Module MAU34804: Annual Examination 2021/22
Worked solutions

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Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MAU34804/`

Notes

1. (a) (i) The correspondence Φ is not upper hemicontinuous at -2. Indeed let

$$V = \{y \in \mathbb{R} : 7 < y < 13\}$$

Then V is open in $\mathbb{R}$ and $\Phi(-2) \subset V$. But $16 \in \Phi(x)$ for all real numbers x satisfying $-2\sqrt{2} < x < -2$, and therefore there cannot exist any positive real number δ with the property that $\Phi(x) \subset V$ for all real numbers x satisfying $|x + 2| < \delta$.

(ii) The correspondence Φ is lower hemicontinuous at -2. Indeed let V be an open set in $\mathbb{R}$ for which $V \cap \Phi(-2) \neq \emptyset$. Then there exists some real number v belonging to V which satisfies $8 < v < 12$. Then $v \in \Phi(x)$ provided that $-\sqrt{v/2} < x < \frac{1}{2}(4 - v)$.

(iii) The correspondence Φ is upper hemicontinuous at 2. Indeed let V be an open set in $\mathbb{R}$ for which $\Phi(2) \subset V$. Then all non-negative real numbers belong to the open set V. Now the elements of $\Phi(x)$ are non-negative for all real numbers x. It follows that $\Phi(x) \in V$ for all real numbers x.

(iv) The correspondence Φ is not lower hemicontinuous at 2. Indeed let $V = \{y \in Y : 0 < y < 8\}$. Then $V \cap \Phi(2) \neq \emptyset$ but $V \cap \Phi(x) = \emptyset$ for all real numbers satisfying $x > 2$.

(b) Let $(\mathbf{p}, \mathbf{q})$ be a point of the complement $X \times Y \setminus \mathrm{Graph}(\Phi)$ of the graph $\mathrm{Graph}(\Phi)$ of Φ in $X \times Y$. Then $\Phi(\mathbf{p})$ is closed in Y and $\mathbf{q} \notin \Phi(\mathbf{p})$. It follows that there exists some positive real number δ_Y such that $|\mathbf{y} - \mathbf{q}| > \delta_Y$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Let

$$V = \{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| > \delta_Y\}$$

and

$$W = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Then V is open in Y and $\Phi(\mathbf{p}) \subset V$. Now the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. It therefore follows from the definition of upper hemicontinuity that the subset W of X is open in X. Moreover $\mathbf{p} \in W$. It follows that there exists some positive real number δ_X such that $\mathbf{x} \in W$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Then $\Phi(\mathbf{x}) \subset V$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Let δ be the minimum of δ_X and δ_Y, and let $(\mathbf{x}, \mathbf{y})$ be a point of $X \times Y$ whose distance from the point $(\mathbf{p}, \mathbf{q})$ is less than δ. Then $|\mathbf{x} - \mathbf{p}| < \delta_X$ and therefore $\Phi(\mathbf{x}) \subset V$.

Also $|\mathbf{y} - \mathbf{q}| < \delta_Y$, and therefore $\mathbf{y} \notin V$. It follows that $\mathbf{y} \notin \Phi(\mathbf{x})$, and therefore $(\mathbf{x}, \mathbf{y}) \notin \mathrm{Graph}(\Phi)$. We conclude from this that the complement of $\mathrm{Graph}(\Phi)$ is open in $X \times Y$. It follows that $\mathrm{Graph}(\Phi)$ itself is closed in $X \times Y$, as required.

2. (a) [Bookwork.] Let $\mathcal{V}$ be collection of open sets in Y that covers $\Phi(K)$. Given any point $\mathbf{p}$ of K, there exists a finite subcollection $\mathcal{W}_{\mathbf{p}}$ of $\mathcal{V}$ that covers the compact set $\Phi(\mathbf{p})$. Let $U_{\mathbf{p}}$ be the union of the open sets belonging to this subcollection $\mathcal{W}_{\mathbf{p}}$. Then $\Phi(\mathbf{p}) \subset U_{\mathbf{p}}$. Now it follows from the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ that there exists an open set $N_{\mathbf{p}}$ in X such that $\Phi(\mathbf{x}) \subset U_{\mathbf{p}}$ for all $\mathbf{x} \in N_{\mathbf{p}}$. Moreover, given any $\mathbf{p} \in K$, the finite collection $\mathcal{W}_{\mathbf{p}}$ of open sets in Y covers $\Phi(N_{\mathbf{p}})$. It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers $\Phi(K)$. The result follows.

(b) [Bookwork.] Let $\Phi\colon X \rightrightarrows Y$ is a compact-valued correspondence, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \neq \emptyset$.

First suppose that, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. We must prove that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Now $\Phi(\mathbf{p})$ is a compact subset of Y, because $\Phi\colon X \to Y$ is compact-valued. It follows that there exists some positive real number ε such that $B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$. There then exists some positive number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Conversely suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at the point $\mathbf{p}$. Now $\Phi(\mathbf{p})$ is a non-empty subset of Y. Let some positive number ε be given. Then $B_Y(\Phi(\mathbf{p}), \varepsilon)$

is open in Y and $\Phi(\mathbf{p}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$. It follows from the upper hemicontinuity of Φ at $\mathbf{p}$ that there exists some positive number δ such that $\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows.

3. (a) [Bookwork.] A *simplicial map* $\varphi\colon K \to L$ between simplicial complexes K and L is a function $\varphi\colon \operatorname{Vert} K \to \operatorname{Vert} L$ from the vertex set of K to that of L such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to L whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

(b) [Bookwork.] Let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of simplicial complexes K and L. A simplicial map $s\colon K \to L$ is said to be a *simplicial approximation* to f if, for each $\mathbf{x} \in |K|$, $s(\mathbf{x})$ is an element of the unique simplex of L which contains $f(\mathbf{x})$ in its interior.

(c) [Bookwork.] Every point of $|K|$ belongs to the interior of a unique simplex of K. It follows that the complement $|K|\setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K.

(d) [Bookwork.] Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, because $\mathbf{x}$ is in the interior of σ. It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$.

Conversely let $s\colon \operatorname{Vert} K \to \operatorname{Vert} L$ be a function with the property that $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L\left(s(\mathbf{v}_j)\right)$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s\colon \operatorname{Vert} K \to \operatorname{Vert} L$

represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required.

4. (a) [Bookwork.] Let $f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q}$ for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Given $\mathbf{q} \in \Delta_Q$, let

$$\mu_P(\mathbf{q}) = \sup\{f(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_P\}$$

and let

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P : f(\mathbf{p}, \mathbf{q}) = \mu_P(\mathbf{q})\}.$$

Similarly given $\mathbf{p} \in \Delta_P$, let

$$\mu_Q(\mathbf{p}) = \inf\{f(\mathbf{p}, \mathbf{q}) : \mathbf{q} \in \Delta_Q\}$$

and let

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q : f(\mathbf{p}, \mathbf{q}) = \mu_Q(\mathbf{p})\}.$$

An application of Berge's Maximum Theorem ensures that the functions $\mu_P \colon \Delta_P \to \mathbb{R}$ and $\mu_Q \colon \Delta_Q \to \mathbb{R}$ are continuous, and that the correspondences $P \colon \Delta_Q \rightrightarrows \Delta_P$ and $Q \colon \Delta_P \rightrightarrows \Delta_Q$ are non-empty, compact-valued and upper hemicontinuous. These correspondences therefore have closed graphs. Morever $P(\mathbf{q})$ is convex for all $\mathbf{q} \in \Delta_Q$ and $Q(\mathbf{p})$ is convex for all $\mathbf{p} \in \Delta_P$. Let $X = \Delta_P \times \Delta_Q$, and let $\Phi \colon X \rightrightarrows X$ be defined such that

$$\Phi(\mathbf{p}, \mathbf{q}) = P(\mathbf{q}) \times Q(\mathbf{p})$$

for all $(\mathbf{p}, \mathbf{q}) \in X$. Kakutani's Fixed Point Theorem then ensures that there exists $(\mathbf{p}^*, \mathbf{q}^*) \in X$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$. Then $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$ and therefore

$$f(\mathbf{p}, \mathbf{q}^*) \le f(\mathbf{p}^*, \mathbf{q}^*) \le f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$, as required.

(b) [Bookwork.] The set K is clearly non-empty. We may assume, without loss of generality, that the set K is both compact and convex, because if K were not convex, then it could be replaced by a compact convex set containing it.

Let $\gamma \colon \mathbb{R}^n \to \mathbb{R}$ be the function defined so that, for each $\mathbf{x} \in \mathbb{R}^n$, $\gamma(\mathbf{x})$ is the maximum of the components of $\mathbf{x}$, and let $\mu \colon \mathbb{R}^n \rightrightarrows \Delta$ be the correspondence defined such that

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta : \mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})\}.$$

Now it follows, on applying Berge's Maximum Theorem, that the correspondence $\mu \colon \mathbb{R}^n \rightrightarrows \Delta$ is upper hemicontinuous, and that

$\mu(\mathbf{x})$ is a non-empty compact convex subset of Δ for all $\mathbf{x} \in \mathbb{R}^n$. Moreover $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p}' \,.\, \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Let $\Phi\colon \Delta \times K \rightrightarrows \Delta \times K$ be the correspondence defined such that

$$\Phi(\mathbf{p}, \mathbf{z}) = (\mu(\mathbf{z}), \zeta(\mathbf{p}))$$

for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. The correspondences μ and ζ are upper hemicontinuous and closed-valued, and every upper hemicontinuous closed-valued correspondence has a closed graph. It follows that the correspondence Φ has closed graph. Moreover $\Phi(\mathbf{p}, \mathbf{z})$ is a non-empty closed convex subset of the compact convex set $\Delta \times K$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. It follows from the Kakutani Fixed Point Theorem that there exists $(\mathbf{p}^*, \mathbf{z}^*) \in \Delta \times K$ for which $(\mathbf{p}^*, \mathbf{z}^*) \in \Phi(\mathbf{p}^*, \mathbf{z}^*)$. Then $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$.

Now the conditions of the theorem require that $\mathbf{p}^* \,.\, \mathbf{z} \leq 0$ for all $\mathbf{z} \in \zeta(\mathbf{p}^*)$. Combining this inequality with the definition of the correspondence μ, and noting that $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$, we find that

$$\mathbf{p} \,.\, \mathbf{z}^* \leq \mathbf{p}^* \,.\, \mathbf{z}^* \leq 0$$

for all $\mathbf{p} \in \Delta$. Applying this result when $\mathbf{p}$ is the vertex of Δ whose ith component is equal to 1 and whose other components are zero, we find that $(\mathbf{z}^*)_i \leq 0$ for $i = 1, 2, \ldots, n$, and thus $\mathbf{z}^* \leq \mathbf{0}$, as required.