Course MAU34804

Fixed Point Theorems and Economic Equilibria.

Hilary Term 2022.

Midterm Assignment.

To be submitted on Blackboard on or before 11pm on Wednesday 30th March, 2022.

Assignment submissions should be uploaded to Blackboard as a *single* file in PDF format. (Thus submissions should neither be constituted of multiple files, nor of photographic image files in a format such as JPEG or PNG.) It is recommended that a suitable scanner app be used.

Students are advised to retain a copy of their submission. It is anticipated that the assignment submissions would not be printed at any stage. Nor would they be annotated with hand-written comments: any feedback would most likely take the form of marks and comments forwarded separately to students to be interpreted in conjunction with copies of assignment submissions in their possession.

Students are reminded that they must comply with College policies with regard to plagiarism, which are published on the website located at the following URL:

`http://tcd-ie.libguides.com/plagiarism`

Please complete the cover sheet on the back of this page and attach it to the front of your completed assignment script, in particular signing the declaration with regard to plagiarism. Please make sure also that you include both name and student number on work handed in.

At an alternative to printing off the plagiarism declaration, it is acceptable to copy the wording of the plagiarism declaration on a sheet of paper and include the signed copy of the declaration, with name and student number, as a page included with the submission.

Solutions to problems should be expressed in appropriately concise and correct logical language. Attempted solutions that are incoherent, unclear, overlong or logically confused are unlikely to gain substantial credit.

Module MAU34804—Fixed Point Theorems and Economic Equilibria.
Hilary Term 2022.
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Name (please print): ....

Student number: ....

Date submitted: ....

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at

`http://www.tcd.ie/calendar`

I have also completed the Online Tutorial on avoiding plagiarism *Ready Steady Write*, located at

`http://tcd-ie.libguides.com/plagiarism/ready-steady-write`

Signed:

....

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1. Let S be the simplex in $\mathbb{R}^3$ with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, where

$$\mathbf{v}_0 = (-1,-1,-1), \quad \mathbf{v}_1 = (2,0,0), \quad \mathbf{v}_2 = (0,3,0), \quad \mathbf{v}_3 = (0,0,4).$$

   Also let T be the 3-simplex in $\mathbb{R}^3$ be the simplex whose vertices are the following:

   - the vertex $\mathbf{v}_3$ of T;
   - the midpoint of the edge of T with endpoints $\mathbf{v}_2$ and $\mathbf{v}_3$;
   - the barycentre of the triangular face of S with vertices $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$;
   - the barycentre of the simplex S itself.

   Determine four constraints, each of the form $ax + by + cz \leq d$ for appropriate real constants a, b, c and d, such that the simplex S consists of those points (x, y, z) of $\mathbb{R}^3$ that satisfy all four constraints.

2. Let $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vertices of a 3-simplex T in $\mathbb{R}^3$. (This simplex T is then a tetrahedron.) Also let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a linear functional on $\mathbb{R}^3$. (There then exist real numbers u, v and w such that $f(x, y, z) = ux + vy + wz$ for all $(x, y, z) \in \mathbb{R}^3$.) Let m be the maximum value attained by the linear functional f on the simplex T, and let

$$M = \{(x, y, z) \in T : f(x, y, z) = m\}.$$

   Prove that the subset M of T is contained in the edge of T with endpoints $\mathbf{v}_2$ and $\mathbf{v}_3$ if and only if

$$f(\mathbf{v}_0) < \max(f(\mathbf{v}_2), f(\mathbf{v}_3)) \quad \text{and} \quad f(\mathbf{v}_1) < \max(f(\mathbf{v}_2), f(\mathbf{v}_3)).$$

3. Let $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vertices of a 3-simplex T in $\mathbb{R}^3$. Also, for each ordered triple (u, v, w) of real numbers, let $f_{(u,v,w)}\colon \mathbb{R}^3 \to \mathbb{R}$ denote the linear functional on $\mathbb{R}^3$ defined so that $f_{(u,v,w)}(x, y, z) = ux + vy + wz$ for all $(x, y, z) \in \mathbb{R}^3$. Let $m(u, v, w)$ be the maximum value attained by the linear functional $f_{(u,v,w)}$ on the simplex T, and let

$$M(u, v, w) = \{(x, y, z) \in T : f_{(u,v,w)}(x, y, z) = m(u, v, w)\}.$$

   Let E be the edge of the simplex T with endpoints $\mathbf{v}_2$ and $\mathbf{v}_3$. Explain why

$$\{(u, v, w) \in \mathbb{R}^3 : M(u, v, w) \subset E\}$$

is an open set in $\mathbb{R}^3$. Then, given this result, explain why the correspondence $M\colon \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ that sends each ordered triple (u, v, w) of real numbers to the subset $M(u, v, w)$ of the simplex T is upper hemicontinuous at any point (u, v, w) of $\mathbb{R}^3$ for which $M(u, v, w) = E$. (You should not apply Berge's Maximum Theorem.)