2 and 3 Refinable Functions

David Malone

22 December 1999
A 2-refinable function satisfies:

\[ f(x) = \sum_k c_k f(2x - k). \]

The equation is a dilation equation.

People are usually interested in compactly supported \( L^1 \) solutions for wavelets with \( \sum c_k = 2 \). In this case there is at most one solution.
$L^2$ solutions are not unique. They are in correspondence with $L^2(\pm[1, 2))$ or $L^\infty(\pm[1, 2))$. However, adding more scales gave uniqueness in operator results.

What functions are 2 and 3 refinable? ie.

$$f(x) = \sum_k c_k f(2x-k) = \sum_k d_k f(3x-k).$$

Assumptions:

- Only finitely many non-zero $c_k$ and $d_k$.
- Functions compactly supported.
Lemma 1 Suppose \( g(x) = \sum d_k g(2x - k) \), and only finitely many of the \( d_k \) are non-zero. Then we can find \( l \) so that when we translate \( g \) by \( l \) to get \( f \) we find:
\[
f(x) = \sum c_k f(2x - k), \quad c_0 \neq 0, \quad c_k = 0 \text{ when } k < 0 \text{ and } c_k = d_{k-l}.
\]

Lemma 2 If \( f \) is compactly supported and satisfies a dilation equation
\[
f(x) = \sum c_k f(2x - k), \quad \text{where } c_0 \neq 0 \text{ and } c_k = 0 \text{ when } k < 0,
\]
then \( f \) is zero almost everywhere in \((-\infty, 0)\).
\[ f(x) = \sum_{k=0}^{n} c_k f(2x - k) \quad x \in \left( \frac{n}{2}, \frac{n+1}{2} \right) \]

\[ f(x) = \frac{f\left(\frac{x}{2}\right)}{c_0} - \sum_{k=1}^{n} c_k f(x - k) \quad x \in [n, n+1). \]
If you do this for two scales, everything lines up and we get:

$$f(x) = c_0 f(2x) = d_0 f(3x),$$
on $[0, 1/2) \cap [0, 1/3)$.

**Theorem 3** Suppose $f$ is 2 and 3 refinable, say:

$$f(x) = \sum_k c_k f(2x-k) = \sum_k d_k f(3x-k),$$

and $c_0 \neq 0$ and $c_k = 0$ when $k < 0$.

Suppose further that $f$ is integrable on some interval $[0, \epsilon]$, then $f(x) = \gamma x^\beta$ on $[0, 1)$ where $\beta = -\log_2 c_0 = -\log_3 d_0$. 

6
In $L^p$ this will be an almost everywhere relationship. So we integrate:

$$F(x) = \int_0^x f(t) \, dt.$$ 

We can show this satisfies:

$$F(2^n 3^m \alpha) = \left( \frac{2}{c_0} \right)^n \left( \frac{3}{d_0} \right)^m F(\alpha)$$

Continuity forces $\log_2 c_0 = \log_3 d_0 = -\beta$. So:

$$F(2^n 3^m \alpha) = (2^n 3^m)^{\beta+1} F(\alpha)$$

One continuous function does this:

$$x^{\beta+1} \frac{F(\alpha)}{\alpha^{\beta+1}}$$
Giving \( f \) on \([0, 1/3]\):

\[
f(x) = \frac{d}{dx} F(x) = (\beta + 1)x^\beta \frac{F(\alpha)}{\alpha^{\beta+1}}.
\]

From here it is easy to show \( f \) must have the form:

\[
f(x) = \sum_{l=0}^{n} a_l (x - l)^\beta,
\]

on \([n, n + 1)\). For this to be compactly supported \( \beta \in \mathbb{N} \). This means that \( f \) must be a B-spline.
The fact that $f$ behaves like $x^\beta$ on $[0,1)$ is interesting, and actually holds in a much looser sense if you have function which just solves one dilation equation.

There is a measure of smoothness called the Hölder exponent, where $f \in C^{n+s}$ if $f$ is $n$ times differentiable and:

$$|f(x + h) - f(x)| < k|h|^s.$$ 

Now, $x^\beta \in C^{n+s}$ for $n + s < \beta$, and it can be shown that for $f \in C^{n+s}$ then $n + s < \beta$.

I have examples which can produce $f$ which are almost this smooth, and in some simple cases this gives the correct smoothness.
<table>
<thead>
<tr>
<th>N</th>
<th>Coefficients</th>
<th>TLoW</th>
<th>SRCISS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$</td>
<td>c_0</td>
<td>- \log_2</td>
</tr>
<tr>
<td>2</td>
<td>0.683</td>
<td>0.550</td>
<td>0.339</td>
</tr>
<tr>
<td>3</td>
<td>0.470</td>
<td>1.087</td>
<td>0.036</td>
</tr>
<tr>
<td>4</td>
<td>0.325</td>
<td>1.617</td>
<td>0.913</td>
</tr>
<tr>
<td>5</td>
<td>0.2264</td>
<td>2.1429</td>
<td>1.177</td>
</tr>
<tr>
<td>6</td>
<td>0.1577</td>
<td>2.6644</td>
<td>1.432</td>
</tr>
<tr>
<td>7</td>
<td>0.1109</td>
<td>3.1831</td>
<td>1.682</td>
</tr>
</tbody>
</table>