partial_l \Gamma^i_{jk}(p), \quad (13.147) $$

and

$$ \nabla_m R^i_{jkl}(p) = \partial_m R^i_{jkl}(p) = \partial_m \partial_k \Gamma^i_{jl}(p) - \partial_m \partial_l \Gamma^i_{jk}(p), \quad (13.148) $$

This can be used to prove the Bianchi identities

$$ \nabla_m R^m_{nijkl} + \nabla_l R^m_{nijk} + \nabla_k R^m_{nijl} = 0. \quad (13.149) $$

Notice also that (Check this!)

$$ \partial_k \Gamma^i_{jl}(p) = \frac{1}{3} (R^i_{jkl}(p) + R^i_{ikl}(p)) \quad (13.150) $$

If the connection is symmetric and compatible with the metric, then $\partial_k \hat{g}_{ij}(p) = 0$ and the metric has the expansion (Check this!)

$$ \hat{g}_{ij}(p) = g_{ij}(p) + \frac{1}{3} R^k_{ijkl}(p) \hat{x}^k \hat{x}^l + O(|x|^3). \quad (13.151) $$

Since $g_{ij}(p)$ is a symmetric matrix it can be diagonalised by an orthogonal transformation of $(\hat{x}^i)$: $\hat{x}^i = A^i_j y^j$. Then by rescaling $y^i$: $y^i = \lambda_i z^i$, one can get the expansion

$$ g_{ij}(z) = \delta_{ij} + \frac{1}{3} R^k_{ijkl}(0) z^k z^l + O(|z|^3), \quad (13.152) $$

where $R_{ijkl}(0)$ is the Riemann curvature tensor in $(z^i)$ chart.