FLOWS IN LIQUID FOAMS

A finite element approach

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Facing a problem of fluid flow in a physical system, we can either :

- **calculate the full solution of the Navier-Stokes equations and associated boundary conditions**
  - Poiseuille flow in pipes
  - Couette flow under shear

- **determine an approximate solution**
  - by scaling arguments
  - by appropriate approximations (lubrification,…)
  - by **FINITE ELEMENT models**

About the Bretherton problem :

Bretherton power law comes from a lubrication approximation. What about tackling the problem with a **Finite Element** calculation ?

« ... Generations of postdocs and postgrads have failed to do this ! Good luck ! »

D. W. from Australia

Let's give the Finite Element technique his best

OUTLINE

PART I : INTRODUCTION TO FINITE ELEMENT

PART II : SOME APPLICATIONS IN FOAM PHYSICS

PART III : SOME TECHNICAL FEATURES

1 – Navier-Stokes equation and the strong formulation
Virtual Power Principle and the weak formulation
2 – Implementation of boundary conditions
3 – Moving meshes
4 – Implementation of the surface tension

TUTORIAL : INTRODUCTION TO COMSOL Multiphysics

GENERAL OVERVIEW

FINITE ELEMENT

=

Resolution of a
Partial Differential Problem

$$\frac{\partial}{\partial t} + \nabla.\mathbf{\Gamma} = \mathbf{F}$$

$$\nabla.\mathbf{\Gamma} = \mathbf{F}$$

\+ **Boundary Conditions**

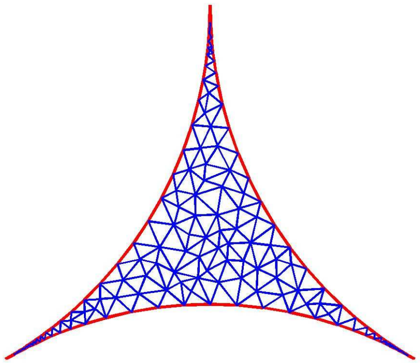
APPLICATION FIELDS

Fluid Mechanics
Structural Mechanics
Electromagnetism
Acoustics
Heat transfer
Chemistry

DEDICATED SOFTWARES :

FEMLAB (COMSOL Multiphysics)
QUICKFIELDS
Fluent
Nastran, Ansys,...

NUMERICAL ASPECTS OF FINITE ELEMENT TECHNIQUES



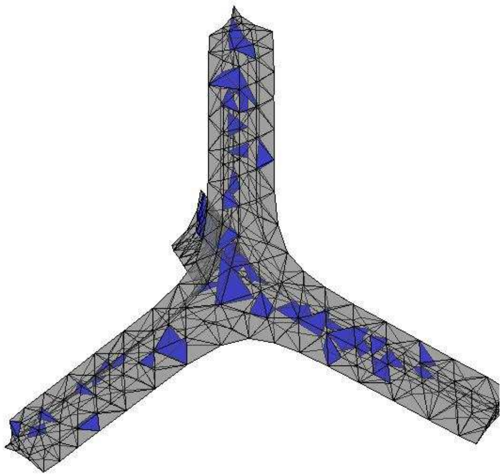
FINITE ELEMENT =

Numerical estimation of the solution of the Partial Differential Equation (PDE)

NUMERICAL ESTIMATE :

Discretization of the geometry in elements

Estimation at node and interpolation (degrees of freedom)



RESOLUTION OF THE PDE

Write the PDE in a matrix format :

$$M\ddot{X} + C\dot{X} + KX = F$$

$$KX = F$$

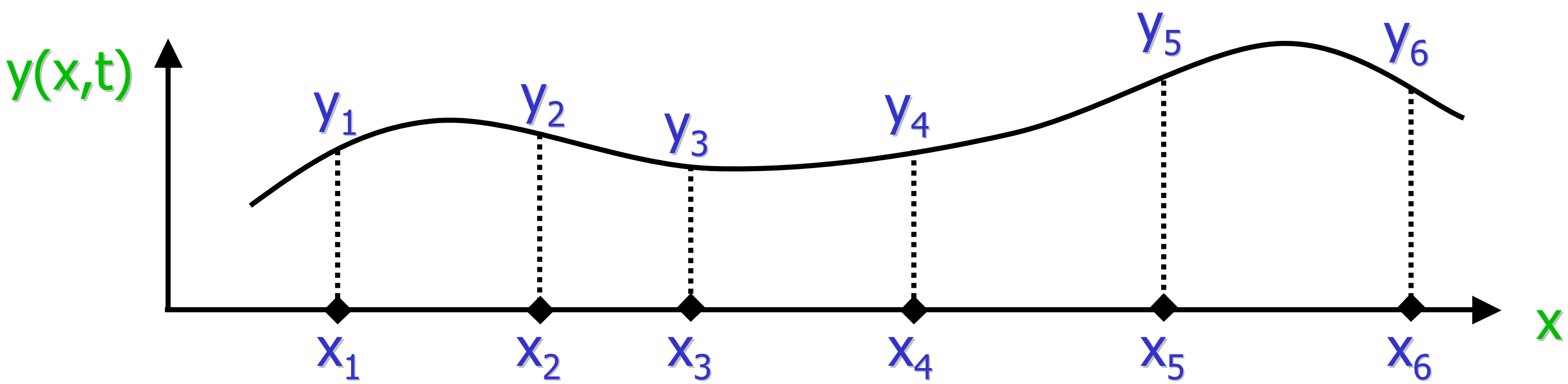
Matrix inversion and time integration

NUMERICAL ASPECTS OF FINITE ELEMENT TECHNIQUES

From the Partial Differential Equation to the Matrix Equation :

Vibrating String :

$$\boxed{\frac{\partial^2 \mathbf{y}}{\partial^2 t} - \frac{\partial^2 \mathbf{y}}{\partial^2 \mathbf{x}} = 0 \Rightarrow \ddot{\mathbf{Y}} + \mathbf{KY} = \mathbf{0}}$$



$$y = \frac{\partial y}{\partial x} = \begin{pmatrix} \frac{y_2 - y_1}{x_2 - x_1} \\ \frac{y_3 - y_2}{x_3 - x_2} \\ \vdots \end{pmatrix} = \begin{bmatrix} \frac{1}{x_2 - x_1} & -\frac{1}{x_2 - x_1} & 0 & 0 \\ 0 & \frac{1}{x_3 - x_2} & -\frac{1}{x_3 - x_2} & \ddots \\ 0 & \ddots & \ddots & \ddots \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}$$

$$\Delta y = \frac{\partial^2 y}{\partial^2 x} = \begin{pmatrix} \frac{(y_2 - y_1)(x_3 - x_2) - (y_3 - y_2)(x_2 - x_1)}{(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)} \\ \vdots \end{pmatrix} = \begin{bmatrix} * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix} \begin{pmatrix} y \\ y \\ y \\ \vdots \end{pmatrix}$$

HOW TO USE FEMLAB

DIRECT IMPLEMENTATION OF THE PDE

$$\boxed{\frac{\partial}{\partial t} + \nabla.\Gamma = \mathbf{F}} \quad + \quad \boxed{\text{Boundary Conditions}}$$

NUMERICAL RESOLUTION AS A BLACK BOX

Discretization (elements, node)

Computation of the matrices

Matrix inversion and time integration

APPLICATIONS IN FOAMS

- **Flow in Plateau borders with Liquid-Vapor surfaces subjected to surface viscosity**
- **Flow in vertices with Liquid-Vapor surfaces subjected to surface viscosity**
- **Drainage in a 2D foam**
- **Wall slip of bubbles : "The Bretherton Problem"**

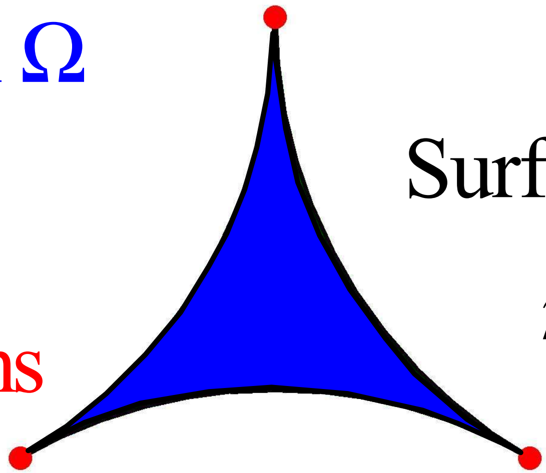
DRAINAGE IN LIQUID FOAMS – PLATEAU BORDERS

Viscous Flow in Ω

$$\eta_B \Delta \mathbf{U} = \rho \mathbf{g}$$

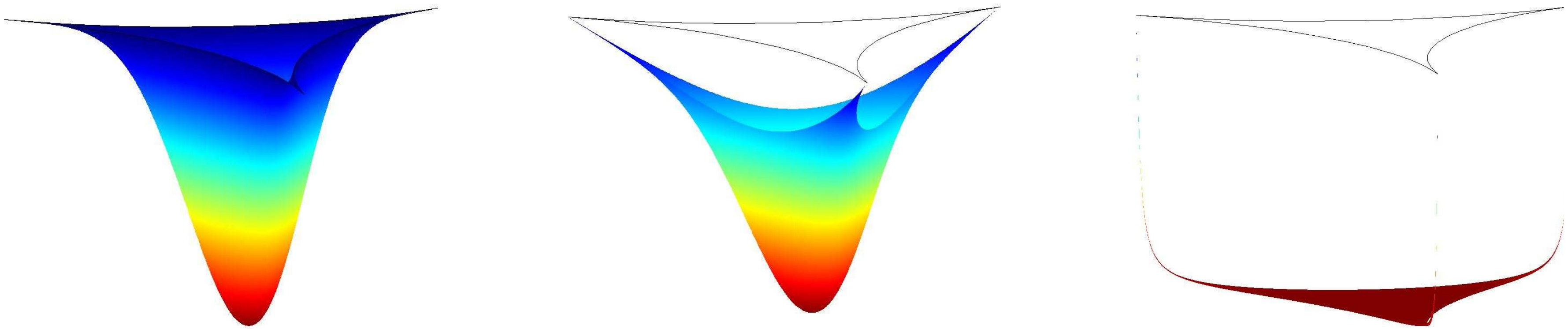
No flow in films

$$\mathbf{U} = \mathbf{0}$$



Surface viscosity on $\partial\Omega$

$$\eta_S \Delta_S \mathbf{U} = \eta_B \frac{\partial \mathbf{U}}{\partial \mathbf{n}}$$



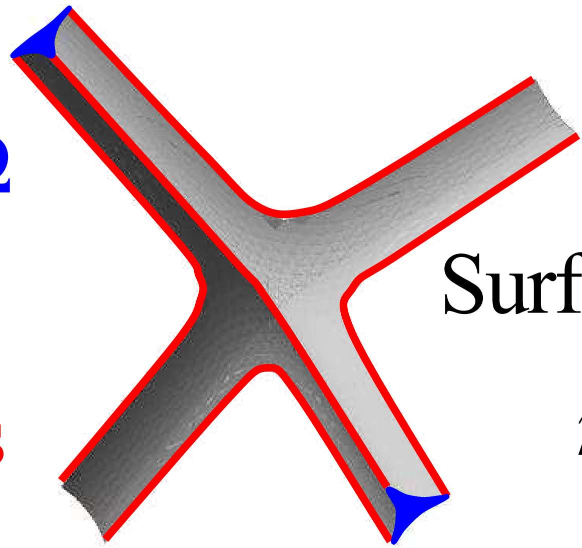
DRAINAGE IN LIQUID FOAMS – VERTICES

Viscous Flow in Ω

$$\eta_B \Delta \mathbf{U} - \nabla p = \mathbf{f}$$

No flow in films

$$\mathbf{U} = \mathbf{0}$$



Surface viscosity on $\partial\Omega$

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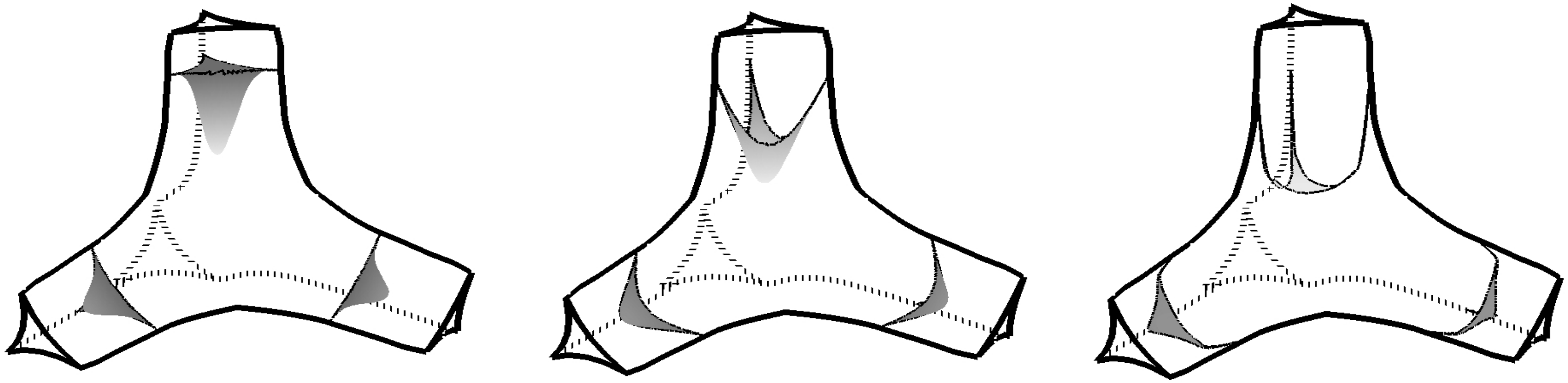
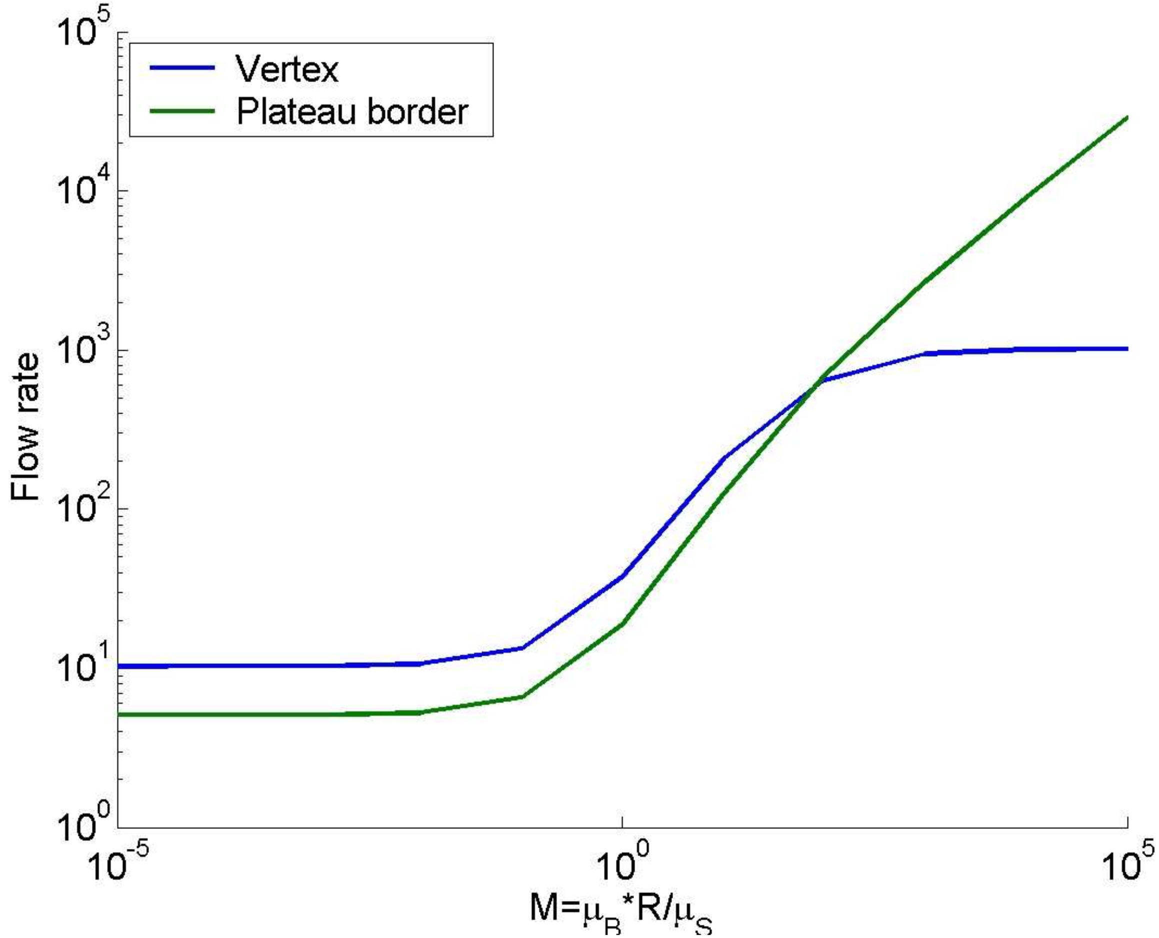


DRAINAGE IN LIQUID FOAMS



DRAINAGE IN 2D FOAMS

CONSTITUTIVE LAWS

$$\frac{\partial \varepsilon}{\partial t} + \nabla \bullet (\varepsilon \mathbf{v}) = f(\mathbf{r}, t)$$

$$\mathbf{v} = \frac{k}{\eta_B}\left(-\nabla p + \rho \mathbf{g}\right)$$

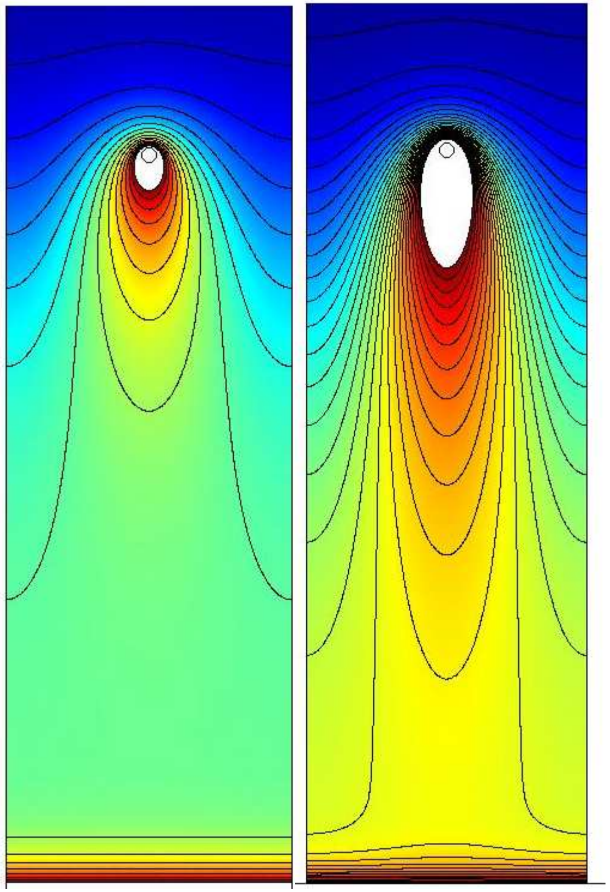
$$p = p_0 - \frac{\gamma}{r(\varepsilon)}$$

PDE SYSTEM

$$\frac{\partial \varepsilon}{\partial t} - \nabla \bullet (\Gamma) = f(\mathbf{r}, t)$$

$$\Gamma = \nabla \varepsilon + \varepsilon^{\frac{3}{2}} \mathbf{e}_z$$

+ **Boundary Conditions**

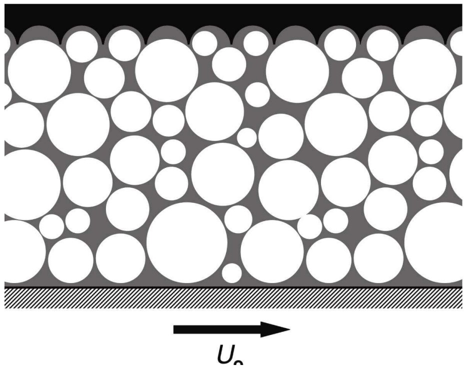


Low flow rate High Flow rate

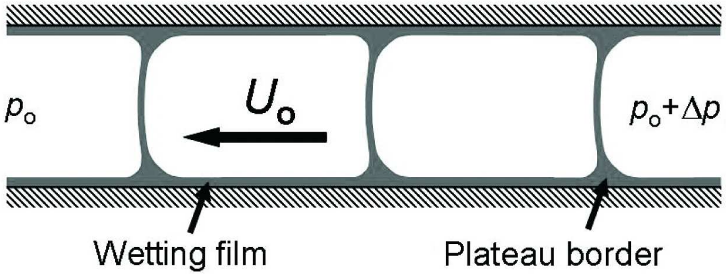
WALL SLIP OF BUBBLES

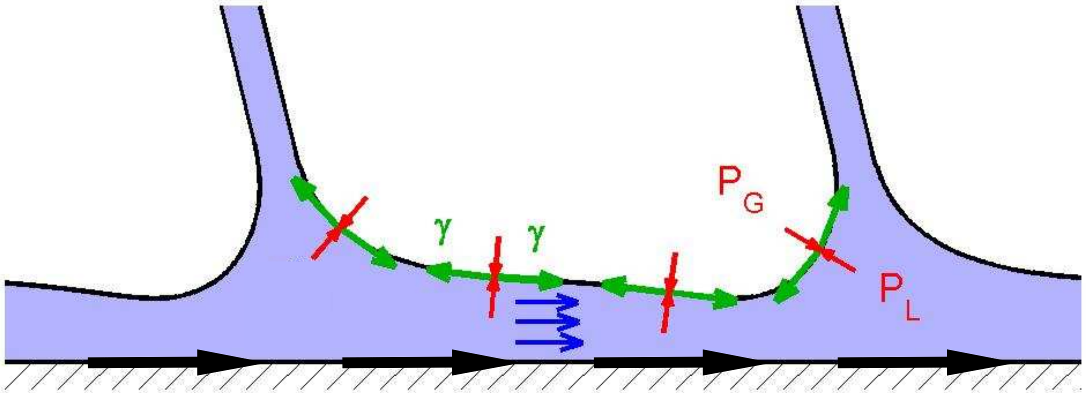
THE BRETHERTON PROBLEM

Rheometer experiments



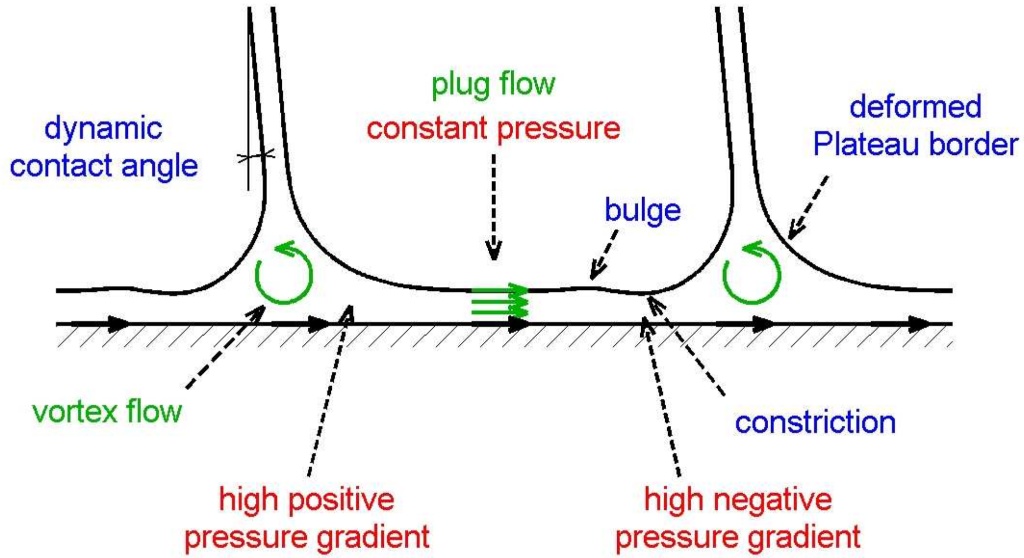
Train of bubbles

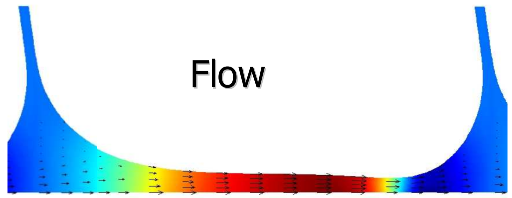


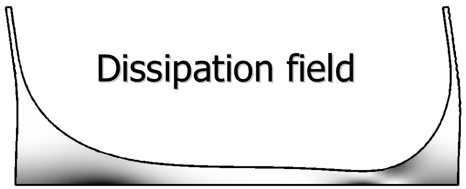


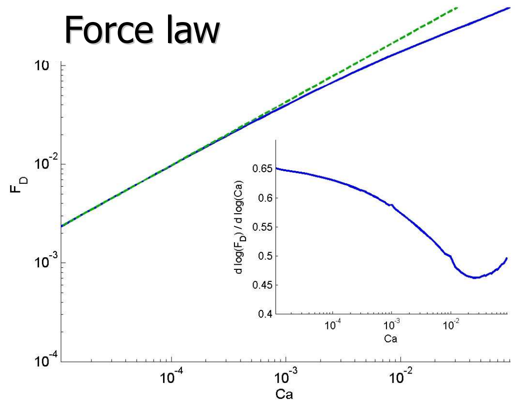
WALL SLIP OF BUBBLES

RESULTS









STILL TO BE DONE :

- **Different boundary conditions (slip, no slip, surface viscosity)**
- **Volume constraint (constant static pressure, constant volume)**
- **2D models, 3D models**
- **Correlation with experiments**

TROU NORMAND

POSSIBILITIES OF FINITE ELEMENT TECHNIQUES :

Numeric solution for

- **Non linear Partial Differential Equations**
- **PDE with specific boundary conditions**
- **PDE with moving geometries**

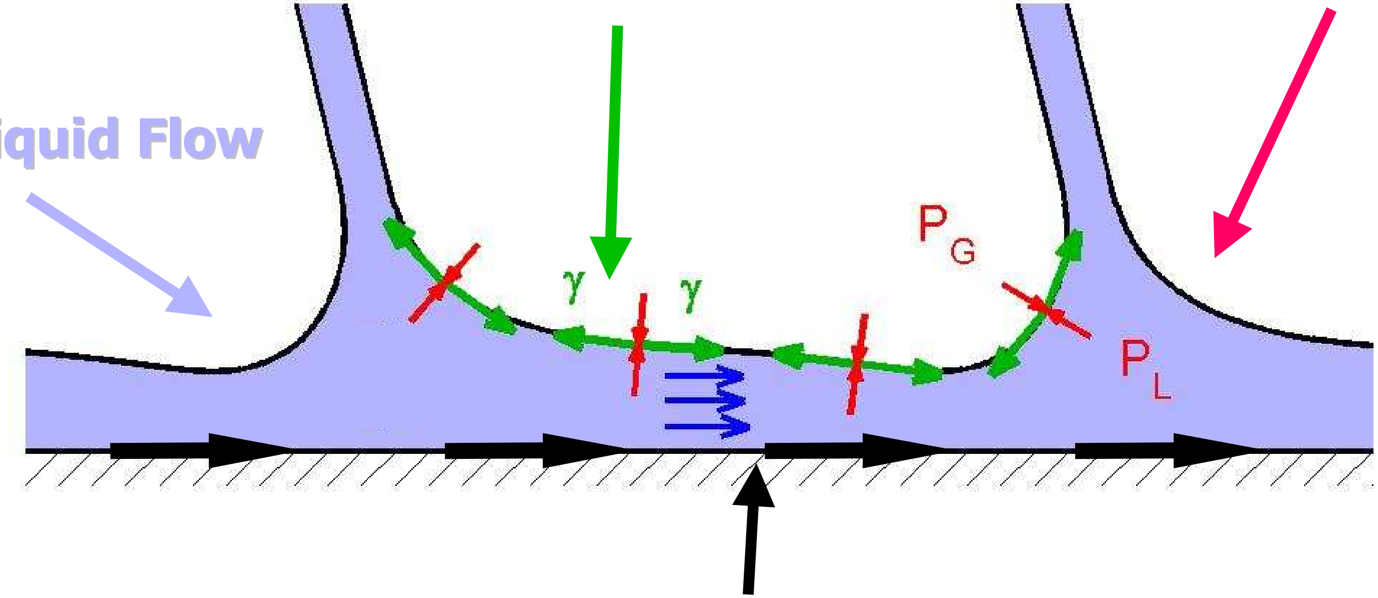
TECHNICAL FEATURES

THE BRETHERTON PROBLEM step by step :

3. Moving geometries

4. Surface tension

1. Bulk liquid Flow



2. Flow boundary conditions

INCOMPRESSIBLE FLUID FLOW

MASS CONSERVATION

$$\nabla \bullet \underline{\mathbf{U}} = 0$$

MOMENTUM EQUATION : TWO EQUIVALENT FUNDAMENTAL LAWS

Navier Stokes Equation => Strong Formulation

$$\underline{\nabla} \bullet \underline{\underline{\boldsymbol{\sigma}}} = \eta_B \Delta \underline{\mathbf{U}} - \underline{\nabla} p = \underline{\mathbf{f}} \qquad + \quad \text{boundary conditions}$$

Virtual Power Law => Weak Formulation

$$\forall \underline{\tilde{\mathbf{U}}} \text{ kinematically admissible : } \int_\Omega \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{\varepsilon}(\tilde{\mathbf{U}})}} = \int_\Omega \underline{\mathbf{f}}.\underline{\tilde{\mathbf{U}}} + \int_{\partial\Omega} \underline{\mathbf{n}}.\underline{\underline{\boldsymbol{\sigma}}}.\underline{\tilde{\mathbf{U}}}$$

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \eta_B\left(\underline{\nabla \mathbf{U}} + {}^t\underline{\nabla \mathbf{U}}\right) \qquad \underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\left(\underline{\nabla \mathbf{U}} + {}^t\underline{\nabla \mathbf{U}}\right)$$

BOUNDARY CONDITIONS

DIRICHLET BC :

$$\left\{ \underline{\mathbf{U}} = 0, \quad \underline{\mathbf{U}} = \underline{\mathbf{U}}_0 \right\} \Rightarrow \quad \mathrm{R}(\underline{\mathbf{U}}) = 0$$

NEUMANN BC :

$$\left\{ \frac{\partial \underline{\mathbf{U}}}{\partial \mathbf{n}} = 0, \frac{\partial \underline{\mathbf{U}}}{\partial \mathbf{n}} = \frac{\underline{\mathbf{U}}}{\delta} \right\} \Rightarrow -\underline{\mathbf{n}}.\underline{\underline{\boldsymbol{\sigma}}} = \mathrm{G}(\underline{\mathrm{U}})$$

MIXED NEUMANN-DIRICHLET BOUNDARY CONDITIONS :

$$\mathrm{R}(\underline{\mathbf{U}}) = 0$$

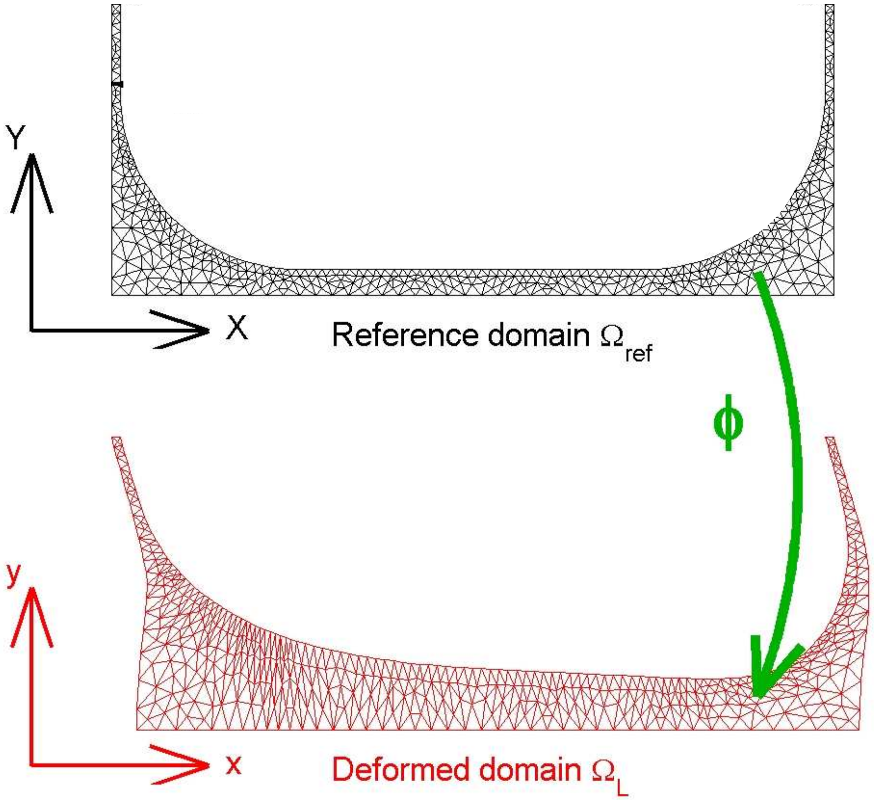
$$-\underline{\mathbf{n}}.\underline{\underline{\boldsymbol{\sigma}}} = \mathrm{G}(\underline{\mathbf{U}}) + \left(\frac{\partial \mathrm{R}}{\partial \underline{\mathbf{U}}} \right)^T \mu$$

PDE BC :

$$\left\{ \eta_S \Delta_{\mathrm{S}} \mathbf{U} = \eta_B \frac{\partial \mathbf{U}}{\partial \mathbf{n}}, \nabla \bullet \sigma = \mathrm{f} \right\} \Rightarrow$$

Modification of the weak formulation of¨Navier Stokes equations

MOVING MESH



$$NS\left\{\underline{\mathbf{U}}(x,y)\right\}=0$$

$$\Downarrow$$

$$NS\left\{\phi\left(\underline{\mathbf{U}}(X,Y)\right)\right\}=0$$

SURFACE TENSION

CURVATURE EQUATION

$$\gamma \nabla \bullet \underline{\mathbf{n}} = p$$

WEAK FORMULATION

Minimization of the bulk + surface energy for a virtual displacement of the interface :

$$0 = -(P_G + \underline{n}.\underline{\underline{\sigma}}.\underline{n}) \int \tilde{\beta} \frac{\partial \alpha}{\partial s} ds + \gamma \int \frac{\partial \beta}{\partial s} \frac{\partial \tilde{\beta}}{\partial s} \left\{ \left(\frac{\partial \alpha}{\partial s} \right)^2 + \left(\frac{\partial \beta}{\partial s} \right)^2 \right\}^{-\frac{1}{2}} ds$$

$$+\gamma \left[\tilde{\beta} \frac{\partial \beta}{\partial s} \left\{ \left(\frac{\partial \alpha}{\partial s} \right)^2 + \left(\frac{\partial \beta}{\partial s} \right)^2 \right\}^{-\frac{1}{2}} \right]_{s=0}^{s=1}$$

Non linear solution
Correction of the Navier Stokes weak formulation

SUMMARY

Physical Aspects

Strong Form

Mixed BC

Weak Form

Weak BC

Moving Geometry

Surface Tension

Technical Requirement