Domain and Range

Find the domain and Range of the function.

1. $f(x,y) = \sqrt{xy}$

Answ.

$\text{Dom}(f) = \{(x,y) \mid (x \geq 0 \text{ and } y \geq 0) \text{ and } (x \leq 0 \text{ and } y \leq 0);\ x,y \in \mathbb{R}\}$

= the first and third quadrant including the axes

$\text{Range}(f) = [0, \infty]$

2. $f(x,y) = \dfrac{1}{x+y}$

Answ. $\text{Dom}(f) = \{(x,y) \mid y \neq -x;\ x,y \in \mathbb{R}\}$

= the set of all points (x,y) not on the line $y = -x$

$\text{Range}(f) = (-\infty, 0) \cup (0, \infty)$

3. $f(x,y) = \dfrac{e^x - e^y}{e^x + e^y}$

Answ. $\text{Dom}(f) = \{(x,y) \mid \forall\, x,y \in \mathbb{R}\}$ = the entire plane

$\text{Range}(f) = (-1, 1)$

4. $f(x,y) = \ln(x,y)$

Answ.

$\text{Dom}(f) = \{(x,y) \mid (x > 0 \text{ and } y > 0) \text{ and } (x < 0 \text{ and } y < 0);\ x,y \in \mathbb{R}\}$

$\text{Range}(f) = (-\infty, \infty)$

Identify and Sketch

1. Write down the equations of :

a. the line

b. the parabola

c. the ellipse

d. the hyperbola

e. the circle

f. the sphere

g. the cylinder

h. the cone

2. Identify and sketch the surface

2.1 $x^2 + 4y^2 - 16z^2 = 0$ Answ: Elliptic cone

2.2 $x - 4y^2 = 0$ Answ: Parabolic cylinder

2.3 $x^2 + y^2 + z^2 - 4 = 0$ Answ: Sphere of radius 2 centered at the origin

2.4 $x^2 + y^2 - z = 0$ Answ: Paraboloid

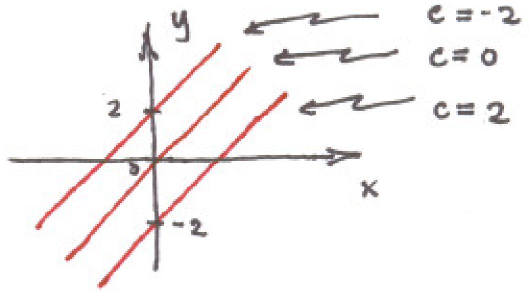
2.5 $2y + 3 - z = 0$ Answ: Plane

Level Curves

Identify the level curves $f(x,y) = c$ and sketch the curves corresponding to the indicated values of c.

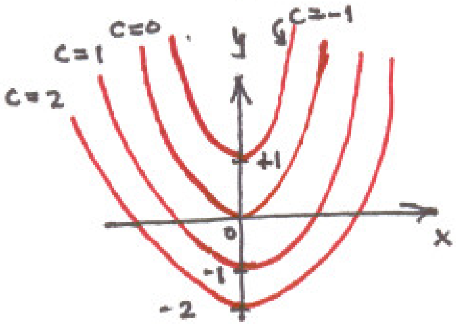
1. $f(x,y) = x - y$; $c = -2, 0, 2$

   Answ: the lines of slope 1 : $y = x - c$

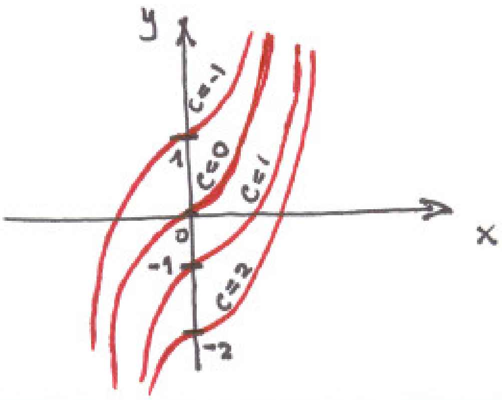


2. $f(x,y) = x^2 - y$; $c = -1, 0, 1, 2$

   Answ: parabolas : $y = x^2 - c$



3. $f(x,y) = x^3 - y$; $c = -1, 0, 1, 2$



Partial Derivatives

Calculate the partial derivatives

1. $f(x,y) = 3x^2 - xy + y$ Answ: $\frac{\partial f}{\partial x} = 6x - y$, $\frac{\partial f}{\partial y} = 1 - x$

2. $\varrho = \sin\phi \cos\theta$ Answ: $\frac{\partial \varrho}{\partial \phi} = \cos\phi\cos\theta$, $\frac{\partial \varrho}{\partial \theta} = -\sin\phi\sin\theta$

3. $f(x,y) = e^{x-y} - e^{y-x}$ Answ: $\frac{\partial f}{\partial x} = e^{x-y} + e^{y-x}$

$$\frac{\partial f}{\partial y} = -e^{x-y} - e^{y-x}$$

4. $g(x,y) = \dfrac{Ax + By}{Cx + Dy}$ Answ: $\frac{\partial g}{\partial x} = \dfrac{(AD - BC)y}{(Cx + Dy)^2}$

$$\frac{\partial g}{\partial y} = \frac{(BC - AD)x}{(Cx + Dy)^2}$$

5. $u = xy + yz + zx$ Answ: $\frac{\partial u}{\partial x} = y + z$, $\frac{\partial u}{\partial y} = x + z$

$$\frac{\partial u}{\partial z} = x + y$$

6. $f(x,y,z) = z\sin(x-y)$ Answ: $\frac{\partial f}{\partial x} = z\cos(x-y)$, $\frac{\partial f}{\partial y} = -z\cos(x-y)$

$$\frac{\partial f}{\partial z} = \sin(x-y)$$

7. $f(x,y) = x^2 y \sec(xy)$ Answ: $\frac{\partial f}{\partial x} = 2x\sec(xy) + x^2y^2\sec(xy)\tan(xy)$

$$\frac{\partial f}{\partial y} = x^2\sec(xy) + x^3y\sec(xy)\tan(xy)$$

8. $h(x,y) = \dfrac{x}{x^2+y^2}$ Answ: $\dfrac{\partial h}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$, $\dfrac{\partial h}{\partial y} = -\dfrac{2xy}{(x^2+y^2)^2}$

9. $f(x,y) = \dfrac{x\sin y}{y\cos x}$ Answ: $\dfrac{\partial f}{\partial x} = \dfrac{\sin y\,(\cos x + x\sin x)}{y\cos^2 x}$

$$\frac{\partial f}{\partial y} = \frac{x\,(y\cos y - \sin y)}{y^2\cos x}$$

10. $f(x,y,z) = z\tan^{-1}\left(\frac{y}{x}\right)$ Answ: $\dfrac{\partial f}{\partial x} = -\dfrac{yz}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = \frac{xz}{x^2+y^2} \qquad \frac{\partial f}{\partial z} = \tan^{-1}\left(\frac{y}{x}\right)$$

11. $f(x,y,z) = z^{xy^2}$ Answ: $\dfrac{\partial f}{\partial x} = (y^2\ln z)\,z^{xy^2}$

$$\frac{\partial f}{\partial y} = (2xy\ln z)\,z^{xy^2}$$

$$\frac{\partial f}{\partial z} = xy^2 z^{xy^2-1}$$

12. Find $\left.\dfrac{\partial f}{\partial x}\right|_{(1,2)}$ and $\left.\dfrac{\partial f}{\partial y}\right|_{(1,2)}$ given that $f(x,y) = \dfrac{x}{x+y}$

Answ: $f_x(1,2) = \dfrac{2}{9}$, $f_y(1,2) = -\dfrac{1}{9}$

Gradients

- Find the gradient of the following functions

1. $f(x,y) = 3x^2 - xy + y$ Answ: $(6x-y)\hat{i} + (1-x)\hat{j}$

2. $f(x,y) = xe^{xy}$ Answ: $e^{xy}\left[(xy+1)\hat{i} + x^2\hat{j}\right]$

3. $f(x,y) = 2xy^2 \sin(x^2+1)$ Answ: $\left[2y^2\sin(x^2+1) + 4x^2y^2\cos(x^2+1)\right]\hat{i} +$
$+ 4xy\sin(x^2+1)\hat{j}$

4. $f(x,y) = e^{x-y} - e^{y-x}$ Answ: $(e^{x-y} + e^{y-x})(\hat{i} - \hat{j})$

5. $f(x,y) = \frac{1}{xy}$ Answ: $-\frac{1}{x^2y^2}\hat{i} - 2\frac{1}{xy^3}\hat{j}$

6. $f(x,y,z) = x^2y + y^2z + z^2x$ Answ: $(z^2 + 2xy)\hat{i} + (x^2 + 2yz)\hat{j} -$
$-(y^2 + 2xz)\hat{k}$

7. $f(x,y,z) = x^2ye^{-z}$ Answ: $e^{-z}(2xy\,\hat{i} + x^2\hat{j} - x^2y\,\hat{k})$

8. $f(x,y,z) = e^{x+2y}\cos(z^2+1)$ Answ: $e^{x+2y}\left[\cos(z^2+1)(\hat{i} + 2\hat{j}) -\right.$
$\left.2z\sin(z^2+1)\hat{k}\right]$

- Find the gradient vector at the point P

1. $f(x,y) = 2x^2 - 3xy + 4y^2$ at $P(2,3)$ Answ: $\nabla f = -\hat{i} + 18\hat{j}$

2. $f(x,y) = \ln(x^2+y^2)$ at $P(2,1)$ Answ: $\frac{4}{5}\hat{i} + \frac{2}{3}\hat{j}$

3. $f(x,y) = x\sin(xy)$ at $P(1,\pi/2)$ Answ: $\hat{i}$

Partial Differential Equations

Show that the given function satisfies the corresponding partial differential equations

1. $u = \dfrac{x^2y^2}{x+y}$; $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$

2. $u = x^2y + y^2z + z^2x$; $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x+y+z)^2$

The Laplace equation in two dimensions is defined as

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Show that the given functions satisfy the Laplace equation in two dimensions.

1. $f(x,y) = x^3 - 3xy^2$

2. $f(x,y) = \ln\sqrt{x^2+y^2}$

The Wave equation is defined by the partial differential equation

$$\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} = 0 \qquad c = \text{constant}$$

Show that the given functions satisfy the wave equation

1. $f(x,t) = (Ax+B)(ct+D)$ $\qquad$ 2. $f(x,t) = \ln(x+ct)$

Directional Derivative

Find the directional derivative at the point P in the direction indicated.

1. $f(x,y) = x^2 + 3y^2$ at $P(1,1)$ in the direction $\hat{i} - \hat{j}$

   Answ: $-2\sqrt{2}$

2. $f(x,y) = xe^y - ye^x$ at $P(1,0)$ in the direction

   $\vec{v} = 3\hat{i} + 4\hat{j}$ Answ: $\frac{1}{5}(7-4e)$

3. $f(x,y) = \dfrac{ax+by}{x+y}$ at $P(1,1)$ in the direction of

   $\vec{v} = \hat{i} - \hat{j}$ Answ: $\frac{1}{4}\sqrt{2}\,(a-b)$

4. $f(x,y) = \ln(x^2+y^2)$ at $P(0,1)$ in the direction of

   $\vec{v} = 8\hat{i} + \hat{j}$ Answ: $\dfrac{2}{\sqrt{65}}$

5. $f(x,y,z) = xy + yz + zx$ at $P(1,-1,1)$ in the direction of

   $\vec{v} = \hat{i} + 2\hat{j} + \hat{k}$ Answ: $\frac{2}{3}\sqrt{6}$

DIRECTIONS OF MOST RAPID INCREASE AND DECREASE

Find the direction in which the function increase and decrease most rapidly at P_0. Then find the ~~directions~~ derivatives of the functions in these directions

1. $f(x,y) = x^2 + xy + y^2$, $P_0(-1,1)$ Answ $\vec{u} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ increase
   $(D_{\vec{u}} f)_{P_0} = \sqrt{2}$

2. $f(x,y,z) = \frac{x}{y} - yz$ $P_0(4,1,1)$ Answ: $\vec{u} = \frac{1}{3\sqrt{3}}\hat{i} - \frac{5}{3\sqrt{3}}\hat{j} - \frac{1}{3\sqrt{3}}\hat{k}$
   $(D_{\vec{u}} f)_{P_0} = 3\sqrt{3}$

3. $f(x,y,z) = \ln(xy) + \ln(yz) + \ln(xz)$ $P_0(1,1,1)$ Answ: increase
   $\vec{u} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$
   $(D_{\vec{u}} f)_{P_0} = 2\sqrt{3}$

TANGENT PLANES AND NORMAL LINES

Find equations for the tangent plane and the normal line at the given point on the surface

4. $x^2 + y^2 + z^2 = 3$ $P_0(1,1,1)$ Answ: a) $x + y + z = 3$
   b) $x = 1 + 2t$, $y = 1 + 2t$, $z = 1 + 2t$

5. $2z - x^2 = 0$ $P_0(2,0,2)$ Answ: a) $2x - z - 2 = 0$

6. $x + y + z = 1$ $P_0(0,1,0)$ Answ: a) $x + y + z - 1 = 0$ b) $x = t$, $y = 1 + t$, $z = t$

Find an equation for the plane that is tangent to the given surface at the given point

7. $z = \ln(x^2 + y^2)$ $P_0(1,0,0)$ Answ: $2x - z - 2 = 0$

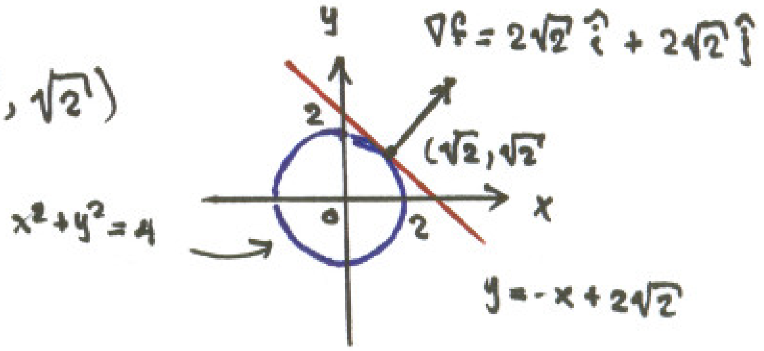
8. $z = \sqrt{y - x}$ $P_0(1,2,1)$ Answ: $x - y + 2z - 1 = 0$

TANGENT LINES TO CURVES

Sketch the curve $f(x,y) = c$ together with ∇f and the tangent line at the given point. Then write an equation for the tangent line

1. $x^2 + y^2 = 4$, $(\sqrt{2}, \sqrt{2})$

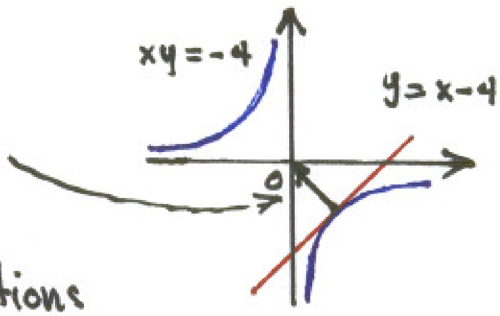
   ANSW:



2. $xy = -4$, $(2,-2)$

   Answ:

   $\nabla f = -2\hat{i} + 2\hat{j}$



Finding Linearizations

Find the linearization $L(x,y)$ of the function at each point

1. $f(x,y) = x^2 + y^2 + 1$ at $(0,0)$, b) $(1,1)$

   Answ: a) $L(x,y) = 1$ b) $L(x,y) = 2x + 2y - 1$

2. $f(x,y) = 3x - 4y + 5$ at a) $(0,0)$, b$(1,1)$

   Answ: a) $L(x,y) = 3x - 4y + 5$ b) $L(x,y) = 3x - 4y + 5$

3. $f(x,y) = e^x \cos y$ at a) $(0,0)$, b) $(0, \frac{\pi}{2})$

   Answ: a) $L(x,y) = 1 + x$ b) $L(x,y) = -y + \frac{\pi}{2}$

UPPER BOUNDS FOR ERRORS IN LINEAR APROXIMATIONS

Find the linearization $L(x,y)$ of the function $f(x,y)$ at P_0. Then find an upper bound for the magnitud $|E|$ of the error in the approximation $f(x,y) \approx L(x,y)$ over the rectangle R.

1. $f(x,y) = x^2 - 3xy + 5$ at $P_0(2,1)$ $R: |x-2| \le 0.1,\ |y-1| \le 0.1$

   Answ : $7 + x - 6y$; 0.06

2. $f(x,y) = 1 + y + x\cos y$ at $P_0(0,0)$ $R: |x| \le 0.2,\ |y| \le 0.2$
   (Use $|\cos y| \le 1$ and $|\sin y| \le 1$ in estimating E.)

   Answ : $L(x,y) = x + y + 1$; 0.08

3. $f(x,y) = e^x \cos y$ at $P_0(0,0)$, $R: |x| \le 0.1,\ |y| \le 0.1$
   (Use $e^x \le 1.11$ and $|\cos y| \le 1$ in estimating E.)

   Answ: $L(x,y) = 1 + x$; 0.0222

Finding Cubic Aproximations

Use ~~fo~~ Taylor's formula for $f(x,y)$ at the origin to calculate the cubic approximation of f near the origin

1. $f(x,y) = xe^y$ Answ: $x + xy + \frac{1}{2}xy^2$

2. $f(x,y) = y\sin x$ Answ: cubic approx: xy

FINDING LOCAL EXTREMA

Find all the local maxima, local minima and saddle points of the functions:

1. $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

   Answ: $f(-3,3) = -5$, local minimum

2. $f(x,y) = x^2 + xy + 3x + 2y + 5$

   Answ: $f(-2,1)$, saddle point

3. $f(x,y) = \dfrac{1}{x^2 + y^2 - 1}$

   Answ: $f(0,0) = -1$, local maximum

FINDING ABSOLUTE EXTREMA

Find the absolute maxima and minima of the functions on the given domains

1. $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant.

   Answ: Abs. maximum: 1 at $(0,0)$; abs. minimum: -5 at $(1,2)$

2. $f(x,y) = (4x - x^2)\cos y$ on the rectangular plate $1 \le x \le 3$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$

   Answ: Abs. Max: 4 at $(2,0)$; Abs. Min: $\frac{3\sqrt{2}}{2}$ at $\left\{ \begin{array}{l} (3,-\frac{\pi}{4}), (3,\frac{\pi}{4}) \\ (1,-\frac{\pi}{4}), (1,\frac{\pi}{4}) \end{array} \right.$

LAGRANGE MULTIPLIERS

1. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x,y) = xy$ has its extreme values.

   Answ: $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

2. Find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

   with sides parallel to the coordinate axis. Answ: $L = 4\sqrt{2}$, $W = 3\sqrt{2}$

3. Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint

$$x^2 - 2x + y^2 - 4y = 0$$

   Answ: $f(0,0) = 0$ is minimum, $f(2,4) = 20$ is maximum

4. Find the maximum and minimum values of

$$f(x,y,z) = x - 2y + 5z$$

   on the sphere $x^2 + y^2 + z^2 = 30$

   Answ: $f(1,-2,5) = 30$ is maximum

   $f(-1,2,-5) = -30$ is minimum

Double Integrals

Evaluate the following integrals

1. $\int_0^1 \int_0^2 (x+3)\,dy\,dx$; Answ: 7

2. $\int_2^4 \int_0^1 x^2 y\,dx\,dy$; Answ: 2

3. $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\,dy\,dx$; Answ: 2

4. $\int_0^{\ln 2} \int_0^1 xy e^{y^2 x}\,dy\,dx$; Answ: $\frac{1-\ln 2}{2}$

5. $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx$; Answ: $1-\ln 2$

Evaluate the integral over the given region. (Sketch the region First)

1. $\iint_R 4xy^3\,dA$, $R = \{(x,y) \mid -1 \le x \le 1 ; -2 \le y \le 2\}$

Answ: 0

2. $\iint_R x\sqrt{1-x^2}\,dA$, $R = \{(x,y) \mid 0 \le x \le 1 ,\ 2 \le y \le 3\}$

Answ: $\frac{1}{3}$

3. Find the volume under the surface $z = 2x + y$ and over the rectangle $R = \{(x,y) \mid 3 \le x \le 5 ,\ 1 \le y \le 2\}$

Answ: 19

Evaluate the integral

1. $\displaystyle\int_0^1\int_{x^2}^{x} xy^2\,dy\,dx$; Answ: $\frac{1}{40}$

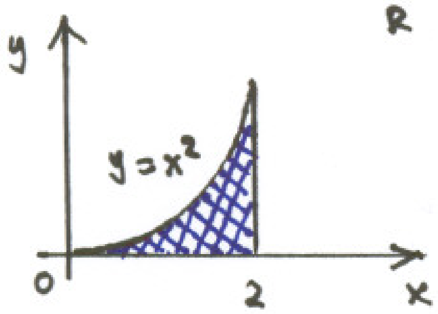
2. $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}} y\,dx\,dy$; Answ: 9

3. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}}\int_0^{x^3} \sin\left(\frac{y}{x}\right)dy\,dx$; Answ: $\frac{\pi}{2}$

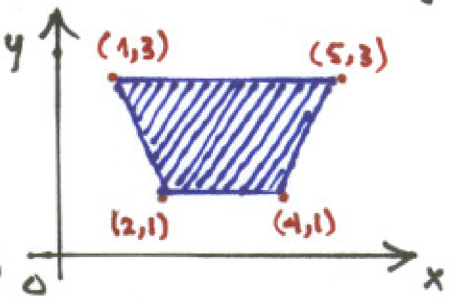
4. $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\left(\frac{y}{x}\right)dy\,dx$; Answ: 1

5. In each part find $\displaystyle\iint_R xy\,dA$ over the shaded region R.

a)



b)



Answ: a) $\frac{16}{3}$ b) 38

6. Express the integral as an equivalent integral with the order of integration reversed.

a) $\displaystyle\int_0^2\int_0^{\sqrt{2}} f(x,y)\,dy\,dx$, Answ: $\displaystyle\int_0^{\sqrt{2}}\int_{y^2}^{2} f(x,y)\,dx\,dy$

b) $\displaystyle\int_0^1\int_{\sin^{-1}y}^{\pi/2} f(x,y)\,dx\,dy$ Answ: $\displaystyle\int_0^{\pi/2}\int_0^{\sin x} f(x,y)\,dy\,dx$

7. Evaluate the integral by first reversing the order of integration

a) $\displaystyle\int_0^1\int_{4x}^{4} e^{-y^2}\,dy\,dx$

Answ: $\dfrac{1-e^{-16}}{8}$

b) $\displaystyle\int_0^4\int_{\sqrt{y}}^{2} e^{x^3}\,dx\,dy$

Answ: $\dfrac{e^8-1}{3}$

Double Integrals in Polar Coordinates

Evaluate the integral

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta \, dr\, d\theta$ Answ: $\frac{1}{6}$

2. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2 \, dr\, d\theta$ Answ: $\frac{2}{9}a^3$

3. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta \, dr\, d\theta$ Answ: 0

Find the area of the region described

1. Cardioid : $r = 1-\cos\theta$ Answ: $\frac{3\pi}{2}$

2. The region in the first quadrant bounded by $r=1$ and $r=\sin 2\theta$, with $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$ Answ: $\frac{\pi}{16}$

3. The region inside the circle $r=4\sin\theta$ and outside the circle $r=2$
   Answ: $\frac{4\pi}{3}+2\sqrt{3}$

Explain, geometrically, how can we find that in polar coordinates $dA = r\,dr\,d\theta$. Answ: In the notes.

Evaluating Polar Coordinates

Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral

1. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$ Answ: $\frac{\pi}{2}$

2. $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$ Answ: $\pi/8$

3. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$ Answ: πa^2

4. $\int_{0}^{6}\int_{0}^{y} x\,dx\,dy$ Answ: 36

MASS AND FIRST MOMENTS

If $\delta(x,y)$ is the density $M = \iint_R \delta(x,y)\,dA$ is the mass, $M_x = \iint_R y\,\delta(x,y)\,dA$ & $M_y = \iint_R x\,\delta(x,y)\,dA$ the first moments and $\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$ the coordinates of the center of mass. Find all these quantities for the plate covering the triangular region bounded by the x-axis and the lines $x=1$ and $y=2x$ in the first quadrant. The plate density at (x,y) is $\delta(x,y) = 6x+6y+6$. Answ: $M=14$, $M_x=11$, $M_y=10$, $\bar{x}=\frac{5}{7}$, $\bar{y}=\frac{11}{14}$

TRIPLE INTEGRALS

Evaluate the integrals

1. $\displaystyle\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$ Answ: 1

2. $\displaystyle\int_1^e\int_1^e\int_1^e \frac{1}{xyz}\,dx\,dy\,dz$ Answ: 1

3. $\displaystyle\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx$ Answ: 7/6

4. Rewrite the integral as an equivalent iterated integral in the order shown.

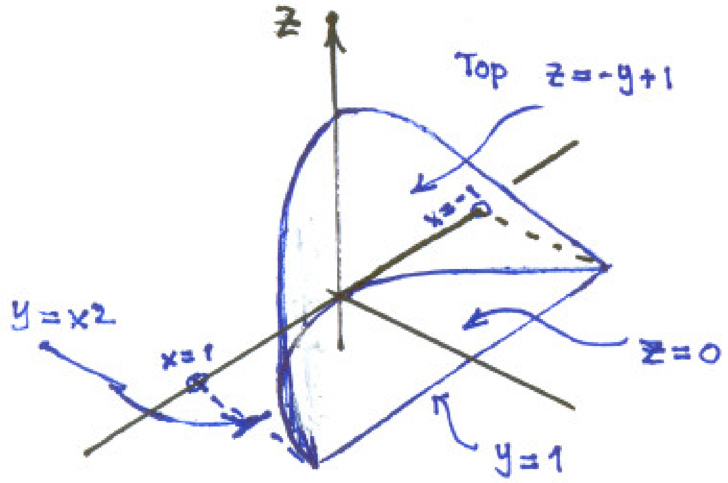
$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz\,dy\,dx$$

a) $dy\,dz\,dx$

b) $dy\,dx\,dz$

c) $dx\,dy\,dz$

Hint: from the limits of integration sketch the region



Answ:

a) $\displaystyle\int_{-1}^{1}\int_0^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$

b) $\displaystyle\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$

c) $\displaystyle\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$

Evaluating integrals in Cylindrical Coordinates

Evaluate the integrals

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta$ Answ: $\dfrac{4\pi(\sqrt{2}-1)}{3}$

2. $\displaystyle\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\, r\, dr\, d\theta$ Answ: $\dfrac{17\pi}{5}$

3. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\, dz\, r\, dr\, d\theta$ Answ: $\pi(6\sqrt{2}-8)$

SPHERICAL COORDINATE INTEGRALS

Evaluate the integral

4. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ Answ: π^2

5. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ Answ: $\dfrac{\pi}{3}$

6. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ Answ: 5π

JACOBIANS AND COORDINATE TRANSFORMATIONS

1. Solve the system $u = x - y$ $\quad v = 2x + y$ for x and y in terms of u and v. Then find the value of the Jacobian.

   Answ: $x = \frac{u+v}{3}$, $y = \frac{v-2u}{3}$, $J = \frac{1}{3}$

2. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0,0)$, $(1,1)$ and $(1,-2)$ in the xy-plane. Sketch the transformed region in the uv-plane

   Answ: Triangular region with boundaries $u = 0$ $\quad v = 0$ and $u + v = 3$

3. Find the Jacobian for the transformation

   (a) $x = u\cos v$
   $y = u\sin v$

   (b) $x = u\sin v$
   $y = u\cos v$

   Answ:
   (a) $J = u\cos^2 v + u\sin^2 v = u$
   (b) $J = -u\sin^2 v - u\cos^2 v = -u$

4. Let Ω be the parallelogram bounded by

   $x + y = 0$ $\qquad x - y = 0$
   $x + y = 1$ $\qquad x - y = 2$

   Evaluate $\iint_\Omega (x^2 - y^2)\,dx\,dy$

   Answ: $\frac{1}{2}$