# Module MAU34802: Hilary Semester Examination 2021 Worked solutions

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28 March, 2021

1. (**Unseen problem**)

   (a) The transport matrix $X$ must be of the form $X = X^0 + \lambda Y$, where the matrix $Y$ is of the form

$$Y = \begin{pmatrix} y_{1,1} & y_{1,2} & 0 & y_{1,4} \\ 0 & y_{2,2} & y_{2,3} & 0 \\ 0 & y_{3,2} & y_{3,3} & 0 \end{pmatrix}.$$

Moreover the rows and columns of the matrix $Y$ must sum to zero. Therefore we must have $y_{1,1} = 0$ and $y_{1,4} = 0$, from which it follows that $y_{1,2} = 0$. Then, scaling so that $y_{2,2} = 1$, we find that

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Thus the matrix $X$ is of the form

$$X^0 + \lambda Y = \begin{pmatrix} 23 & 1 & 0 & 13 \\ 0 & 41 + \lambda & 7 - \lambda & 0 \\ 0 & 6 - \lambda & 9 + \lambda & 0 \end{pmatrix}.$$

for an appropriate value of $\lambda$. Moreover

$$T(X) = T(X^0) - 52\lambda.$$

Consequently we should take a positive value of $\lambda$ which is small enough to ensure that the solution $X$ remains feasible and becomes basic. Accordingly we must take $\lambda = 6$

$$X = \begin{pmatrix} 23 & 1 & 0 & 13 \\ 0 & 47 & 1 & 0 \\ 0 & 0 & 15 & 0 \end{pmatrix}.$$

We must check that this basic solution is indeed feasible.

The expected basis $B$ is given by

$$B = \{(1,1), (1,2), (1,4), (2,2), (2,3), (3,3),$$

We must check that this is indeed a basis. Suppose that we are given real numbers $y_1, y_2, y_3$ and $z_1, z_2, z_3, z_4$ for which

$$y_1 + y_2 + y_3 = z_1 + z_2 + z_3 + z_4.$$

We seek $w_{1,1}$, $w_{1,2}$, $w_{1,4}$, $w_{2,2}$, $w_{2,3}$ and $w_{3,3}$ such that

$$\begin{aligned} w_{1,1}+w_{1,2}+w_{1,4} &= y_1, \\ w_{2,2}+w_{2,3} &= y_2, \\ w_{3,3} &= y_3, \\ w_{1,1} &= z_1, \\ w_{1,2}+w_{2,2} &= z_2, \\ w_{2,3}+w_{3,3} &= z_3, \\ w_{1,4} &= z_4. \end{aligned}$$

Clearly $w_{3,3} = y_3$, $w_{1,1} = z_1$, $w_{1,4} = z_4$, $w_{2,3} = z_3 - y_3$, $w_{2,2} = y_2 - z_3 + y_3$ and $w_{1,2} = z_2 - y_2 + z_3 - y_3$. Accordingly the equations with right hand sides equal to $y_2$, $y_3$, $z_1$, $z_2$, $z_3$ and $z_4$ are satisfied. It remains to check that the equation with right hand side equal to $z_1$ is also satisfied with these values of the $w_{i,j}$ quantities. Now

$$w_{1,1}+w_{1,2}+w_{1,4} = z_1 + z_2 - y_2 + z_3 - y_3 + z_4 = y_1.$$

Thus the required quantities $w_{i,j}$ have been found for any real quantities $y_i$ and $z_i$ satisfying the stated condition. We conclude that the set $B$ is indeed a basis for the transportation problem, and thus we have found the required basic feasible solution. (Note valid alternative expressions

$$\begin{aligned} w_{1,2} &= z_2 + z_3 - y_2 - y_3 = y_1 + z_1 - z_4 \\ w_{2,2} &= y_2 + y_3 - z_3 = z_1 + z_2 + z_4 - y_1 \end{aligned}$$

that arise from the requirement that $\sum_{i=1}^{3} y_i = \sum_{j=1}^{4} z_j$.)

(b) (**Routine, seen similar**)

Filling in the appropriate tableau in a routine fashion so as to determine $u_i$, $v_j$ and $q_{i,j}$ such that ensure that $c_{i,j} + u_i = v_j + q_{i,j}$ for all $i$ and $j$ and $q_{i,j} = 0$ whenever $(i,j) \in B$, we obtain the following filled-out tableau:

| $c_{i,j} \searrow q_{i,j}$ | 1 | 2 | 3 | 4 | $u_i$ |
|---|---|---|---|---|---|
| 1 | 73 • | 47 • | 57 | 36 • | |
| | 0 | 0 | $-2$ | 0 | 0 |
| 2 | 37 | 62 • | 74 • | 25 | |
| | $-51$ | 0 | 0 | $-26$ | $-15$ |
| 3 | 62 | 71 | 31 • | 59 | |
| | 17 | 52 | 0 | 51 | 28 |
| $v_j$ | 73 | 47 | 59 | 36 | |

Accordingly the cost of a feasible solution $(\overline{x}_{i,j})$ is

$$T(X) - 2\overline{x}_{1,3} - 51\overline{x}_{2,1} - 26\overline{x}_{2,4} + 17\overline{x}_{3,1} + 52_{3,2} + 41\overline{x}_{3,4}.$$

Accordingly the basic feasible solution $T(X)$ could be undercut by bringing into commission transport along any of the routes $(1,3)$, $(2,1)$ and $(2,4)$. Accordingly this basic feasible solution is not optimal.

(c) No, there cannot exist any optimal solution of this transportation problem within the set $\mathcal{S}$ of feasible solutions. For if there were to exist an optimal solution, then the standard procedure for constructing a basic feasible solution from a feasible solution without increasing cost would produce a basic optimal solution belonging to $\mathcal{S}$. But we have shown that unique basic feasible solution in $\mathcal{S}$ with cost less than that of $X^0$ is not optimal. Therefore there can exist no other.

2. (**Unseen problem**)

(a) Letting $B = \{1, 2\}$ and

$$M_B = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

and letting $D$ be the value of the determinant of this matrix, we find that

$$DM_B^{-1} = \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}.$$

Consequently

$$Dr_{1,1} = a_{2,2}, \quad Dr_{1,2} = -a_{1,2}, \quad Dr_{2,1} = -a_{2,1}, \quad Dr_{2,2} = a_{1,1}.$$

Let $\hat{B} = \{1, 2, 4\}$ and

$$\hat{M}_{\hat{B}} = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 \\ -c1 & -c_2 & 1 \end{pmatrix}$$

and letting $D$ be the value of the determinant of this matrix, we find that

$$\det \hat{M}_{\hat{B}} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = D = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}.$$

Calculating the inverse of the matrix $\hat{M}_{\hat{B}}$ by dividing the adjugate matrix by the determinant, we find that

$$D\hat{r}_{1,1} = \begin{vmatrix} a_{2,2} & 0 \\ -c_2 & 1 \end{vmatrix} = a_{2,2} = Dr_{1,1},$$

$$D\hat{r}_{2,1} = - \begin{vmatrix} a_{2,1} & 0 \\ -c_1 & 1 \end{vmatrix} = -a_{2,1} = Dr_{2,1},$$

$$\begin{aligned} D\hat{r}_{3,1} &= \begin{vmatrix} a_{2,1} & a_{2,2} \\ -c_1 & -c_2 \end{vmatrix} = c_1 a_{2,2} - c_2 a_{2,1} \\ &= D(c_1 r_{1,1} + c_2 r_{2,1}) = Dp_1, \end{aligned}$$

$$D\hat{r}_{1,2} = - \begin{vmatrix} a_{1,2} & 0 \\ -c_2 & 1 \end{vmatrix} = -a_{1,2} = Dr_{1,2},$$

$$D\hat{r}_{2,2} = \begin{vmatrix} a_{1,1} & 0 \\ -c_1 & 1 \end{vmatrix} = a_{1,1} = Dr_{2,2},$$

$$\begin{aligned} D\hat{r}_{3,2} &= -\begin{vmatrix} a_{1,1} & a_{1,2} \\ -c_1 & -c_2 \end{vmatrix} = -c_1 a_{1,2} + c_2 a_{1,1} \\ &= D(c_1 r_{1,2} + c_2 r_{2,2}) = Dp_2. \end{aligned}$$

$$D\hat{r}_{1,3} = \begin{vmatrix} a_{1,2} & 0 \\ a_{2,2} & 0 \end{vmatrix} = 0.$$

$$D\hat{r}_{2,3} = -\begin{vmatrix} a_{1,1} & 0 \\ a_{2,1} & 0 \end{vmatrix} = 0.$$

$$D\hat{r}_{3,3} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = 1.$$

Consequently

$$\hat{M}_{\hat{B}}^{-1} = \begin{pmatrix} \hat{r}_{1,1} & \hat{r}_{1,2} & \hat{r}_{1,3} \\ \hat{r}_{2,1} & \hat{r}_{2,2} & \hat{r}_{2,3} \\ \hat{r}_{3,1} & \hat{r}_{3,2} & \hat{r}_{3,3} \end{pmatrix} = \begin{pmatrix} r_{1,1} & r_{1,2} & 0 \\ r_{2,1} & r_{2,2} & 0 \\ p_1 & p_2 & 1 \end{pmatrix}.$$

(b) Now

$$\mathbf{b} = b_1 \mathbf{e}^{(1)} + b_2 \mathbf{e}^{(2)} = (b_1 r_{1,1} + b_2 r_{1,2})\mathbf{a}^{(1)} + (b_1 r_{2,1} + b_2 r_{2,2})\mathbf{a}^{(2)}.$$

Consequently

$$s_1 = b_1 r_{1,1} + b_2 r_{1,2}, \quad s_2 = b_1 r_{2,1} + b_2 r_{2,2}.$$

Therefore

$$\begin{aligned} \hat{\mathbf{b}} &= b_1 \hat{\mathbf{e}}^{(1)} + b_2 \hat{\mathbf{e}}^{(2)} \\ &= (b_1 \hat{r}_{1,1} + b_2 \hat{r}_{2,1})\mathbf{a}^{(1)} + (b_1 \hat{r}_{1,2} + b_2 \hat{r}_{2,2})\mathbf{a}^{(2)} \\ &\qquad + (b_1 \hat{r}_{1,3} + b_2 \hat{r}_{2,3})\mathbf{a}^{(4)} \\ &= (b_1 r_{1,1} + b_2 r_{2,1})\mathbf{a}^{(1)} + (b_1 r_{2,1} + b_2 r_{2,2})\mathbf{a}^{(2)} \\ &\qquad + (b_1 p_1 + b_2 p_2)\mathbf{a}^{(4)} \end{aligned}$$

Now

$$b_1 p_1 + b_2 p_2 = b_1 c_1 r_{1,1} + b_1 c_2 r_{2,1} + b_2 c_1 r_{1,2} + b_2 c_2 r_{2,2} = c_1 s_1 + c_2 s_2 = C.$$

Consequently

$$\hat{\mathbf{b}} = s_1 \hat{\mathbf{a}}^{(1)} + s_2 \hat{\mathbf{a}}^{(2)} + C \hat{\mathbf{a}}^{(4)},$$

and consequently $\hat{s}_1 = s_1$, $\hat{s}_2 = s_2$ and $\hat{s}_3 = C$.

(c) Now

$$\begin{aligned}\mathbf{a}^{(j)} &= a_{1,j}\mathbf{e}^{(1)} + a_{2,j}\mathbf{e}^{(2)} \\ &= (a_{1,j}r_{1,1} + a_{2,j}r_{1,2})\mathbf{a}^{(1)} + (a_{1,j}r_{2,1} + a_{2,j}r_{2,2})\mathbf{a}^{(2)}.\end{aligned}$$

Consequently

$$t_{1,j} = a_{1,j}r_{1,1} + a_{2,j}r_{1,2}, \quad t_{2,j} = a_{1,j}r_{2,1} + a_{2,j}r_{2,2}.$$

Therefore

$$\begin{aligned}\hat{\mathbf{a}}^{(j)} &= a_{1,j}\hat{\mathbf{e}}^{(1)} + a_{2,j}\hat{\mathbf{e}}^{(2)} - c_j\hat{\mathbf{e}}^{(4)} \\ &= (a_{1,j}\hat{r}_{1,1} + a_{2,j}\hat{r}_{1,2} - c_j\hat{r}_{1,3})\mathbf{a}^{(1)} \\ &\qquad + (a_{1,j}\hat{r}_{2,1} + a_{2,j}\hat{r}_{2,2} - c_j\hat{r}_{2,3})\mathbf{a}^{(2)} \\ &\qquad + (a_{1,j}\hat{r}_{3,1} + a_{2,j}\hat{r}_{3,2} - c_j\hat{r}_{3,3})\mathbf{a}^{(4)} \\ &= (a_{1,j}r_{1,1} + a_{2,j}r_{1,2})\mathbf{a}^{(1)} + (a_{1,j}r_{2,1} + a_{2,j}r_{2,2})\mathbf{a}^{(2)} \\ &\qquad + (a_{1,j}p_1 + a_{2,j}p_2 - c_j)\mathbf{a}^{(4)}\end{aligned}$$

Now

$$\begin{aligned}a_{1,j}p_1 + a_{2,j}p_2 - c_j &= a_{1,j}c_1r_{1,1} + a_{1,j}c_2r_{2,1} \\ &\qquad + a_{2,j}c_1r_{1,2} + a_{2,j}c_2r_{2,2} - c_j \\ &= c_1t_{1,j} + c_2t_{2,j} - c_j = -q_j\end{aligned}$$

Consequently

$$\hat{\mathbf{a}}^{(j)} = t_{1,j}\hat{\mathbf{a}}^{(1)} + t_{2,j}\hat{\mathbf{a}}^{(2)} - q_j\hat{\mathbf{a}}^{(4)},$$

and consequently

$$\hat{t}_{1,j} = t_{1,j}, \quad \hat{t}_{2,j} = t_{2,j}, \quad \hat{t}_{3,j} = -q_j.$$

Note furthermore that, because $\{1, 2\}$ is the chosen basis, $t_{1,1} = t_{2,2} = 1$, $t_{1,2} = t_{2,1} = 0$ and $q_1 = q_2 = 0$.

3. (**Unseen problem**)

(a) The points of $\mathbb{R}^3$ that lie on the plane containing the points with position vectors $\mathbf{0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are those points which have position vectors $\mathbf{x}$ that satisfy

$$(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{x} = 0.$$

The two sides of this plane are thus those for which

$$(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{x} < 0 \quad \text{or} \quad (\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{x} > 0.$$

Now the point $\mathbf{x}$ must lie on the same side of the plane as the point $\mathbf{v}_3$, and moreover the question states that $(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{v}_3 > 0$. Accordingly we must take $\varepsilon = +1$.

Now it follows from standard 3-dimensional vector algebra $(\mathbf{v}_1 \times \mathbf{v}_3) \,.\, \mathbf{v}_2 = -(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{v}_3$. Accordingly, arguing on the same lines, we should take $\eta = -1$.

(b) The set $C$ is indeed a convex cone. Indeed let $\mathbf{x}$ and $\mathbf{y}$ be position vectors of points of $C$, and let $\lambda$ and $\mu$ be real numbers with $\lambda \geq 0$ and $\mu \geq 0$. Then

$$b(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{x} \geq 0, \quad c(\mathbf{v}_1 \times \mathbf{v}_3) \,.\, \mathbf{x} \geq 0.$$

$$b(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{y} \geq 0, \quad c(\mathbf{v}_1 \times \mathbf{v}_3) \,.\, \mathbf{y} \geq 0,$$

and consequently

$$b(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, (\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda b(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{x} + \mu b(\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{y} \geq 0,$$

$$c(\mathbf{v}_1 \times \mathbf{v}_3) \,.\, (\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda b(\mathbf{v}_1 \times \mathbf{v}_3) \,.\, \mathbf{x} + \mu c(\mathbf{v}_1 \times \mathbf{v}_3) \,.\, \mathbf{y} \geq 0.$$

Thus the set $C$ is indeed a convex cone in $\mathbb{R}^3$.

(c) The tetrahedron $T$ is not a convex cone in $\mathbf{R}^3$. Let $\mathbf{x}$ be the position vector of a point in the interior of the tetrahedron. Then the origin $\mathbf{0}$ and the point $\lambda\mathbf{x}$ lies on opposite sides of the plane containing the points $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ when the positive real number $\lambda$ is sufficiently large.

(d) Let $f_3\colon \mathbb{R}^3 \to \mathbb{R}$ and $f_3\colon \mathbb{R}^3 \to \mathbb{R}$ be defined so that

$$f_3(\mathbf{x}) = (\mathbf{v}_1 \times \mathbf{v}_2) \,.\, \mathbf{x} \quad \text{and} \quad f_2(\mathbf{x}) = -(\mathbf{v}_1 \times \mathbf{v}_3) \,.\, \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^3$. Then $f_2(\mathbf{x}^*) = 0$, $f_3(\mathbf{x}^*) = 0$ and moreover $f_2(\mathbf{x}) \geq 0$ and $f_3(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in T$. Moreover affine linear functionals $f_0$ and $f_1$ can be constructed so that $f_0(\mathbf{x}^*) > 0$, $f_1(\mathbf{x}^*) > 0$ and

$$T = \{\mathbf{x} \in X : f_j(\mathbf{x}) \geq 0 \text{ for } j = 0, 1, 2, 3\}.$$

(Indeed we can take

$$f_1(\mathbf{x}) = (\mathbf{v}_2 \times \mathbf{v}_3) \,.\, \mathbf{x}$$

and

$$f_0(\mathbf{x}) = ((\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_1) \,.\, (\mathbf{v}_1 - \mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^3$. However the details of the construction of these affine linear functions $f_0$ and $f_1$ is unimportant for the problem in hand.) Applying the Karush-Kuhn-Tucker Theorem (in the form of Theorem 5.21 of the module core notes, we find that if the function $g$ attains a maximum at the point $\mathbf{x}^*$ (which lies in the interior of the line segment at which the faces $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_3$ of the tetrahedron intersect) then there exist non-negative real constants $\lambda$ and $\mu$ such that

$$(\nabla g)_{\mathbf{x}^*} + \lambda(\nabla f_3)_{\mathbf{x}^*} + \mu(\nabla f_2)_{\mathbf{x}^*} = 0.$$

But $(\nabla f_3)_{\mathbf{x}^*} = \mathbf{v}_1 \times \mathbf{v}_2$ and $(\nabla f_2)_{\mathbf{x}^*} = -\mathbf{v}_1 \times \mathbf{v}_3$. Thus

$$(\nabla g)_{\mathbf{x}^*} + \lambda \mathbf{v}_1 \times \mathbf{v}_2 - \mu \mathbf{v}_1 \times \mathbf{v}_3 = 0,$$

as required.