Course 2E01 (Frolov), Multivariable Calculus
Tutorial Sheet 1

Due: at the end of the tutorial session Tuesday/Thursday, 4/6 October 2011

Name:

Consider the vector function (with values in $\mathbf{R}^3$)

$$\mathbf{r}(t) = \ln(2-t^3)\,\mathbf{i} + (1+t^3)\,\mathbf{j} - \frac{(t^3-2)^2}{4}\,\mathbf{k}$$

1. Find the domain $\mathcal{D}(\mathbf{r})$ of the vector function $\mathbf{r}(t)$.

Solution: The domain $\mathcal{D}(\mathbf{r})$ of $\mathbf{r}(t)$ is the intersection of domains of its component functions. Since $\mathcal{D}(\ln(2-t^3)) = (-\infty, \sqrt[3]{2}\,)$, $\mathcal{D}(1+t^3) = (-\infty,\infty)$ and $\mathcal{D}(-\frac{(t^3-2)^2}{4}) = (-\infty,\infty)$, one gets

$$\mathcal{D}(\mathbf{r}) = (-\infty, \sqrt[3]{2}\,),$$

that is the vector function $\mathbf{r}(t)$ is defined for $t < \sqrt[3]{2}$.

2. Find
(a) the derivative $d\mathbf{r}/dt$,
(b) the norm $||d\mathbf{r}/dt||$
(c) the unit tangent vector $\mathbf{T}$ for all values of t in $\mathcal{D}(\mathbf{r})$.
Simplify the expressions obtained.
Hint: use the formula $a^2 + 2ab + b^2 = (a+b)^2$.
Solution:
(a)

$$\frac{d\mathbf{r}}{dt} = \left(-\frac{3t^2}{2-t^3},\ 3t^2,\ 3t^2 - \frac{3t^5}{2}\right).$$

(b) The magnitude of this vector is

$$\begin{aligned}
||\frac{d\mathbf{r}}{dt}|| &= \sqrt{\left(-\frac{3t^2}{2-t^3}\right)^2 + (3t^2)^2 + \left(3t^2 - \frac{3t^5}{2}\right)^2} = 3t^2\sqrt{\left(\frac{1}{2-t^3}\right)^2 + 1 + \left(1 - \frac{t^3}{2}\right)^2} \\
&= 3t^2\sqrt{\left(\frac{1}{2-t^3}\right)^2 + 2\,\frac{1}{2-t^3}\,\frac{2-t^3}{2} + \left(\frac{2-t^3}{2}\right)^2} \\
&= 3t^2\sqrt{\left(\frac{1}{2-t^3} + \frac{2-t^3}{2}\right)^2} = 3t^2\left(\frac{1}{2-t^3} + \frac{2-t^3}{2}\right),
\end{aligned}$$

because $t^3 < 2$.

(c) The unit tangent vector is

$$\mathbf{T} = \frac{\frac{d\mathbf{r}}{dt}}{||\frac{d\mathbf{r}}{dt}||} = \left(-\frac{2}{t^6-4t^3+6}, -\frac{2\,(t^3-2)}{t^6-4t^3+6}, \frac{t^6-4t^3+4}{t^6-4t^3+6}\right).$$

3. Find the vector equation of the line tangent to the graph of $\mathbf{r}(t)$ at the point $P_0(0, 2, -\frac{1}{4})$ on the curve.

Solution: The point $P_0(0, 2, -\frac{1}{4})$ on the curve corresponds to $t = 1$. We find

$$\mathbf{r}_0 = \mathbf{r}(1) = 2\mathbf{j} - \frac{1}{4}\mathbf{k}, \quad \mathbf{v}_0 = \frac{d\mathbf{r}}{dt}(1) = -3\mathbf{i} + 3\mathbf{j} + \frac{3}{2}\mathbf{k}.$$

Thus the tangent line equation is

$$\mathbf{r} = \mathbf{r}_0 + (t-1)\,\mathbf{v}_0 = 3(1-t)\,\mathbf{i} + (3t-1)\,\mathbf{j} + \frac{1}{4}(6t-7)\,\mathbf{k}.$$

Note that the same line is also described by the following equation which is obtained from the one above by the rescaling and shift of the parameter t: $t \to \frac{t}{3} + 1$

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}_0 = -t\,\mathbf{i} + (t+2)\,\mathbf{j} + \frac{1}{4}(2t-1)\,\mathbf{k}.$$

4. Find the arc length of the graph of $\mathbf{r}(t)$ if $-1 \le t \le 1$.

Solution: The arc length of the graph of $\mathbf{r}(t)$ is given by the definite integral

$$\begin{aligned} L = \int_{-1}^{1} \|\frac{d\mathbf{r}}{dt}\|\,dt &= \int_{-1}^{1} 3t^2 \left(\frac{1}{2-t^3} + \frac{2-t^3}{2}\right) dt = \int_{-1}^{1} \left(\frac{1}{2-v} + \frac{2-v}{2}\right) dv \\ &= \left(-\ln(2-v) - \frac{(2-v)^2}{4}\right)\Big|_{-1}^{1} = 2 + \ln 3, \end{aligned}$$

where we have made the substitution $v = t^3$.

5. Find a positive change of parameter from t to s where s is an arc length parameter of the curve having $\mathbf{r}(1)$ as its reference point.

Solution: The arc length parameter s can be found as follows

$$\begin{aligned} s = \int_{1}^{t} \|\frac{d\mathbf{r}}{du}\|\,du = &= \int_{1}^{t} 3u^2 \left(\frac{1}{2-u^3} + \frac{2-u^3}{2}\right) du = \int_{1}^{t^3} \left(\frac{1}{2-v} + \frac{2-v}{2}\right) dv \\ &= \left(-\ln(2-v) - \frac{(2-v)^2}{4}\right)\Big|_{1}^{t^3} = -\frac{t^6 - 4t^3 + 3}{4} - \ln(2-t^3), \end{aligned}$$

where we have made the substitution $v = u^3$.