Theorem 0.1 (Rolles Theorem) *Let f be a continuous real valued function defined on some interval $[a,b]$ & differentiable on all (a,b). If $f(a) = f(b) = 0$ then $\exists$ some $s \in [a,b]$ s.t. $f'(s) = 0$.*

Proof f is continuous on $[a,b]$ therefore assumes absolute max and min values on $[a,b]$. These can only occur at :

1. Points on $[a,b]$ where $f'(x)$ doesn't exist.
2. The end points $a \& b$.
3. Some internal point s where $f'(s) = 0$

1. Void by hypothesis (f is continuous).
2. If either the end points $a \& b$ are a max or min then f is a constant function and s can be taken anywhere in $[a,b]$.
3. If a max or min occurs at some internal point s in $[a,b]$ then $f'(s) = 0$ and we have a point for the theorem. ∎

Theorem 0.2 (Mean Value Theorem) *Let f be a continuous real valued function defined on some interval $[a,b]$ & differentiable on all (a,b). Then $\exists$ some $s \in [a,b]$ s.t. $a < s < b$ & $\frac{f(b)-f(a)}{b-a} = f'(s)$*

Proof Let $g(x) : [a,b] \Longrightarrow R$ be defined by

$$g(x) = f(x) - \frac{b-x}{b-a} f'(b) - \frac{x-a}{b-a} f'(a)$$

Applying Rolles Theorem to g on $[a,b]$ $\exists$ some $s \in [a,b]$ s.t. $g'(s) = 0$

$$\Rightarrow g'(s) = f'(s) - \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = f'(s)$$

as required ∎

Theorem 0.3 (Cauchy Mean Value Theorem) *Let f & g be continuous real valued functions defined on some interval $[a,b]$ and differentiable on all (a,b). Then $\exists$ some $s \in R$ s.t. $(a < s < b)$ &*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(s)}{g'(s)}$$

when $g'(s)$ & $g(b) - g(a)$ are both not 0.

Proof Considder $h : [a,b] \Longrightarrow R$ defined by

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

then

$$h(a) = f(a)g(b) - f(b)g(a) = h(b)$$

and h satisfies *Rolles Theorem*
The result follows imediately. ∎

Theorem 0.4 (Taylors Theorem) *Let f be a continuous real valued function defined on some interval containing $s, s+h$ $(s, h \in R)$. Then*

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h)$$

for some $\theta \in R$ s.t. $(0 < \theta < 1)$.

Proof Let $p : [a,b] \Longrightarrow R$ be defined by

$$p(t) = f(s+th) - f(s) - \sum_{n=1}^{k-1} \frac{t^n h^n}{n!} f^{(n)}(s)$$

by calculation it is clear $p^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k-1$

$$\Longrightarrow \; if \; q(t) = p(t) - p(1)t^k$$

$\forall t \in [0,1]$ then $q^{(n)}(0) = 0$ & $q(1) = 0$

Applying *Rolles Theorem* to q on $[0,1]$ we deduce the existence of $t_1 \in R$ s.t. $(o < t_1 < 1)$ & $q'(t_1) = 0$.

Subsequently applying *Rolles Theorem* to q' on $[0, t_1]$ we deduce the existence of $t_2 \in R$ s.t. $(0 < t_2 < t_1)$ & $q^{ii}(t_2) = 0$

Continuing in this fashion, applying *Rolles Theorem* to $q^{ii}, q^{iii}, \ldots, q^{(k-1)}$ we deduce the existence of $t_1, t_2, \ldots, t_k)$ s.t $(0 < t_k < \ldots < t_1 < 1)$ & $q^{(n)}(t_k) = 0$, Let $\theta = t_k$. Then $(0 < t_k < 1)$ &

$$0 = \frac{1}{k!} q^{(k)}(\theta) = \frac{1}{k!} p^{(k)}(\theta) - p(1) = \frac{h^k}{k!} f^{(k)}(s + \theta h) - p(1)$$

$$\Longrightarrow f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + p(1)$$

$$\Longrightarrow f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h)$$ ∎

Theorem 0.5 *Let* $a\&b \in \mathcal{R}$,$(a < b)$. *Then any continuous real valued function on* $[a, b]$ *is Riemann Integrable.*

Proof Let f be a continuous real valued function on $[a, b]$. Then f is bounded above and below on $[a, b]$.

Let $\epsilon > 0$ be given. Then $\exists$ *some* $\delta > 0$ s.t. $|f(x) - f(y)| < \epsilon \; \forall \; x, y \in [a, b]$ s.t. $|x - y| < \delta$

Choose some partion P on $[a, b]$ s.t. the length of each subinterval is $< |\delta|$. Where $P = \{x_0, x_1, \ldots, x_n\}$ and $a = x_0 < x_1 < \ldots < x_n = b$.

If $x_{i-1} < x < x_i$ then $|x_i - x| < \delta$

$$\implies f(x_i) - \epsilon < f(x) < f(x_i) + \epsilon$$

$$\implies f(x_i) - \epsilon < m_i < M_i < f(x_i) + \epsilon$$

Where $m_i = inf\{f(x) : x_{i-1} < x < x_i\}$ amd $M_i = sup\{f(x) : x_{i-1} < x < x_i\}$

$$\implies \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \epsilon(b-a) \leq L(P, f) \leq U(P, f) \leq \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \epsilon(b-a)$$

Where L & U are the upper and lower sums of P on $[a, b]$.

$$\implies 0 \leq \mathcal{U}\int_a^n f(x)dx - \mathcal{L}\int_a^b f(x)dx \leq U(P, f) - L(P, f) \leq 2\epsilon(b-a)$$

Solving the inequality for any $\epsilon > 0$

$$\mathcal{L}\int_a^b f(x)dx \leq \mathcal{U}\int_a^b f(x)dx$$

$\implies f$ is Riemann Integrable. ■

Theorem 0.6 (Fundamental Theorem of Calculus) *Let* f *be a continuous real valued function on* $[a, b]$ *where* $(a < b)$ *then is*

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = f(x)$$

Proof Let $F(s) = \int_a^s f(t)dt$. Now f is continuous at $x \implies$ given any $\epsilon > 0$ $\exists$ some $\delta > 0$ s.t. $|f(t) - f(x)| < \frac{1}{2}\epsilon \; \forall t, x \in [a, b]$ s.t. $|t - x| < \delta$ now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))dt$$

if $0 < |h| < \delta$ & $x + h \in [a, b]$ then

$$|\int_x^{x+h} (f(t) - f(x))dt| < \frac{1}{2}\epsilon|h|$$

$$\Longrightarrow |\frac{F(x+h) - F(x)}{h} - f(x)| < \frac{1}{2}\epsilon < \epsilon$$

$$\Longrightarrow F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \qquad \blacksquare$$

Theorem 0.7 *Let $f_1, f_2, \ldots$ be a sequence of continuous real valued functions that converge uniformly to some real valued funcion f on $[a, b]$ Then*

$$\lim_{n \to +\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Proof Let $\epsilon > 0$ be given. Choose some ϵ_0 s.t. $0 < \epsilon_0(b - a) < \epsilon$. Then $\exists$ some $N \in \mathcal{N}$ s.t. $|f_n(x) - f(x)| < \epsilon_0$ ($\forall x \in [a, b]$ & $n \geq N$) now

$$-\int_a^b |f_n(x) - f(x)|dx \leq \int_a^b f_n(x)dx - \int_a^b f(x)dx \leq \int_a^b |f_n(x) - f(x)|dx$$

$$\Longrightarrow |\int_a^b f_n(x)dx - \int_a^b f(x)dx| \leq \int_a^b |f_n(x) - f(x)|dx \leq \epsilon_0(b-a) < \epsilon\, whenn \geq N$$

The Result therefore follows. $\blacksquare$

Theorem 0.8 (Bolzano Weierstrass Theorem) *Every bounded series of complex numbers has a convergent sub-sequence.*

Proof Let $z_1, z_2, \ldots, z_n$ be a bounded series of complex numbers & let $z_n = x_n + iy_n$

The Bolzano Weierstrass Theorem for sequences of real numbers guarantees the existence of a sub-seqence $z_{n1}, z_{n2}, \ldots$ of the given series s.t. the real parts $x_1, x_2, \ldots$ converge.

A further application of this theorem allows us to replace this sub-sequence with a further subsequence to ensure that the imaginary parts $y_1, y_2, \ldots$ converge aswell.

If

$$\lim_{n \to +\infty} z_n = l \;\&\; \lim_{n \to +\infty} x_n = \lambda \;\&\; \lim_{n \to +\infty} y_n = \mu$$

then $l = \lambda + \mu$

$\Longrightarrow\ z_{n1}, z_{n2}, \ldots\ is\ a\ convergent\ subsequence\ of\ z_1, z_2, \ldots, z_n \qquad \blacksquare$

Theorem 0.9 (Cauchy's Criterion for Convergence) *An infinite series of complex numbers is convergent IFF it is a Cauchy Sequence*

Proof 1^{st} show that convergent sequences are Cauchy Sequences

Let $z_1, z_2, \ldots$ be a convergent sequence of complex numbers &
Let $\lim_{n\to+\infty} z_n = l$
Let $\epsilon > 0$ be given. Then $\exists$ some $N in \mathcal{N}$ s.t. $|z_n - l| < \frac{1}{2}\epsilon \; when (n \geq N)$. If $m \& n \geq N$ then $|z_m - l| < \frac{1}{2}\epsilon$ and $|z_n - l| < \frac{1}{2}\epsilon$

$$\implies |z_m - z_n| = |(z_m - l) - (z_n - l)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

$$\implies |z_m - z_n| < \epsilon \implies z_1, z_2, \ldots \; isaCauchySequence.$$

2^{nd} show that and Cauchy Sequence is convergent

Cauchy Sequences are bounded therefore have convergent subsequences, ie $z_1, z_2, \ldots$ has a convergent sub-sequence $z_{n1}, z_{n2}, \ldots$ & $\lim_{j\to+\infty} z_{nj} = l$
We claim $\lim_{n\to+\infty} z_n = l$
Let $\epsilon > 0$ be given. Then $\exists$ some $N \in \mathcal{N}$ s.t. $|z_m - z_n| < \frac{1}{2}\epsilon \; \forall \; m \& \; n \geq N)$
Let j be chosen large enough s.t. $n_j \geq N \& \; |z_{nj} - l| < \frac{1}{2}\epsilon$

$$\implies |z_n - l| \leq |z_n - z_{nj}| + |z_{nj} - l| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

$$\implies |z_n - l| < \epsilon \implies \lim_{n\to+\infty} z_n = l \quad \blacksquare$$

Theorem 0.10 *Let D be a subset of $\mathcal{C}$ & let $f_1, f_2, \ldots$ be a sequence of of continuous functions mapping $D \Rightarrow \mathcal{C}$ which is uniformly convergent on D to some $f : D \Rightarrow \mathcal{C}$. Then f is continuous.*

Proof Let $w \in D$. R.T.P f is continuous at w.
Let $\epsilon > 0$ be given then $\exists$ some $\delta > 0$ s.t. $|f(z) - f(w)| < \epsilon \; \forall \; z, w \in D$ s.t. $|z - w| < \delta$.
Now we can find some $N \in \mathcal{N}$ s.t. $|f_n(z) - f(z)| < \frac{1}{3}\epsilon \, \forall \, n \geq N$ $(n \, is \, independant \, of \, N)$.
Choose any n s.t. $n \geq N$ now we can find some $\delta > 0$ s.t.

$$|f_n(z) - f_n(w)| < \frac{1}{3}\epsilon \; \forall \; z, w \in D |z - w| < \delta$$

But then

$$|f(z) - f(w)| < |f(z) - f_n(z)| + |f_n(z) - f_n(w)| + |f_n(w) - f(w)| < \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon$$

$\implies |f(z) - f(w)| < \epsilon$ when $|z - w| < \delta$
$\implies f$ is continuous at w. $\blacksquare$

Theorem 0.11 (Alternating Series Test) *Let $a_1, a_2, \ldots$ be non-negative real numbers. Suppose $a_1 \geq a_2 \geq \ldots$ & $lim_{n\to+\infty} a_n = 0$ Then,*

$$\sum_{n=1}^{+\infty}(-1)^{n-1}a_n \text{ is convergent}$$

Proof For each $m \in \mathcal{N}$ let $s_m = \sum_{n=1}^{m}(-1)^{n-1}a_n$
now $s_{2k+1} = s_{2k} - a_{2k} - a_{2k+1} \leq s_{2k-1}$
and $s_{2k+2} = s_{2k} + a_{2k+1} - a_{2k+2} \geq s_{2k}\ \forall\ k \in \mathcal{N}$

Therefore the sub-sequence $s_1, s_3, s_5, \ldots$ is non-increasing,

And the sub-sequence $s_2, s_4, s_6, \ldots,$ is non-decreasing.

But $s_2 \leq s_{2k} \leq s_{2k-1} \leq s_1$. Therefore these subsequences are bounded, thus convergent, and also share same limit. Because

$$\lim_{k\to+\infty} s_{2k} - \lim_{k\to+\infty} s_{2k-1} = \lim_{k\to+\infty}(s_{2k} - s_{2k-1}) = \lim_{k\to+\infty} a_{2k} = 0$$

We Claim $\sum_{n=1}^{+\infty}(-1)^{n-1}a_n = s$, $s = \lim_{k\to+\infty} s_{2k} = \lim_{k\to+\infty} s_{2k-1}$
Let $\epsilon > 0$ be given. Then $\exists\ K_1, K_2 \in \mathcal{N}$ s.t.

$$|s_{2k} - s| < \epsilon \text{ when } k \geq K_1 \ \&\ |s_{2k-1} - s| < \epsilon \text{ when } k \geq K_2$$

Choose N s.t. $N \geq 2K_1 - 1$ & $N \geq 2K - 2$ Then

$$|s_m - s| < \epsilon \text{ when } m \geq N$$

$\implies \sum_{n=1}^{+\infty}(-1)^{n-1}a_n = \lim_{n\to+\infty} s_m = s$ ∎

Theorem 0.12 (Cauchy Product) *The Cauchy Product $\sum_{n=1}^{+\infty} c_n$ of two absolutely convergent infinite series $\sum_{n=1}^{+\infty} a_n$ & $\sum_{n=1}^{+\infty} b_n$ is itself absolutely convergent and*

$$\sum_{n=1}^{+\infty} c_n = \left(\sum_{n=1}^{+\infty} a_n\right)\left(\sum_{n=1}^{+\infty} b_n\right)$$

Proof OMITTED

Theorem 0.13 *Let $\sum_{n=0}^{+\infty}(z - z_0)^n$ be a Power Series with radius of convergence R_0 & let $s(z)$ be the sum of the power series at those complex numbers z for which the series converges. Then*

1. If $R_0 = +\infty$ then $s(z)$ is a continuous function defined on the complex plane.

2. If $R_0 < +\infty$ then $s(z)$ is a continuous function defined over the entire disk $\{z \in \mathcal{C} : |z - z_0| < R_0\}$ bounded by the circle of convergence of the series.

Proof Let $z_1 \in \mathcal{C}$ satisfy $|z_1 - z_0| < R_0$. Then we can choose R st.

$$|z_1 - z_0| < R < R_0 \ (R < +\infty)$$

$\Longrightarrow \ \exists$ *some* $w \in \mathcal{C}$ s.t. $R < |w| < R_0$ & $\sum_{n=1}^{+\infty} a_n w^n$ *converges*
Choose some $A > 0$ $A \in \mathcal{R}$ s.t. $|a_n w^n| \leq \ A \ \forall \ n$
Set $\rho = \frac{R}{|w|}$ & $M_n = A\rho^n$
If $|z - z_0| < R$ then $|a_n(z - z_0)^n| \leq |a_n|R^n \leq A\rho^n = M_n \ \forall \ n$
Also $\sum_{n=1} +\infty M_n$ converges to $\frac{A}{1-\rho}$
By the Weierstrass M-test the P-series $\sum_{n=0}^{+\infty} a_n(z - z_0)^n$ converges uniformly on disk $\{z \in \mathcal{C} : |z - z_0| < R\}$

$\Longrightarrow$ *the restriction of the function s to the disk is continuous & in particular is continuous arround z.*

$\Longrightarrow$ *we deduce s is continuous in all* $\mathcal{C}$ *when* $R_0 = +\infty$

$\Longrightarrow$ *and is continuous inside the circle of convergence if* $R_0 < +\infty$ ∎