Course MAU34804—Hilary Term 2022.

1. *Let S be the simplex in $\mathbb{R}^3$ with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, where*

$$\mathbf{v}_0 = (-1,-1,-1), \quad \mathbf{v}_1 = (2,0,0), \quad \mathbf{v}_2 = (0,3,0), \quad \mathbf{v}_3 = (0,0,4).$$

Also let T be the 3-simplex in $\mathbb{R}^3$ be the simplex whose vertices are the following:

- *the vertex $\mathbf{v}_3$ of S;*
- *the midpoint of the edge of S with endpoints $\mathbf{v}_2$ and $\mathbf{v}_3$;*
- *the barycentre of the triangular face of S with vertices $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$;*
- *the barycentre of the simplex S itself.*

Determine four constraints, each of the form $ax + by + cz \leq d$ for appropriate real constants a, b, c and d, such that the simplex T consists of those points (x, y, z) of $\mathbb{R}^3$ that satisfy all four constraints.

The vertices of the simplex T, in the order listed, are $\mathbf{w}_0$, $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$, where

$$\begin{aligned} \mathbf{w}_0 &= \mathbf{v}_3 = (0,0,4), \\ \mathbf{w}_1 &= \tfrac{1}{2}(\mathbf{v}_2 + \mathbf{v}_3) = (0, \tfrac{3}{2}, 2), \\ \mathbf{w}_2 &= \tfrac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = (\tfrac{2}{3}, 1, \tfrac{4}{3}), \end{aligned}$$

and

$$\mathbf{w}_3 = \tfrac{1}{4}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}).$$

Now the plane through vertices $\mathbf{w}_1$ $\mathbf{w}_2$, $\mathbf{w}_3$ is perpendicular to the vector $\mathbf{m}_0$ where

$$\begin{aligned} \mathbf{m}_0 &= (\mathbf{w}_2 - \mathbf{w}_1) \times (\mathbf{w}_3 - \mathbf{w}_1) = (\tfrac{2}{3}, -\tfrac{1}{2}, -\tfrac{2}{3}) \times (\tfrac{1}{4}, -1, -\tfrac{5}{4}) \\ &= (-\tfrac{1}{24}, \tfrac{2}{3}, -\tfrac{13}{24}). \end{aligned}$$

The plane through $\mathbf{w}_1$ $\mathbf{w}_2$, $\mathbf{w}_3$ can then be identified, and consequently the half-space bounded by this plane and containing the point $\mathbf{w}_0$ can be identified. This half-space is the half-space

$$x - 16y + 13z \geq 2.$$

(This may be verified immediately.)

Next the plane through vertices $\mathbf{w}_0$ $\mathbf{w}_2$, $\mathbf{w}_3$ is perpendicular to the vector $\mathbf{m}_1$ where

$$\begin{aligned} \mathbf{m}_1 &= (\mathbf{w}_2 - \mathbf{w}_0) \times (\mathbf{w}_3 - \mathbf{w}_0) = (\tfrac{2}{3}, 1, -\tfrac{8}{3}) \times (\tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{13}{4}) \\ &= (-\tfrac{23}{12}, \tfrac{18}{12}, \tfrac{1}{12}). \end{aligned}$$

The plane through $\mathbf{w}_0$ $\mathbf{w}_2$, $\mathbf{w}_3$ can then be identified, and consequently the half-space bounded by this plane and containing the point $\mathbf{w}_1$ can be identified. This half-space is the half-space

$$-23x + 18y + z \geq 4.$$

Next the plane through vertices $\mathbf{w}_0$ $\mathbf{w}_1$, $\mathbf{w}_3$ is perpendicular to the vector $\mathbf{m}_2$ where

$$\begin{aligned} \mathbf{m}_2 &= (\mathbf{w}_1 - \mathbf{w}_0) \times (\mathbf{w}_3 - \mathbf{w}_0) = (0, \tfrac{3}{2}, -2) \times (\tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{13}{4}) \\ &= (-\tfrac{35}{8}, -\tfrac{1}{2}, -\tfrac{3}{8}). \end{aligned}$$

The plane through $\mathbf{w}_0$ $\mathbf{w}_1$, $\mathbf{w}_3$ can then be identified, and consequently the half-space bounded by this plane and containing the point $\mathbf{w}_2$ can be identified. This half-space is the half-space

$$31x + 4y + 3z \geq 12.$$

Next the plane through vertices $\mathbf{w}_0$ $\mathbf{w}_1$, $\mathbf{w}_2$ is perpendicular to the vector $\mathbf{m}_3$ where

$$\begin{aligned} \mathbf{m}_3 &= (\mathbf{w}_1 - \mathbf{w}_0) \times (\mathbf{w}_2 - \mathbf{w}_0) = (0, \tfrac{3}{2}, -2) \times (\tfrac{2}{3}, 1, -\tfrac{8}{3}) \\ &= (-2, -\tfrac{4}{3}, -1). \end{aligned}$$

The plane through $\mathbf{w}_0$ $\mathbf{w}_1$, $\mathbf{w}_2$ can then be identified, and consequently the half-space bounded by this plane and containing the point $\mathbf{w}_3$ can be identified. This half-space is the half-space

$$6x + 4y + 3z \leq 12.$$

Consequently the tetrahedron T is the intersection of the four half spaces

$$-x + 16y - 13z \leq -2, \quad 23x - 18y - z \leq -4,$$

$$-31x - 4y - 3z \leq -12, \quad 6x + 4y + 3z \leq 12.$$

2. *Let* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *be vertices of a 3-simplex* T *in* $\mathbb{R}^3$*. (This simplex* T *is then a tetrahedron.) Also let* $f\colon \mathbb{R}^3 \to \mathbb{R}$ *be a linear functional on* $\mathbb{R}^3$*. (There then exist real numbers* u*,* v *and* w *such that* $f(x, y, z) = ux + vy + wz$ *for all* $(x, y, z) \in \mathbb{R}^3$*.) Let* m *be the maximum value attained by the linear functional* f *on the simplex* T*, and let*

$$M = \{(x, y, z) \in T : f(x, y, z) = m\}.$$

Prove that the subset M *of* T *is contained in the edge of* T *with endpoints* $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *if and only if*

$$f(\mathbf{v}_0) < \max(f(\mathbf{v}_2), f(\mathbf{v}_3)) \quad \textit{and} \quad f(\mathbf{v}_1) < \max(f(\mathbf{v}_2), f(\mathbf{v}_3)).$$

We note that $M = \{\mathbf{v}_2\}$ if and only if $f(\mathbf{v}_2)$ exceeds $f(\mathbf{v}_0)$, $f(\mathbf{v}_1)$ and $f(\mathbf{v}_3)$. (This is a consequence of the linearity of the function f.) Similarly $M = \{\mathbf{v}_3\}$ if and only if $f(\mathbf{v}_3)$ exceeds $f(\mathbf{v}_0)$, $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$. Also M is the edge of T with endpoints $\mathbf{v}_2$ and $\mathbf{v}_3$ if and only if $f(\mathbf{v}_2) = f(\mathbf{v}_3)$ and the common value of $f(\mathbf{v}_2)$ and $f(\mathbf{v}_3)$ exceeds $f(\mathbf{v}_0)$ and $f(\mathbf{v}_1)$. Indeed let $\mathbf{x}$ be a point of the simplex T with barycentric coordinates t_0, t_1, t_2, t_3, so that t_0, t_1, t_2 and t_3 are all non-negative,

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 \quad \text{and} \quad t_0 + t_1 + t_2 + t_3 = 0.$$

The linearity of f then ensures that $f(\mathbf{x}) = m$, where m is the maximum of $f(\mathbf{v}_j)$ for $j = 0, 1, 2, 3$, if and only if $t_j = 0$ whenever $f(\mathbf{v}_j) < m$. The observations above cover all cases in which the set M is contained in the edge of T with endpoints $\mathbf{v}_2$ and $\mathbf{v}_3$. The result follows.

3. *Let* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *be vertices of a 3-simplex* T *in* $\mathbb{R}^3$*. Also, for each ordered triple* (u, v, w) *of real numbers, let* $f_{(u,v,w)}\colon \mathbb{R}^3 \to \mathbb{R}$ *denote the linear functional on* $\mathbb{R}^3$ *defined so that* $f_{(u,v,w)}(x, y, z) = ux + vy + wz$ *for all* $(x, y, z) \in \mathbb{R}^3$*. Let* $m(u, v, w)$ *be the maximum value attained by the linear functional* $f_{(u,v,w)}$ *on the simplex* T*, and let*

$$M(u, v, w) = \{(x, y, z) \in T : f_{(u,v,w)}(x, y, z) = m(u, v, w)\}.$$

Let E *be the edge of the simplex* T *with endpoints* $\mathbf{v}_2$ *and* $\mathbf{v}_3$*. Explain why*

$$\{(u, v, w) \in \mathbb{R}^3 : M(u, v, w) \subset E\}$$

is an open set in $\mathbb{R}^3$*. Then, given this result, explain why the correspondence* $M\colon \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ *that sends each ordered triple* (u, v, w) *of real numbers to the subset* $M(u, v, w)$ *of the simplex* T *is upper hemicontinuous at any point* (u, v, w) *of* $\mathbb{R}^3$ *for which* $M(u, v, w) = E$*. (You should not apply Berge's Maximum Theorem.)*

Let $g_j: \mathbb{R}^3 \to \mathbf{R}$ be defined for $j = 0, 1, 2, 3$ so that $g_j(u, v, w) = f_{(u,v,w)}(\mathbf{v}_j)$. Note that

$$g_j(u, v, w) = ux_j + vu_j + wz_j$$

for $j = 0, 1, 2, 3$, where $(x_j, y_j, z_j) = \mathbf{v}_j$. It follows that the functions g_j are continuous on $\mathbb{R}^3$. The maximum of two continuous real-valued functions is continuous, and accordingly the function mapping $(u, v, w) \in \mathbb{R}^3$ to $\max(g_2(u, v, w), g_3(u, v, w))$ is continuous on $\mathbb{R}^3$, and therefore the functions mapping $(u, v, w) \in \mathbb{R}^3$ to

$$\max(g_2(u, v, w), g_3(u, v, w)) - g_0(u, v, w)$$

and

$$\max(g_2(u, v, w), g_3(u, v, w)) - g_1(u, v, w)$$

are continuous functions. Let

$$W_0 = \{(u, v, w) \in \mathbb{R}^3 : \max(g_2(u, v, w), g_3(u, v, w)) - g_0(u, v, w) > 0\}$$

and

$$W_1 = \{(u, v, w) \in \mathbb{R}^3 : \max(g_2(u, v, w), g_3(u, v, w)) - g_1(u, v, w) > 0\}$$

Then W_0 and W_1, being the preimages of open intervals under continuous maps are open in $\mathbb{R}^3$, and therefore $W_0 \cap W_1$ is open in $\mathbb{R}^3$.

Let $W = W_0 \cap W_1$. Then W is open in $\mathbb{R}^3$ and it follows from the result of the previous question that

$$W = \{(u, v, w) \in \mathbb{R}^3 : M(u, v, w) \subset E\}$$

Thus, given any any point (u, v, w) of $\mathbb{R}^3$ for which $M(u, v, w) = E$, and given any open set V for which $M(u, v, w) \subset V$, there exists an open set W in $\mathbb{R}^3$, namely the set W defined above, with the properties that $(u, v, w) \in W$ and $M(u', v', w') \subset V$ for all $(u', v', w') \in W$. It follows that the correspondence M is upper hemicontinuous at (u, v, w), as required.