Module MAU34802: The Theory of Linear Programming
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Section 4: The Simplex Method

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Contents

4 The Simplex Method

4.1 Vector Inequalities and Notational Conventions

Let $\mathbf{v}$ be an element of the real vector space $\mathbb{R}^n$. We denote by $(\mathbf{v})_j$ the jth component of the vector $\mathbf{v}$. The vector $\mathbf{v}$ can be represented in the usual fashion as an n-tuple $(v_1, v_2, \ldots, v_n)$, where $v_j = (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$. However where an n-dimensional vector appears in matrix equations it will usually be considered to be an $n \times 1$ column vector. The row vector corresponding to an element $\mathbf{v}$ of $\mathbb{R}^n$ will be denoted by $\mathbf{v}^T$ because, considered as a matrix, it is the transpose of the column vector representing $\mathbf{v}$. We denote the zero vector (in the appropriate dimension) by $\mathbf{0}$.

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors belonging to the real vector space $\mathbb{R}^n$ for some positive integer n. We write $\mathbf{x} \leq \mathbf{y}$ (and $\mathbf{y} \geq \mathbf{x}$) when $(\mathbf{x})_j \leq (\mathbf{y})_j$ for $j = 1, 2, \ldots, n$. Also we write $\mathbf{x} \ll \mathbf{y}$ (and $\mathbf{y} \gg \mathbf{x}$) when $(\mathbf{x})_j < (\mathbf{y})_j$ for $j = 1, 2, \ldots, n$.

These notational conventions ensure that $\mathbf{x} \geq \mathbf{0}$ if and only if $(\mathbf{x})_j \geq 0$ for $j = 1, 2, \ldots, n$.

The scalar product of two n-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ can be represented as the matrix product $\mathbf{u}^T\mathbf{v}$. Thus

$$\mathbf{u}^T\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $u_j = (\mathbf{u})_j$ and $v_j = (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$.

Given an $m \times n$ matrix A, where m and n are positive integers, we denote by $(A)_{i,j}$ the coefficient in the ith row and jth column of the matrix A.

4.2 Feasible and Optimal Solutions

A general linear programming problem is one that seeks values of real variables $x_1, x_2, \ldots, x_n$ that maximize or minimize some *objective function*

$$c_1x_1 + c_2x_2 + \cdots c_nx_n$$

that is a linear functional of $x_1, x_2, \ldots, x_n$ determined by real constants $c_1, c_2, \ldots, c_n$, where the variables $x_1, x_2, \ldots, x_n$ are subject to a finite number of *constraints* that each place bounds on the value of some linear functional of the variables. These constraints can then be numbered from 1 to m, for an appropriate value of m, such that, for each value of i between 1 and m, the ith constraint takes the form of an equation or inequality that can be expressed in one of the following three forms:—

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n = b_i,$$

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n \geq b_i,$$

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n \leq b_i$$

for appropriate values of the real constants $a_{i,1}, a_{i,2}, \ldots, a_{i,n}$ and b_i. In addition some, but not necessarily all, of the variables $x_1, x_2, \ldots, x_n$ may be required to be non-negative. (Of course a constraint requiring a variable to be non-negative can be expressed by an inequality that conforms to one of the three forms described above. Nevertheless constraints that simply require some of the variables to be non-negative are usually listed separately from the other constraints.)

Definition Consider a general linear programming problem with n real variables $x_1, x_2, \ldots, x_n$ whose objective is to maximize or minimize some objective function subject to appropriate constraints. A *feasible solution* of this linear programming problem is specified by an n-dimensional vector $\mathbf{x}$ whose components satisfy the constraints but do not necessarily maximize or minimize the objective function.

Definition Consider a general linear programming problem with n real variables $x_1, x_2, \ldots, x_n$ whose objective is to maximize or minimize some objective function subject to appropriate constraints. A *optimal solution* of this linear programming problem is specified by an n-dimensional vector $\mathbf{x}$ that is a feasible solution that optimizes the value of the objective function amongst all feasible solutions to the linear programming problem.

4.3 Programming Problems in Dantzig Standard Form

Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, and let $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ be vectors of dimensions m and n respectively. We consider the following linear programming problem:—

> *Determine an n-dimensional vector $\mathbf{x}$ so as to minimize $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

We refer to linear programming problems presented in this form as being in *Dantzig standard form.* We refer to the $m \times n$ matrix A, the m-dimensional vector $\mathbf{b}$ and the n-dimensional vector $\mathbf{c}$ as the *constraint matrix*, *target vector* and *cost vector* for the linear programming problem.

Remark Nomenclature in Linear Programming textbooks varies. Problems presented in the above form are those to which the basic algorithms of George B. Dantzig's *Simplex Method* are applicable. In the series of textbooks by

George B. Dantzig and Mukund N. Thapa entitled *Linear Programming*, such problems are said to be in *standard form*. In the textbook *Introduction to Linear Programming* by Richard B. Darst, such problems are said to be *standard-form LP*. On the other hand, in the textbook *Methods of Mathematical Economics* by Joel N. Franklin, such problems are said to be in *canonical form*, and the term *standard form* is used for problems which match the form above, except that the vector equality $A\mathbf{x} = \mathbf{b}$ is replaced by a vector inequality $A\mathbf{x} \geq \mathbf{b}$. Accordingly the term *Danztig standard form* is used in these notes both to indicate that such problems are in *standard form* at that term is used by textbooks of which Dantzig is the author, and also to emphasize the connection with the contribution of Dantzig in creating and popularizing the *Simplex Method* for the solution of linear programming problems.

A linear programming problem in Dantzig standard form specified by an $m \times n$ constraint matrix A of rank m, an m-dimensional target vector $\mathbf{b}$ and an n-dimensional cost vector $\mathbf{c}$ has the objective of finding values of real variables $x_1, x_2, \ldots, x_n$ that minimize the value of the *cost*

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

subject to constraints

$$\begin{aligned} A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n &= b_1, \\ A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n &= b_2, \\ &\vdots \\ A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n &= b_m \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \ldots, \quad x_n \geq 0.$$

In the above programming problem, the function sending the n-dimensional vector $\mathbf{x}$ to the corresponding *cost* $\mathbf{c}^T\mathbf{x}$ is the objective function for the problem. A feasible solution to the problem consists of an n-dimensional vector $(x_1, x_2, \ldots, x_n)$ whose components satisfy the above constraints but do not necessarily minimize cost. An optimal solution is a feasible solution whose cost does not exceed that of any other feasible solution.

4.4 Basic Feasible Solutions

We define the notion of a *basis* for a linear programming problem in Dantzig standard form.

Definition Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:

> *find* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

For each integer j between 1 and n, let $\mathbf{a}^{(j)}$ denote the m-dimensional vector determined by the jth column of the matrix A, so that $(\mathbf{a}^{(j)})_i = (A)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. A *basis* for this linear programming problem is a set consisting of m distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n for which the corresponding vectors

$$\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$$

constitute a basis of the vector space $\mathbb{R}^m$.

We next define what is meant by saying that a feasible solution of a programming problem Dantzig standard form is a *basic feasible solution* for the programming problem.

Definition Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:—

> *find* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

A feasible solution $\mathbf{x}$ for this programming problem is said to be *basic* if there exists a basis B for the linear programming problem such that $(\mathbf{x})_j = 0$ when $j \notin B$.

Lemma 4.1 *Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:*

> *find* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

Let $\mathbf{a}^{(j)}$ denote the vector specified by the jth column of the matrix A for $j = 1, 2, \ldots, n$. Let $\mathbf{x}$ be a feasible solution of the linear programming problem. Suppose that the m-dimensional vectors $\mathbf{a}^{(j)}$ for which $(\mathbf{x})_j > 0$ are linearly independent. Then $\mathbf{x}$ is a basic feasible solution of the linear programming problem.

Proof Let $\mathbf{x}$ be a feasible solution to the programming problem, let $x_j = (\mathbf{x})_j$ for all $j \in J$, where $J = \{1, 2, \ldots, n\}$, and let $K = \{j \in J : x_j > 0\}$. If the vectors $\mathbf{a}^{(j)}$ for which $j \in K$ are linearly independent then basic linear algebra ensures that further vectors $\mathbf{a}^{(j)}$ can be added to the linearly independent set $\{\mathbf{a}^{(j)} : j \in K\}$ so as to obtain a finite subset of $\mathbb{R}^m$ whose elements constitute a basis of that vector space (see Proposition 2.2). Thus exists a subset B of J satisfying $K \subset B \subset J$ such that the m-dimensional vectors $\mathbf{a}^{(j)}$ for which $j \in B$ constitute a basis of the real vector space $\mathbb{R}^m$. Moreover $(\mathbf{x})_j = 0$ for all $j \in J \setminus B$. It follows that $\mathbf{x}$ is a basic feasible solution to the linear programming problem, as required. ∎

Theorem 4.2 *Let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, let $\mathbf{b} \in \mathbb{R}^m$ be an m-dimensional column vector, let $\mathbf{c} \in \mathbb{R}^n$ be an n-dimensional column vector. Consider the following programming problem in Dantzig standard form:*

> *find $\mathbf{x} \in \mathbb{R}^n$ so as to minimize $\mathbf{c}^T\mathbf{x}$ subject to constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

If there exists a feasible solution to this programming problem then there exists a basic feasible solution to the problem. Moreover if there exists an optimal solution to the programming problem then there exists a basic optimal solution to the problem.

Proof Let $J = \{1, 2, \ldots, n\}$, and let $\mathbf{a}^{(j)}$ denote the vector specified by the jth column of the matrix A for all $j \in J$.

Let $\mathbf{x}$ be a feasible solution to the programming problem, let $x_j = (\mathbf{x})_j$ for all $j \in J$, and let $K = \{j \in J : x_j > 0\}$. Suppose that $\mathbf{x}$ is not basic. Then the vectors $\mathbf{a}^{(j)}$ for which $j \in K$ must be linearly dependent. We show that there then exists a feasible solution with fewer non-zero components than the given feasible solution $\mathbf{x}$.

Now there exist real numbers y_j for $j \in K$, not all zero, such that $\sum_{j \in K} y_j \mathbf{a}^{(j)} = \mathbf{0}$, because the vectors $\mathbf{a}^{(j)}$ for $j \in K$ are linearly dependent. Let $y_j = 0$ for all $j \in J \setminus K$, and let $\mathbf{y} \in \mathbb{R}^n$ be the n-dimensional vector satisfying $(\mathbf{y})_j = y_j$ for $j = 1, 2, \ldots, n$. Then

$$A\mathbf{y} = \sum_{j \in J} y_j \mathbf{a}^{(j)} = \sum_{j \in K} y_j \mathbf{a}^{(j)} = \mathbf{0}.$$

It follows that $A(\mathbf{x} - \lambda\mathbf{y}) = \mathbf{b}$ for all real numbers λ, and thus $\mathbf{x} - \lambda\mathbf{y}$ is a feasible solution to the programming problem for all real numbers λ for which $\mathbf{x} - \lambda\mathbf{y} \geq \mathbf{0}$.

Now $\mathbf{y}$ is a non-zero vector. Replacing $\mathbf{y}$ by $-\mathbf{y}$, if necessary, we can assume, without loss of generality, that at least one component of the vector $\mathbf{y}$ is positive. Let

$$\lambda_0 = \text{minimum}\left(\frac{x_j}{y_j} : j \in K \text{ and } y_j > 0\right),$$

and let j_0 be an element of K for which $\lambda_0 = x_{j_0}/y_{j_0}$. Then $\dfrac{x_j}{y_j} \geq \lambda_0$ for all $j \in J$ for which $y_j > 0$. Multiplying by the positive number y_j, we find that $x_j \geq \lambda_0 y_j$ and thus $x_j - \lambda_0 y_j \geq 0$ when $y_j > 0$. Also $\lambda_0 > 0$ and $x_j \geq 0$, and therefore $x_j - \lambda_0 y_j \geq 0$ when $y_j \leq 0$. Thus $x_j - \lambda_0 y_j \geq 0$ for all $j \in J$. Also $x_{j_0} - \lambda_0 y_{j_0} = 0$, and $x_j - \lambda_0 y_j = 0$ for all $j \in J \setminus K$. Let $\mathbf{x}' = \mathbf{x} - \lambda_0\mathbf{y}$. Then $\mathbf{x}' \geq \mathbf{0}$ and $A\mathbf{x}' = \mathbf{b}$, and thus $\mathbf{x}'$ is a feasible solution to the linear programming problem with fewer non-zero components than the given feasible solution.

Suppose in particular that the feasible solution $\mathbf{x}$ is optimal. Now there exist both positive and negative values of λ for which $\mathbf{x} - \lambda\mathbf{y} \geq \mathbf{0}$. If it were the case that $\mathbf{c}^T\mathbf{y} \neq 0$ then there would exist values of λ for which both $\mathbf{x} - \lambda\mathbf{y} \geq \mathbf{0}$ and $\lambda\mathbf{c}^T\mathbf{y} > 0$. But then $\mathbf{c}^T(x - \lambda\mathbf{y}) < \mathbf{c}^T\mathbf{x}$, contradicting the optimality of $\mathbf{x}$. It follows that $\mathbf{c}^T\mathbf{y} = 0$, and therefore $\mathbf{x} - \lambda\mathbf{y}$ is an optimal solution of the linear programming problem for all values of λ for which $\mathbf{x} - \lambda\mathbf{y} \geq \mathbf{0}$. The previous argument then shows that there exists a real number λ_0 for which $\mathbf{x} - \lambda_0\mathbf{y}$ is an optimal solution with fewer non-zero components than the given optimal solution $\mathbf{x}$.

We have shown that if there exists a feasible solution $\mathbf{x}$ which is not basic then there exists a feasible solution with fewer non-zero components than $\mathbf{x}$. It follows that if a feasible solution $\mathbf{x}$ is chosen such that it has the smallest possible number of non-zero components then it is a basic feasible solution of the linear programming problem.

Similarly we have shown that if there exists an optimal solution $\mathbf{x}$ which is not basic then there exists an optimal solution with fewer non-zero components than $\mathbf{x}$. It follows that if an optimal solution $\mathbf{x}$ is chosen such that it has the smallest possible number of non-zero components then it is a basic optimal solution of the linear programming problem. ∎

4.5 A Simplex Method Example

Example We consider the following linear programming problem:—

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:
$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11$*;*
$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6$*;*
$x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5$.

The constraints require that x_1, x_2, x_3, x_4, x_5 be non-negative real numbers satisfying the matrix equation

$$\begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Thus we are required to find a (column) vector $\mathbf{x}$ with components x_1, x_2, x_3, x_4 and x_5 satisfying the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Let

$$\mathbf{a}^{(1)} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \quad \mathbf{a}^{(2)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{a}^{(3)} = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

$$\mathbf{a}^{(4)} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^{(5)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

For a feasible solution to the problem we must find non-negative real numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)} + x_3\mathbf{a}^{(3)} + x_4\mathbf{a}^{(4)} + x_5\mathbf{a}^{(5)} = \mathbf{b}.$$

An optimal solution to the problem is a feasible solution that minimizes

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5$$

amongst all feasible solutions to the problem, where $c_1 = 3$, $c_2 = 4$, $c_3 = 2$, $c_4 = 9$ and $c_5 = 5$.

Let $\mathbf{c}$ denote the column vector whose ith component is c_i respectively. Then

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix},$$

and an optimal solution is a feasible solution that minimizes $\mathbf{c}^T\mathbf{x}$ amongst all feasible solutions to the problem. We refer to the quantity $\mathbf{c}^T\mathbf{x}$ as the *cost* of the feasible solution $\mathbf{x}$.

Let $I = \{1, 2, 3, 4, 5\}$. A basis for this optimization problem is a subset $\{j_1, j_2\}$ of I, where $j_1 \neq j_2$, for which the corresponding vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}$ constitute a basis of $\mathbb{R}^2$. By inspection we see that each pair of vectors taken from the list $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}, \mathbf{a}^{(4)}, \mathbf{a}^{(5)}$ consists of linearly independent vectors, and therefore each pair of vectors from this list constitutes a basis of $\mathbb{R}^2$. It follows that every subset of I with exactly two elements is a basis for the optimization problem.

A feasible solution $(x_1, x_2, x_3, x_4, x_5)$ to this optimization problem is a basic feasible solution if there exists a basis B for the optimization problem such that $x_j = 0$ when $j \neq B$.

In the case of the present problem, all subsets of $\{1, 2, 3, 4, 5\}$ with exactly two elements are bases for the problem. It follows that a feasible solution to the problem is a basic feasible solution if and only if the number of non-zero components of the solution does not exceed 2.

We take as given the following initial basic feasible solution $x_1 = 1$, $x_2 = 2$, $x_3 = x_4 = x_5 = 0$. One can readily verify that $\mathbf{a}^{(1)} + 2\mathbf{a}^{(2)} = \mathbf{b}$. This initial basic feasible solution is associated with the basis $\{1, 2\}$. The cost of this solution is 11.

We apply the procedures of the *simplex method* to test whether or not this basic feasible solution is optimal, and, if not, determine how to improve it.

The basis $\{1, 2\}$ determines a 2×2 minor M_B of A consisting of the first two columns of A. Thus

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

We now determine the components of the vector $\mathbf{p} \in \mathbb{R}^2$ whose transpose $\begin{pmatrix} p_1 & p_2 \end{pmatrix}$ satisfies the matrix equation

$$\begin{pmatrix} c_1 & c_2 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} M_B.$$

Now

$$M_B^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}.$$

It follows that

$$\begin{aligned} \mathbf{p}^T &= \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \end{pmatrix} M_B^{-1} \\ &= -\frac{1}{7}\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{13}{7} & -\frac{11}{7} \end{pmatrix}. \end{aligned}$$

We next compute a vector $\mathbf{q} \in \mathbb{R}^5$, where $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$. Solving the equivalent matrix equation for the transpose $\mathbf{q}^T$ of the column vector $\mathbf{q}$, we find that

$$\begin{aligned}
\mathbf{q}^T &= \mathbf{c}^T - \mathbf{p}^T A \\
&= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} \frac{13}{7} & -\frac{11}{7} \end{pmatrix} \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 4 & \frac{19}{7} & \frac{3}{7} & -\frac{5}{7} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -\frac{5}{7} & \frac{60}{7} & \frac{40}{7} \end{pmatrix}.
\end{aligned}$$

We denote the jth component of the vector j by q_j.

Now $q_3 < 0$. We show that this implies that the initial basic feasible solution is not optimal, and that it can be improved by bringing 3 (the index of the third column of A) into the basis.

Suppose that $\overline{\mathbf{x}}$ is a feasible solution of this optimization problem. Then $A\overline{\mathbf{x}} = \mathbf{b}$, and therefore

$$\mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\mathbf{x} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}}.$$

The initial basic feasible solution $\mathbf{x}$ satisfies

$$\mathbf{q}^T\mathbf{x} = \sum_{j=1}^{5} q_j x_j = 0,$$

because $q_1 = q_2 = 0$ and $x_3 = x_4 = x_5 = 0$. This comes about because the manner in which we determined first $\mathbf{p}$ then $\mathbf{q}$ ensures that $q_j = 0$ for all $j \in B$, whereas the components of the basic feasible solution $\mathbf{x}$ associated with the basis B satisfy $x_j = 0$ for $j \notin B$. We find therefore that $\mathbf{p}^T\mathbf{b}$ is the cost of the initial basic feasible solution.

The cost of the initial basic feasible solution is 11, and this is equal to the value of $\mathbf{p}^T\mathbf{b}$. The cost $\mathbf{c}^T\overline{\mathbf{x}}$ of any other basic feasible solution satisfies

$$\mathbf{c}^T\overline{\mathbf{x}} = 11 - \tfrac{5}{7}\overline{x}_3 + \tfrac{60}{7}\overline{x}_4 + \tfrac{40}{7}\overline{x}_5,$$

where $\overline{x}_j$ denotes the jth component of $\overline{\mathbf{x}}$.

We seek to determine a new basic feasible solution $\overline{\mathbf{x}}$ for which $\overline{x}_3 > 0$, $\overline{x}_4 = 0$ and $\overline{x}_5 = 0$. The cost of such a basic feasible solution will then be less than that of our initial basic feasible solution.

In order to find our new basic feasible solution we determine the relationships between the coefficients of a feasible solution $\overline{\mathbf{x}}$ for which $\overline{x}_4 = 0$ and $\overline{x}_5 = 0$. Now such a feasible solution must satisfy

$$\overline{x}_1\mathbf{a}^{(1)} + \overline{x}_2\mathbf{a}^{(2)} + \overline{x}_3\mathbf{a}^{(3)} = \mathbf{b} = x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)},$$

where x_1 and x_2 are the non-zero coefficients of the initial basic feasible solution. Now the vectors $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ constitute a basis of the real vector space $\mathbb{R}^2$. It follows that there exist real numbers $t_{1,3}$ and $t_{2,3}$ such that $\mathbf{a}^{(3)} = t_{1,3}\mathbf{a}^{(1)} + t_{2,3}\mathbf{a}^{(2)}$. It follows that

$$(\overline{x}_1 + t_{1,3}\overline{x}_3)\mathbf{a}^{(1)} + (\overline{x}_2 + t_{2,3}\overline{x}_3)\mathbf{a}^{(2)} = x_1\mathbf{a}^{(1)} + x_2\mathbf{a}^{(2)}.$$

The linear independence of $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ then ensures that $\overline{x}_1+t_{1,3}\overline{x}_3 = x_1$ and $\overline{x}_2 + t_{2,3}\overline{x}_3 = x_2$. Thus if $\overline{x}_3 = \lambda$, where $\lambda \geq 0$ then

$$\overline{x}_1 = x_1 - \lambda t_{1,3}, \quad \overline{x}_2 = x_2 - \lambda t_{2,3}.$$

Thus, once $t_{1,3}$ and $t_{2,3}$ have been determined, we can determine the range of values of λ that ensure that $\overline{x}_1 \geq 0$ and $\overline{x}_2 \geq 0$.

In order to determine the values of $t_{1,3}$ and $t_{2,3}$ we note that

$$\begin{aligned} \mathbf{a}^{(1)} &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{a}^{(2)} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

and therefore

$$\begin{aligned} \mathbf{a}^{(3)} &= t_{3,1}\mathbf{a}^{(1)} + t_{3,2}\mathbf{a}^{(2)} = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} t_{3,1} \\ t_{3,2} \end{pmatrix} \\ &= M_B \begin{pmatrix} t_{3,1} \\ t_{3,2} \end{pmatrix}, \end{aligned}$$

where

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} t_{3,1} \\ t_{3,2} \end{pmatrix} = M_B^{-1}\mathbf{a}^{(3)} = -\frac{1}{7} \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} \\ \frac{1}{7} \end{pmatrix}.$$

Thus $t_{3,1} = \frac{5}{7}$ and $t_{3,2} = \frac{1}{7}$.

We now determine the feasible solutions $\overline{\mathbf{x}}$ of this optimization problem that satisfy $\overline{x}_3 = \lambda$ and $\overline{x}_4 = \overline{x}_5 = 0$. we have already shown that

$$\overline{x}_1 = x_1 - \lambda t_{1,3}, \quad \overline{x}_2 = x_2 - \lambda t_{2,3}.$$

Now $x_1 = 1$, $x_2 = 2$, $t_{1,3} = \frac{5}{7}$ and $t_{2,3} = \frac{1}{7}$. It follows that $\overline{x}_1 = 1 - \frac{5}{7}\lambda$ and $\overline{x}_2 = 2 - \frac{1}{7}\lambda$. Now the components of a feasible solution must satisfy $\overline{x}_1 \geq 0$

and $\overline{x}_2 \geq 0$. it follows that $0 \leq \lambda \leq \frac{7}{5}$. Moreover on setting $\lambda = \frac{7}{5}$ we find that $\overline{x}_1 = 0$ and $\overline{x}_2 = \frac{9}{5}$. We thus obtain a new basic feasible solution $\overline{\mathbf{x}}$ associated to the basis $\{2, 3\}$, where

$$\overline{\mathbf{x}}^T = \begin{pmatrix} 0 & \frac{9}{5} & \frac{7}{5} & 0 & 0 \end{pmatrix}.$$

The cost of this new basic feasible solution is 10.

We now let B' and $\mathbf{x}'$ denote the new basic and new associated basic feasible solution respectively, so that $B' = \{2, 3\}$ and

$$\mathbf{x}'^T = \begin{pmatrix} 0 & \frac{9}{5} & \frac{7}{5} & 0 & 0 \end{pmatrix}.$$

We also let $M_{B'}$ be the 2×2 minor of the matrix A with columns indexed by the new basis B, so that

$$M_{B'} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad M_{B'}^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}.$$

We now determine the components of the vector $\mathbf{p}' \in \mathbb{R}^2$ whose transpose $\begin{pmatrix} p'_1 & p'_2 \end{pmatrix}$ satisfies the matrix equation

$$\begin{pmatrix} c_2 & c_3 \end{pmatrix} = \begin{pmatrix} p'_1 & p'_2 \end{pmatrix} M_{B'}.$$

We find that

$$\begin{aligned} \begin{pmatrix} p'_1 & p'_2 \end{pmatrix} &= \begin{pmatrix} c_2 & c_3 \end{pmatrix} M_{B'}^{-1} \\ &= \frac{1}{5} \begin{pmatrix} 4 & 2 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 \end{pmatrix}. \end{aligned}$$

We next compute the components of the vector $\mathbf{q}' \in \mathbb{R}^5$ so as to ensure that

$$\begin{aligned} \mathbf{q}'^T &= \mathbf{c}^T - \mathbf{p}'^T A \\ &= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 2 & -2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 11 & 7 \end{pmatrix}. \end{aligned}$$

The components of the vector $\mathbf{q}'$ determined using the new basis $\{2, 3\}$ are all non-negative. This ensures that the new basic feasible solution is an optimal solution.

Indeed let $\overline{\mathbf{x}}$ be a feasible solution of this optimization problem. Then $A\overline{\mathbf{x}'} = \mathbf{b}$, and therefore

$$\mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}'^T A\overline{\mathbf{x}} + \mathbf{q}'^T\mathbf{x}' = \mathbf{p}'^T\mathbf{b} + \mathbf{q}'^T\overline{\mathbf{x}}.$$

Moreover $\mathbf{p}'^T\mathbf{b} = 10$. It follows that

$$\mathbf{c}^T\overline{\mathbf{x}} = 10 + \overline{x}_1 + 11\overline{x}_4 + 7\overline{x}_5 \geq 10,$$

and thus the new basic feasible solution $\mathbf{x}'$ is optimal.

We summarize the result we have obtained. The optimization problem was the following:—

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:
$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11$;
$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6$;
$x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5$.

We have found the following basic optimal solution to the problem:

$$x_1 = 0, \quad x_2 = \frac{9}{5}, \quad x_3 = \frac{7}{5}, \quad x_4 = 0, \quad x_5 = 0.$$

We now investigate all bases for this linear programming problem in order to determine which bases are associated with basic feasible solutions.

The problem is to find $\mathbf{x} \in \mathbb{R}^5$ that minimizes $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

and

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix}.$$

For each two-element subset B of $\{1, 2, 3, 4, 5\}$ we compute M_B, M_B^{-1} and $M_B^{-1}\mathbf{b}$, where M_B is the 2×2 minor of the matrix A whose columns are indexed by the elements of B. We find the following:—

B	M_B	M_B^{-1}	$M_B^{-1}\mathbf{b}$	$\mathbf{c}^T M_B^{-1}\mathbf{b}$
$\{1,2\}$	$\begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$	$-\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	11
$\{1,3\}$	$\begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$	$-\begin{pmatrix} 3 & -4 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} -9 \\ 14 \end{pmatrix}$	1
$\{1,4\}$	$\begin{pmatrix} 5 & 7 \\ 4 & 8 \end{pmatrix}$	$\frac{1}{12}\begin{pmatrix} 8 & -7 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} \frac{23}{6} \\ -\frac{7}{6} \end{pmatrix}$	1
$\{1,5\}$	$\begin{pmatrix} 5 & 3 \\ 4 & 4 \end{pmatrix}$	$\frac{1}{8}\begin{pmatrix} 4 & -3 \\ -4 & 5 \end{pmatrix}$	$\begin{pmatrix} \frac{13}{4} \\ -\frac{7}{4} \end{pmatrix}$	1
$\{2,3\}$	$\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$	$\frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{9}{5} \\ \frac{7}{5} \end{pmatrix}$	10

B	M_B	M_B^{-1}	$M_B^{-1}\mathbf{b}$	$\mathbf{c}^T M_B^{-1}\mathbf{b}$
$\{2,4\}$	$\begin{pmatrix} 3 & 7 \\ 1 & 8 \end{pmatrix}$	$\frac{1}{17}\begin{pmatrix} 8 & -7 \\ -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{46}{17} \\ \frac{7}{17} \end{pmatrix}$	$\frac{247}{17}$
$\{2,5\}$	$\begin{pmatrix} 3 & 3 \\ 1 & 4 \end{pmatrix}$	$\frac{1}{9}\begin{pmatrix} 4 & -3 \\ -1 & 3 \end{pmatrix}$	$\begin{pmatrix} \frac{26}{9} \\ \frac{7}{9} \end{pmatrix}$	$\frac{139}{9}$
$\{3,4\}$	$\begin{pmatrix} 4 & 7 \\ 3 & 8 \end{pmatrix}$	$\frac{1}{11}\begin{pmatrix} 8 & -7 \\ -3 & 4 \end{pmatrix}$	$\begin{pmatrix} \frac{46}{11} \\ -\frac{9}{11} \end{pmatrix}$	1
$\{3,5\}$	$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$	$\frac{1}{7}\begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$	$\begin{pmatrix} \frac{26}{7} \\ -\frac{9}{7} \end{pmatrix}$	1
$\{4,5\}$	$\begin{pmatrix} 7 & 3 \\ 8 & 4 \end{pmatrix}$	$\frac{1}{4}\begin{pmatrix} 4 & -3 \\ -8 & 7 \end{pmatrix}$	$\begin{pmatrix} \frac{13}{2} \\ -\frac{23}{2} \end{pmatrix}$	1

From this data, we see that there are four basic feasible solutions to the problem. We tabulate them below:—

B	$\mathbf{x}$	Cost
$\{1,2\}$	$(1,2,0,0,0)$	11
$\{2,3\}$	$(0,\frac{9}{5},\frac{7}{5},0,0)$	10
$\{2,4\}$	$(0,\frac{46}{17},0,\frac{7}{17},0)$	$\frac{247}{17} = 14.529\ldots$
$\{2,5\}$	$(0,\frac{26}{9},0,0,\frac{7}{9})$	$\frac{139}{9} = 15.444\ldots$

4.6 A Linear Tableau Example

Example Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ subject to constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 1 & 2 & 3 \\ 4 & 2 & 5 & 1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 13 \\ 13 \\ 20 \end{pmatrix},$$

$$\mathbf{c}^T = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix}.$$

As usual, we denote by $A_{i,j}$ the coefficient of the matrix A in the ith row and jth column, we denote by b_i the ith component of the m-dimensional vector $\mathbf{b}$, and we denote by c_j the jth component of the n-dimensional vector $\mathbf{c}$.

We let $\mathbf{a}^{(j)}$ be the m-dimensional vector specified by the jth column of the matrix A for $j = 1, 2, 3, 4, 5$. Then

$$\mathbf{a}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{a}^{(2)} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{a}^{(3)} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix},$$

$$\mathbf{a}^{(4)} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}^{(5)} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}.$$

A *basis* B for this linear programming problem is a subset of $\{1, 2, 3, 4, 5\}$ consisting of distinct integers j_1, j_2, j_3 for which the corresponding vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \mathbf{a}^{(j_3)}$ constitute a basis of the real vector space $\mathbb{R}^3$.

Given a basis B for the linear programming programming problem, where $B = \{j_1, j_2, j_3\}$, we denote by M_B the matrix whose columns are specified by the vectors $\mathbf{a}^{(j_1)}$, $\mathbf{a}^{(j_2)}$ and $\mathbf{a}^{(j_3)}$. Thus $(M_B)_{i,k} = A_{i,j_k}$ for $i = 1, 2, 3$ and $k = 1, 2, 3$. We also denote by $\mathbf{c}_B$ the 3-dimensional vector defined such that

$$\mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & c_{j_3} \end{pmatrix}.$$

The ordering of the columns of M_B and $\mathbf{c}_B$ is determined by the ordering of the elements j_1, j_2 and j_3 of the basis. However we shall proceed on the basis that some ordering of the elements of a given basis has been chosen, and the matrix M_B and vector $\mathbf{c}_B$ will be determined so as to match the chosen ordering.

Let $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$, and let $B = \{j_1, j_2, j_3\} = \{1, 2, 3\}$. Then B is a basis of the linear programming problem, and the invertible matrix

M_B determined by $\mathbf{a}^{(j_k)}$ for $k = 1, 2, 3$ is the following 3×3 matrix:—

$$M_B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{pmatrix}.$$

This matrix has determinant -23, and

$$M_B^{-1} = \frac{-1}{23} \begin{pmatrix} 13 & -4 & -7 \\ -6 & -7 & 5 \\ -8 & 6 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{13}{23} & \frac{4}{23} & \frac{7}{23} \\ \frac{6}{23} & \frac{7}{23} & -\frac{5}{23} \\ \frac{8}{23} & -\frac{6}{23} & \frac{1}{23} \end{pmatrix}.$$

Then

$$M_B^{-1}\mathbf{a}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad M_B^{-1}\mathbf{a}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad M_B^{-1}\mathbf{a}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$M_B^{-1}\mathbf{a}^{(4)} = \begin{pmatrix} -\frac{24}{23} \\ \frac{27}{23} \\ \frac{13}{23} \end{pmatrix} \quad \text{and} \quad M_B^{-1}\mathbf{a}^{(5)} = \begin{pmatrix} -\frac{25}{23} \\ \frac{31}{23} \\ \frac{26}{23} \end{pmatrix}.$$

Also

$$M_B^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

It follows that $\mathbf{x}$ is a basic feasible solution of the linear programming problem, where

$$\mathbf{x}^T = \begin{pmatrix} 1 & 3 & 2 & 0 & 0 \end{pmatrix}.$$

The vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}, \mathbf{a}^{(4)}, \mathbf{a}^{(5)}$, $\mathbf{b}$, $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ can then be expressed as linear combinations of $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ with coefficients as recorded in the following tableau:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	·	·	·	·	·	·	·	·	·

There is an additional row at the bottom of the tableau. This row is the *criterion row* of the tableau. The values in this row have not yet been calculated, but, when calculated according to the rules described below, the values in the criterion row will establish whether the current basic feasible solution is optimal and, if not, how it can be improved.

Ignoring the criterion row, we can represent the structure of the remainder of the tableau in block form as follows:—

	$\mathbf{a}^{(1)}$ $\cdots$ $\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$ $\cdots$ $\mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_3)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\cdot$	$\cdot$	$\cdot$

We now employ the principles of the Simplex Method in order to determine whether or not the current basic feasible solution is optimal and, if not, how to improve it by changing the basis.

Let $\mathbf{p}$ be the 3-dimensional vector determined so that

$$\mathbf{p}^T = \mathbf{c}_B^T M_B^{-1}.$$

Then $\mathbf{p}^T M_B = \mathbf{c}_B^T$, and therefore $\mathbf{p}^T \mathbf{a}^{(j_k)} = c_{j_k}$ for $k = 1, 2, 3$. It follows that $(\mathbf{p}^T A)_j = c_j$ whenever $j \in B$. Putting in the relevant numerical values, we find that

$$\mathbf{p}^T M_B = \mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & c_{j_3} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 \end{pmatrix},$$

and therefore

$$\mathbf{p}^T = \begin{pmatrix} 2 & 4 & 3 \end{pmatrix} M_B^{-1} = \begin{pmatrix} \frac{22}{23} & \frac{18}{23} & \frac{-3}{23} \end{pmatrix}.$$

We enter the values of p_1, p_2 and p_3 into the cells of the criterion row in the columns labelled by $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ respectively. The tableau with these values entered is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

The values in the criterion row in the columns labelled by $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ can be calculated from the components of the cost vector $\mathbf{c}$ and the values in these columns of the tableau. Indeed let $r_{i,k} = (M_B^{-1})_{i,k}$ for $i = 1, 2, 3$ and

$k = 1, 2, 3$. Then each $r_{i,k}$ is equal to the value of the tableau element located in the row labelled by $\mathbf{a}^{(j_i)}$ and the column labelled by $\mathbf{e}^{(k)}$. The definition of the vector $\mathbf{p}$ then ensures that

$$p_k = c_{j_1} r_{1,k} + c_{j_2} r_{2,k} + c_{j_3} r_{3,k}$$

for $k = 1, 2, 3$, where, for the current basis, $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$.

The cost C of the current basic feasible solution $\mathbf{x}$ satisfies $C = \mathbf{c}^T\mathbf{x}$. Now $(\mathbf{p}^T A)_j = c_j$ for all $j \in B$, where $B = \{1, 2, 3\}$. Moreover the current basic feasible solution $\mathbf{x}$ satisfies $x_j = 0$ when $j \notin B$, where $x_j = (\mathbf{x})_j$ for $j = 1, 2, 3, 4, 5$. It follows that

$$\begin{aligned} C - \mathbf{p}^T\mathbf{b} &= \mathbf{c}^T\mathbf{x} - \mathbf{p}^T A\mathbf{x} = \sum_{j=1}^{5} (c_j - (\mathbf{p}^T A)_j) x_j \\ &= \sum_{j \in B} (c_j - (\mathbf{p}^T A)_j) x_j = 0, \end{aligned}$$

and thus

$$C = \mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}.$$

Putting in the numerical values, we find that $C = 20$.

We enter the cost C into the criterion row of the tableau in the column labelled by the vector $\mathbf{b}$. The resultant tableau is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	·	·	·	·	·	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

Let s_i denote the value recorded in the tableau in the row labelled by $\mathbf{a}^{(j_i)}$ and the column labelled by $\mathbf{b}$ for $i = 1, 2, 3$. Then the construction of the tableau ensures that

$$\mathbf{b} = s_1\mathbf{a}^{(j_1)} + s_2\mathbf{a}^{(j_2)} + s_3\mathbf{a}^{(j_3)},$$

and thus $s_i = x_{j_i}$ for $i = 1, 2, 3$, where $(x_1, x_2, x_3, x_4, x_5)$ is the current basic feasible solution. It follows that

$$C = c_{j_1} s_1 + c_{j_2} s_2 + c_{j_3} s_3,$$

where, for the current basis, $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$. Thus the cost of the current basic feasible solution can be calculated from the components of the cost vector $\mathbf{c}$ and the values recorded in the rows above the criterion row of the tableau in the column labelled by the vector $\mathbf{b}$.

We next determine a 5-dimensional vector $\mathbf{q}$ such that $\mathbf{c}^T = \mathbf{p}^T A + \mathbf{q}^T$. We find that

$$\begin{aligned}
-\mathbf{q}^T &= \mathbf{p}^T A - \mathbf{c}^T \\
&= \begin{pmatrix} \frac{22}{23} & \frac{18}{23} & \frac{-3}{23} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 1 & 2 & 3 \\ 4 & 2 & 5 & 1 & 4 \end{pmatrix} \\
&\qquad - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 4 & 3 & \frac{99}{23} & \frac{152}{23} \end{pmatrix} - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & \frac{76}{23} & \frac{60}{23} \end{pmatrix}
\end{aligned}$$

Thus

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 0, \quad q_4 = -\tfrac{76}{23}, \quad q_5 = -\tfrac{60}{23}.$$

The 4th and 5th components of the vector $\mathbf{q}$ are negative. It follows that the current basic feasible solution is not optimal. Indeed let $\overline{\mathbf{x}}$ be a feasible solution to the problem, and let $\overline{x}_j = (\overline{\mathbf{x}})_j$ for $j = 1, 2, 3, 4, 5$. Then the cost $\overline{C}$ of the feasible solution $\overline{\mathbf{x}}$ satisfies

$$\begin{aligned}
\overline{C} &= \mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\overline{\mathbf{x}} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}} = C + \mathbf{q}^T\overline{\mathbf{x}} \\
&= C - \frac{76}{23}\overline{x}_4 - \frac{60}{23}\overline{x}_5.
\end{aligned}$$

It follows that the feasible solution $\overline{\mathbf{x}}$ will have lower cost if either $\overline{x}_4 > 0$ or $\overline{x}_5 > 0$.

We enter the value of $-q_j$ into the criterion row of the tableau in the column labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, 3, 4, 5$. The completed tableau associated with basis $\{1, 2, 3\}$ is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	0	0	0	$\frac{76}{23}$	$\frac{60}{23}$	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

We refer to this tableau as the *extended simplex tableau* associated with the basis $\{1, 2, 3\}$.

The general structure of the extended simplex tableau is then as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$	$t_{1,4}$	$t_{1,5}$	s_1	$r_{1,1}$	$r_{1,2}$	$r_{1,3}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$t_{2,3}$	$t_{2,4}$	$t_{2,5}$	s_2	$r_{2,1}$	$r_{2,2}$	$r_{2,3}$
$\mathbf{a}^{(j_3)}$	$t_{3,1}$	$t_{3,2}$	$t_{3,3}$	$t_{3,4}$	$t_{3,5}$	s_3	$r_{3,1}$	$r_{3,2}$	$r_{3,3}$
	$-q_1$	$-q_2$	$-q_3$	$-q_4$	$-q_5$	C	p_1	p_2	p_3

where j_1, j_2 and j_3 are the elements of the current basis, and where the coefficients $t_{i,j}$ s_i and $r_{i,k}$ are determined so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{3} t_{i,j}\mathbf{a}^{(j_i)}, \quad \mathbf{b} = \sum_{i=1}^{3} s_i\mathbf{a}^{(j_i)}, \quad \mathbf{e}^{(k)} = \sum_{i=1}^{3} r_{i,k}\mathbf{a}^{(j_i)}$$

for $j = 1, 2, 3, 4, 5$ and $k = 1, 2, 3$.

The coefficients of the criterion row can then be calculated according to the following formulae:—

$$p_k = \sum_{i=1}^{3} c_{j_i} r_{i,k}, \quad C = \sum_{i=1}^{3} p_i b_i, \quad -q_j = \sum_{i=1}^{3} p_i A_{i,j} - c_j.$$

The extended simplex tableau can therefore be presented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \cdots \quad \mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_3)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{p}^T A - \mathbf{c}^T$	$\mathbf{p}^T\mathbf{b}$	$\mathbf{p}^T$

The values in the criterion row in any column labelled by some $\mathbf{a}^{(j)}$ can also be calculated from the values in the relevant column in the rows above the criterion row.

To see this we note that the value entered into the tableau in the row labelled by $\mathbf{a}^{(j_i)}$ and the column labelled by $\mathbf{a}^{(j)}$ is equal to $t_{i,j}$, where $t_{i,j}$ is the coefficient in the ith row and jth column of the matrix $M_B^{-1}A$. Also $\mathbf{p}^T = \mathbf{c}_B^T M_B^{-1}$, where $(\mathbf{c}_B)_i = c_{j_i}$ for $i = 1, 2, 3$. It follows that

$$(\mathbf{p}^T A)_j = (\mathbf{c}_B^T M_B^{-1} A)_j = \sum_{i=1}^{3} c_{j_i} t_{i,j}.$$

Therefore

$$\begin{aligned} -q_j &= (\mathbf{p}^T A)_j - c_j \\ &= c_{j_1} t_{1,j} + c_{j_2} t_{2,j} + c_{j_3} t_{3,j} - c_j \end{aligned}$$

for $j = 1, 2, 3, 4, 5$.

The coefficients of the criterion row can then be calculated according to the formulae

$$p_k = \sum_{i=1}^{3} c_{j_i} r_{i,k}, \quad C = \sum_{i=1}^{3} c_{j_i} s_i, \quad -q_j = \sum_{i=1}^{3} c_{j_i} t_{i,j} - c_j.$$

The extended simplex tableau can therefore also be presented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \cdots \quad \mathbf{e}^{(3)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_3)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T$	$\mathbf{c}_B^T M_B^{-1}\mathbf{b}$	$\mathbf{c}_B^T M_B^{-1}$

We now carry through procedures for adjusting the basis and calculating the extended simplex tableau associated with the new basis.

We recall that the extended simplex tableau corresponding to the old basis $\{1, 2, 3\}$ is as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	0	0	0	$\frac{76}{23}$	$\frac{60}{23}$	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

We now consider which of the indices 4 and 5 to bring into the basis.

Suppose we look for a basis which includes the vector $\mathbf{a}^{(4)}$ together with two of the vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$. A feasible solution $\overline{\mathbf{x}}$ with $\overline{x}_5 = 0$ will satisfy

$$\overline{\mathbf{x}}^T = \begin{pmatrix} 1 + \frac{24}{23}\lambda & 3 - \frac{27}{23}\lambda & 2 - \frac{13}{23}\lambda & \lambda & 0 \end{pmatrix},$$

where $\lambda = \overline{x}_4$. Indeed $A(\overline{\mathbf{x}} - \mathbf{x}) = \mathbf{0}$, where $\mathbf{x}$ is the current basic feasible solution, and therefore

$$(\overline{x}_1 - 1)\mathbf{a}^{(1)} + (\overline{x}_2 - 3)\mathbf{a}^{(2)} + (\overline{x}_3 - 2)\mathbf{a}^{(3)} + \overline{x}_4 \mathbf{a}^{(4)} = \mathbf{0}.$$

Now

$$\mathbf{a}^{(4)} = -\tfrac{24}{23}\mathbf{a}^{(1)} + \tfrac{27}{23}\mathbf{a}^{(2)} + \tfrac{13}{23}\mathbf{a}^{(3)},$$

It follows that

$$(\overline{x}_1 - 1 - \tfrac{24}{23}\overline{x}_4)\mathbf{a}^{(1)} + (\overline{x}_2 - 3 + \tfrac{27}{33}\overline{x}_4)\mathbf{a}^{(2)} + (\overline{x}_3 - 2 + \tfrac{13}{23}\overline{x}_4)\mathbf{a}^{(3)} = \mathbf{0}.$$

But the vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$ are linearly independent. Thus if $\overline{x}_4 = \lambda$ and $\overline{x}_5 = 0$ then

$$\overline{x}_1 - 1 - \tfrac{24}{23}\lambda = 0, \quad \overline{x}_2 - 3 + \tfrac{27}{23}\lambda = 0, \quad \overline{x}_3 - 2 + \tfrac{13}{23}\lambda = 0,$$

and thus

$$\overline{x}_1 = 1 + \tfrac{24}{23}\lambda, \quad \overline{x}_2 = 3 - \tfrac{27}{23}\lambda, \quad \overline{x}_3 = 2 - \tfrac{13}{23}\lambda.$$

For the solution $\overline{\mathbf{x}}$ to be feasible the components of $\overline{\mathbf{x}}$ must all be non-negative, and therefore λ must satisfy

$$\lambda \leq \min\left(3 \times \tfrac{23}{27},\ 2 \times \tfrac{23}{13}\right).$$

Now $3 \times \frac{23}{27} = \frac{69}{27} \approx 2.56$ and $2 \times \frac{23}{13} = \frac{46}{13} \approx 3.54$. It follows that the maximum possible value of λ is $\frac{69}{27}$. The feasible solution corresponding to this value of λ is a basic feasible solution with basis $\{1, 3, 4\}$, and passing from the current basic feasible solution $\mathbf{x}$ to the new feasible basic solution would lower the cost by $-q_4\lambda$, where $-q_4\lambda = \frac{76}{23} \times \frac{69}{27} = \frac{228}{27} \approx 8.44$.

We examine this argument in more generality to see how to calculate the change in the cost that arises if an index j not in the current basis is brought into that basis. Let the current basis be $\{j_1, j_2, j_3\}$. Then

$$\mathbf{b} = s_1\mathbf{a}^{(j_1)} + s_2\mathbf{a}^{(j_2)} + s_3\mathbf{a}^{(j_3)}$$

and

$$\mathbf{a}^{(j)} = t_{1,j}\mathbf{a}^{(j_1)} + t_{2,j}\mathbf{a}^{(j_2)} + t_{3,j}\mathbf{a}^{(j_3)}.$$

Now if $\overline{\mathbf{x}}$ is a feasible solution, and if $(\overline{\mathbf{x}})_{j'} = 0$ for $j' \notin \{j_1, j_2, j_3, j\}$, then

$$\overline{x}_{j_1}\mathbf{a}^{(j_1)} + \overline{x}_{j_2}\mathbf{a}^{(j_2)} + \overline{x}_{j_3}\mathbf{a}^{(j_3)} + \overline{x}_j\mathbf{a}^{(j)} - \mathbf{b} = \mathbf{0}.$$

Let $\lambda = \overline{x}_j$. Then

$$(\overline{x}_{j_1} + \lambda t_{1,j} - s_1)\mathbf{a}^{(j_1)} + (\overline{x}_{j_2} + \lambda t_{2,j} - s_2)\mathbf{a}^{(j_2)} + (\overline{x}_{j_3} + \lambda t_{3,j} - s_3)\mathbf{a}^{(j_3)} = \mathbf{0}.$$

But the vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \mathbf{a}^{(j_3)}$ are linearly independent, because $\{j_1, j_2, j_3\}$ is a basis for the linear programming problem. It follows that

$$\overline{x}_{j_i} = s_i - \lambda t_{i,j}$$

for $i = 1, 2, 3$.

For a feasible solution we require $\lambda \geq 0$ and $s_i - \lambda t_{i,j} \geq 0$ for $i = 1, 2, 3$. We therefore require

$$0 \leq \lambda \leq \min\left(\frac{s_i}{t_{i,j}} : t_{i,j} > 0\right).$$

We could therefore obtain a new basic feasible solution by ejecting from the current basis an index j_i for which the ratio $\dfrac{s_i}{t_{i,j}}$ has its minimum value, where this minimum is taken over those values of i for which $t_{i,j} > 0$. If we set λ equal to this minimum value, then the cost is then reduced by $-q_j\lambda$.

With the current basis we find that $s_2/t_{4,2} = \frac{69}{27}$ and $s_3/t_{4,3} = \frac{46}{13}$. Now $\frac{69}{27} < \frac{46}{13}$. It follows that we could bring the index 4 into the basis, obtaining a new basis $\{1, 3, 4\}$, to obtain a cost reduction equal to $\frac{228}{27}$, given that $\frac{76}{23} \times \frac{69}{27} = \frac{76}{9} \approx 8.44$.

We now calculate the analogous cost reduction that would result from bringing the index 5 into the basis. Now $s_2/t_{5,2} = \frac{69}{31}$ and $s_3/t_{5,3} = \frac{46}{26}$. Moreover $\frac{46}{26} < \frac{69}{31}$. It follows that we could bring the index 5 into the basis, obtaining a new basis $\{1, 2, 5\}$, to obtain a cost reduction equal to $\frac{60}{23} \times \frac{46}{26} = \frac{120}{26} \approx 4.62$.

We thus obtain the better cost reduction by changing basis to $\{1, 3, 4\}$.

We need to calculate the tableau associated with the basis $\{1, 3, 4\}$. We will initially ignore the change to the criterion row, and calculate the updated values in the cells of the other rows. The current tableau with the values in the criterion row deleted is as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	0	0	$-\frac{24}{23}$	$-\frac{25}{23}$	1	$-\frac{13}{23}$	$\frac{4}{23}$	$\frac{7}{23}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
$\mathbf{a}^{(3)}$	0	0	1	$\frac{13}{23}$	$\frac{26}{23}$	2	$\frac{8}{23}$	$-\frac{6}{23}$	$\frac{1}{23}$
	·	·	·	·	·	·	·	·	·

Let $\mathbf{v}$ be a vector in $\mathbb{R}^3$ and suppose that

$$\mathbf{v} = \mu_1\mathbf{a}^{(1)} + \mu_2\mathbf{a}^{(2)} + \mu_3\mathbf{a}^{(3)} = \mu'_1\mathbf{a}^{(1)} + \mu'_2\mathbf{a}^{(4)} + \mu'_3\mathbf{a}^{(3)}.$$

Now

$$\mathbf{a}^{(4)} = -\tfrac{24}{23}\mathbf{a}^{(1)} + \tfrac{27}{23}\mathbf{a}^{(2)} + \tfrac{13}{23}\mathbf{a}^{(3)}.$$

On multiplying this equation by $\frac{23}{27}$, we find that

$$\tfrac{23}{27}\mathbf{a}^{(4)} = -\tfrac{24}{27}\mathbf{a}^{(1)} + \mathbf{a}^{(2)} + \tfrac{13}{27}\mathbf{a}^{(3)},$$

and therefore

$$\mathbf{a}^{(2)} = \tfrac{24}{27}\mathbf{a}^{(1)} + \tfrac{23}{27}\mathbf{a}^{(4)} - \tfrac{13}{27}\mathbf{a}^{(3)}.$$

It follows that

$$\mathbf{v} = (\mu_1 + \tfrac{24}{27}\mu_2)\mathbf{a}^{(1)} + \tfrac{23}{27}\mu_2\mathbf{a}^{(4)} + (\mu_3 - \tfrac{13}{27}\mu_2)\mathbf{a}^{(3)},$$

and thus

$$\mu_1' = \mu_1 + \tfrac{24}{27}\mu_2, \quad \mu_2' = \tfrac{23}{27}\mu_2, \quad \mu_3' = \mu_3 - \tfrac{13}{27}\mu_2.$$

Now each column of the tableau specifies the coefficients of the vector labelling the column of the tableau with respect to the basis specified by the vectors labelling the rows of the tableau.

The *pivot row* of the old tableau is that labelled by the vector $\mathbf{a}^{(2)}$ that is being ejected from the basis. The *pivot column* of the old tableau is that labelled by the vector $\mathbf{a}^{(4)}$ that is being brought into the basis. The *pivot element* of the tableau is the element or value in both the pivot row and the pivot column. In this example the pivot element has the value $\frac{27}{23}$.

We see from the calculations above that the values in the pivot row of the old tableau are transformed by multiplying them by the reciprocal $\frac{23}{27}$ of the pivot element; the entries in the first row of the old tableau are transformed by adding to them the entries below them in the pivot row multiplied by the factor $\frac{24}{27}$; the values in the third row of the old tableau are transformed by subtracting from them the entries above them in the pivot row multiplied by the factor $\frac{13}{27}$.

Indeed the coefficients $t_{i,j}$, s_i, $r_{i,k}$, $t'_{i,j}$, s'_i and $r'_{i,k}$ are defined for $i = 1, 2, 3$, $j = 1, 2, 3, 4, 5$ and $k = 1, 2, 3$ so that

$$\begin{aligned}
\mathbf{a}^{(j)} &= \sum_{i=1}^{3} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} t'_{i,j}\mathbf{a}^{(j'_i)}, \\
\mathbf{b} &= \sum_{i=1}^{3} s_i\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} s'_i\mathbf{a}^{(j'_i)}, \\
\mathbf{e}^{(k)} &= \sum_{i=1}^{3} r_{i,k}\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} r'_{i,k}\mathbf{a}^{(j'_i)},
\end{aligned}$$

where $j_1 = j'_1 = 1$, $j_3 = j'_3 = 3$, $j_2 = 2$ and $j'_2 = 4$.

The general rule for transforming the coefficients of a vector when changing from the basis $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ to the basis $\mathbf{a}^{(1)}, \mathbf{a}^{(4)}, \mathbf{a}^{(3)}$ ensure that

$$t'_{2,j} = \frac{1}{t_{2,4}}\, t_{2,j},$$

$$
\begin{aligned}
t'_{i,j} &= t_{i,j} - \frac{t_{i,4}}{t_{2,4}}\, t_{2,j} \quad (i = 1, 3). \\
s'_2 &= \frac{1}{t_{2,4}}\, s_2, \\
s'_i &= s_i - \frac{t_{i,4}}{t_{2,4}}\, s_2 \quad (i = 1, 3).
\end{aligned}
$$

$$
\begin{aligned}
r'_{2,k} &= \frac{1}{t_{2,4}}\, r_{2,j}, \\
r'_{i,k} &= r_{i,k} - \frac{t_{i,4}}{t_{2,4}}\, r_{2,k} \quad (i = 1, 3).
\end{aligned}
$$

The quantity $t_{2,4}$ is the value of the pivot element of the old tableau. The quantities $t_{2,j}$, s_2 and $r_{2,k}$ are those that are recorded in the pivot row of that tableau, and the quantities $t_{i,4}$ are those that are recorded in the pivot column of the tableau.

We thus obtain the following tableau:–

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	$\frac{24}{27}$	0	0	$\frac{3}{27}$	$\frac{99}{27}$	$-\frac{9}{27}$	$\frac{12}{27}$	$\frac{3}{27}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
$\mathbf{a}^{(3)}$	0	$-\frac{13}{27}$	1	0	$\frac{13}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$-\frac{11}{27}$	$\frac{4}{27}$
	·	·	·	·	·	·	·	·	·

The values in the column of the tableau labelled by the vector $\mathbf{b}$ give us the components of a new basic feasible solution $\mathbf{x}'$. Indeed the column specifies that

$$\mathbf{b} = \tfrac{99}{27}\mathbf{a}^{(1)} + \tfrac{69}{27}\mathbf{a}^{(4)} + \tfrac{15}{27}\mathbf{a}^{(3)},$$

and thus $A\mathbf{x}' = \mathbf{b}$ where

$$\mathbf{x}'^T = \left(\begin{array}{ccccc} \frac{99}{27} & 0 & \frac{15}{27} & \frac{69}{27} & 0 \end{array} \right).$$

We continue the discussion of how the extended simplex tableau transforms under a change of basis.

We now calculate the new values for the criterion row. The new basis B' is given by $B' = \{j'_1, j'_2, j'_3\}$, where $j'_1 = 1$, $j'_2 = 4$ and $j'_3 = 3$. The values p'_1, p'_2 and p'_3 that are to be recorded in the criterion row of the new tableau in the columns labelled by $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ respectively are determined by the equation

$$p'_k = c_{j'_1} r'_{1,k} + c_{j'_2} r'_{2,k} + c_{j'_3} r'_{3,k}$$

for $k = 1, 2, 3$, where

$$c_{j'_1} = c_1 = 2, \quad c_{j'_2} = c_4 = 1, \quad c_{j'_3} = c_3 = 3,$$

and where $r'_{i,k}$ denotes the ith component of the vector $\mathbf{e}^{(k)}$ with respect to the basis $\mathbf{a}^{(1)}, \mathbf{a}^{(4)}, \mathbf{a}^{(3)}$ of $\mathbb{R}^3$.

We find that

$$\begin{aligned} p'_1 &= c_{j'_1} r'_{1,1} + c_{j'_2} r'_{2,1} + c_{j'_3} r'_{3,1} \\ &= 2 \times \left(-\tfrac{9}{27}\right) + 1 \times \tfrac{6}{27} + 3 \times \tfrac{6}{27} = \tfrac{6}{27}, \\ p'_2 &= c_{j'_1} r'_{1,2} + c_{j'_2} r'_{2,2} + c_{j'_3} r'_{3,2} \\ &= 2 \times \tfrac{12}{27} + 1 \times \tfrac{7}{27} + 3 \times \left(-\tfrac{11}{27}\right) = -\tfrac{2}{27}, \\ p'_3 &= c_{j'_1} r'_{1,3} + c_{j'_2} r'_{2,3} + c_{j'_3} r'_{3,3} \\ &= 2 \times \tfrac{3}{27} + 1 \times \left(-\tfrac{5}{27}\right) + 3 \times \tfrac{4}{27} = \tfrac{13}{27}. \end{aligned}$$

We next calculate the cost C' of the new basic feasible solution. The quantities s'_1, s'_2 and s'_3 satisfy $s'_i = x'_{j_i}$ for $i = 1, 2, 3$, where $(x'_1, x'_2, x'_3, x'_4, x'_5)$ is the new basic feasible solution. It follows that

$$C' = c_{j'_1} s'_1 + c_{j'_2} s'_2 + c_{j'_3} s'_3,$$

where s'_1, s'_2 and s'_3 are determined so that

$$\mathbf{b} = s'_1 \mathbf{a}^{(j'_1)} + s'_2 \mathbf{a}^{(j'_2)} + s'_3 \mathbf{a}^{(j'_3)}.$$

The values of s'_1, s'_2 and s'_3 have already been determined, and have been recorded in the column of the new tableau labelled by the vector $\mathbf{b}$.

We can therefore calculate C' as follows:—

$$\begin{aligned} C' &= c_{j'_1} s'_1 + c_{j'_2} s'_2 + c_{j'_3} s'_3 = c_1 s'_1 + c_4 s'_2 + c_3 s'_3 \\ &= 2 \times \tfrac{99}{27} + \tfrac{69}{27} + 3 \times \tfrac{15}{27} = \tfrac{312}{27}. \end{aligned}$$

Alternatively we can use the identity $C' = \mathbf{p}'^T \mathbf{b}$ to calculate C' as follows:

$$C' = p'_1 b_1 + p'_2 b_2 + p'_3 b_3 = \tfrac{6}{27} \times 13 - \tfrac{2}{27} \times 13 + \tfrac{13}{27} \times 20 = \tfrac{312}{27}.$$

We now enter the values of p'_1, p'_2, p'_3 and C' into the tableau associated with basis $\{1, 4, 3\}$. The tableau then takes the following form:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	$\frac{24}{27}$	0	0	$\frac{3}{27}$	$\frac{99}{27}$	$-\frac{9}{27}$	$\frac{12}{27}$	$\frac{3}{27}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
$\mathbf{a}^{(3)}$	0	$-\frac{13}{27}$	1	0	$\frac{13}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$-\frac{11}{27}$	$\frac{4}{27}$
	·	·	·	·	·	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{27}$	$\frac{13}{27}$

In order to complete the extended tableau, it remains to calculate the values $-q'_j$ for $j = 1, 2, 3, 4, 5$, where q'_j satisfies the equation $-q'_j = \mathbf{p}'^T \mathbf{a}_j - c_j$ for $j = 1, 2, 3, 4, 5$.

Now q'_j is the jth component of the vector $\mathbf{q}'$ that satisfies the matrix equation $-\mathbf{q}'^T = \mathbf{p}'^T A - \mathbf{c}^T$. It follows that

$$\begin{aligned}
-\mathbf{q}'^T &= \mathbf{p}'^T A - \mathbf{c}^T \\
&= \begin{pmatrix} \frac{6}{27} & \frac{-2}{27} & \frac{13}{27} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 1 & 2 & 3 \\ 4 & 2 & 5 & 1 & 4 \end{pmatrix} \\
&\qquad - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & \frac{32}{27} & 3 & 1 & \frac{76}{27} \end{pmatrix} - \begin{pmatrix} 2 & 4 & 3 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{76}{27} & 0 & 0 & -\frac{32}{27} \end{pmatrix}
\end{aligned}$$

Thus

$$q'_1 = 0, \quad q'_2 = \tfrac{76}{27}, \quad q'_3 = 0, \quad q'_4 = 0, \quad q'_5 = \tfrac{32}{27}.$$

The value of each q'_j can also be calculated from the other values recorded in the column of the extended simplex tableau labelled by the vector $\mathbf{a}^{(j)}$. Indeed the vector $\mathbf{p}'$ is determined so as to satisfy the equation $\mathbf{p}'^T \mathbf{a}^{(j')} = c_{j'}$ for all $j' \in B'$. It follows that

$$\mathbf{p}'^T \mathbf{a}^{(j)} = \sum_{i=1}^{3} t'_{i,j} \mathbf{p}'^T \mathbf{a}^{(j'_i)} = \sum_{i=1}^{3} c_{j'_i} t'_{i,j},$$

and therefore

$$-q'_j = \sum_{i=1}^{3} c_{j'_i} t'_{i,j} - c_j.$$

The extended simplex tableau for the basis $\{1, 4, 3\}$ has now been computed, and the completed tableau is as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(1)}$	1	$\frac{24}{27}$	0	0	$\frac{3}{27}$	$\frac{99}{27}$	$-\frac{9}{27}$	$\frac{12}{27}$	$\frac{3}{27}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
$\mathbf{a}^{(3)}$	0	$-\frac{13}{27}$	1	0	$\frac{13}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$-\frac{11}{27}$	$\frac{4}{27}$
	0	$-\frac{76}{27}$	0	0	$-\frac{32}{27}$	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{27}$	$\frac{13}{27}$

The fact that $q'_j \geq 0$ for $j = 1, 2, 3, 4, 5$ shows that we have now found our basic optimal solution. Indeed the cost $\overline{C}$ of any feasible solution $\overline{\mathbf{x}}$ satisfies

$$\overline{C} \;=\; \mathbf{c}^T \overline{\mathbf{x}} = \mathbf{p}'^T A \overline{\mathbf{x}} + \mathbf{q}'^T \overline{\mathbf{x}} = \mathbf{p}'^T \mathbf{b} + \mathbf{q}'^T \overline{\mathbf{x}}$$

$$
\begin{aligned}
&= C' + \mathbf{q}'^T\overline{\mathbf{x}} \\
&= C' + \frac{76}{27}\overline{x}_2 + \frac{32}{27}\overline{x}_5,
\end{aligned}
$$

where $\overline{x}_2 = (\overline{\mathbf{x}})_2$ and $\overline{x}_5 = (\overline{\mathbf{x}})_5$.

Therefore $\mathbf{x}'$ is a basic optimal solution to the linear programming problem, where

$$\mathbf{x}'^T = \left(\begin{array}{ccccc} \frac{99}{27} & 0 & \frac{15}{27} & \frac{69}{27} & 0 \end{array} \right).$$

It is instructive to compare the pivot row and criterion row of the tableau for the basis $\{1, 2, 3\}$ with the corresponding rows of the tableau for the basis $\{1, 4, 3\}$.

These rows in the old tableau for the basis $\{1, 2, 3\}$ contain the following values:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(2)}$	0	1	0	$\frac{27}{23}$	$\frac{31}{23}$	3	$\frac{6}{23}$	$\frac{7}{23}$	$-\frac{5}{23}$
	0	0	0	$\frac{76}{23}$	$\frac{60}{23}$	20	$\frac{22}{23}$	$\frac{18}{23}$	$-\frac{3}{23}$

The corresponding rows in the new tableau for the basis $\{1, 4, 3\}$ contain the following values:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\mathbf{e}^{(3)}$
$\mathbf{a}^{(4)}$	0	$\frac{23}{27}$	0	1	$\frac{31}{27}$	$\frac{69}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$-\frac{5}{27}$
	0	$-\frac{76}{27}$	0	0	$-\frac{32}{27}$	$\frac{312}{27}$	$\frac{6}{27}$	$-\frac{2}{27}$	$\frac{13}{27}$

If we examine the values of the criterion row in the new tableau we find that they are obtained from corresponding values in the criterion row of the old tableau by subtracting off the corresponding elements of the pivot row of the old tableau multiplied by the factor $\frac{76}{27}$. As a result, the new tableau has value 0 in the cell of the criterion row in column $\mathbf{a}^{(4)}$. Thus the same rule used to calculate values in other rows of the new tableau would also have yielded the correct elements in the criterion row of the tableau.

We now investigate the reasons why this is so.

First we consider the transformation of the elements of the criterion row in the columns labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, 3, 4, 5$. Now the coefficients $t_{i,j}$ and $t'_{i,j}$ are defined for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{3} t_{i,j}\mathbf{a}^{(j_i)} = \sum_{i=1}^{3} t'_{i,j}\mathbf{a}^{(j'_i)},$$

where $j_1 = j_1' = 1$, $j_3 = j_3' = 3$, $j_2 = 2$ and $j_2' = 4$. Moreover

$$t'_{2,j} = \frac{1}{t_{2,4}}\, t_{2,j}$$

and

$$t'_{i,j} = t_{i,j} - \frac{t_{i,4}}{t_{2,4}}\, t_{2,j} \quad (i = 1, 3).$$

Now

$$\begin{aligned}
-q_j &= \sum_{i=1}^{3} c_{j_i} t_{i,j} - c_j \\
&= c_1 t_{1,j} + c_2 t_{2,j} + c_3 t_{3,j} - c_j, \\
-q'_j &= \sum_{i=1}^{3} c_{j'_i} t'_{i,j} - c_j. \\
&= c_1 t'_{1,j} + c_4 t'_{2,j} + c_3 t'_{3,j} - c_j.
\end{aligned}$$

Therefore

$$\begin{aligned}
q_j - q'_j &= c_1(t'_{1,j} - t_{1,j}) + c_4 t'_{2,j} - c_2 t_{2,j} + c_3(t'_{3,j} - t_{3,j}) \\
&= \frac{1}{t_{2,4}}\, (-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4})\, t_{2,j} \\
&= \frac{q_4}{t_{2,4}}\, t_{2,j}
\end{aligned}$$

and thus

$$-q'_j = -q_j + \frac{q_4}{t_{2,4}}\, t_{2,j}$$

for $j = 1, 2, 3, 4, 5$.

Next we note that

$$\begin{aligned}
C &= \sum_{i=1}^{3} c_{j_i} s_i = c_1 s_1 + c_2 s_2 + c_3 s_3, \\
C' &= \sum_{i=1}^{3} c_{j'_i} s'_i = c_1 s'_1 + c_4 s'_2 + c_3 s'_3.
\end{aligned}$$

Therefore

$$\begin{aligned}
C' - C &= c_1(s'_1 - s_1) + c_4 s'_2 - c_2 s_2 + c_3(s'_3 - s_3) \\
&= \frac{1}{t_{2,4}}\, (-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4})\, s_2 \\
&= \frac{q_4}{t_{2,4}}\, s_2
\end{aligned}$$

and thus

$$C' = C + \frac{q_4}{t_{2,4}}\, s_2$$

for $k = 1, 2, 3$.

To complete the verification that the criterion row of the extended simplex tableau transforms according to the same rule as the other rows we note that

$$\begin{aligned} p_k &= \sum_{i=1}^{3} c_{j_i} r_{i,k} = c_1 r_{1,k} + c_2 r_{2,k} + c_3 r_{3,k}, \\ p'_k &= \sum_{i=1}^{3} c_{j'_i} r'_{i,k} = c_1 r'_{1,k} + c_4 r'_{2,k} + c_3 r'_{3,k}. \end{aligned}$$

Therefore

$$\begin{aligned} p'_k - p_k &= c_1(r'_{1,k} - r_{1,k}) + c_4 r'_{2,k} - c_2 r_{2,k} + c_3(r'_{3,k} - r_{3,k}) \\ &= \frac{1}{t_{2,4}}\,(-c_1 t_{1,4} + c_4 - c_2 t_{2,4} - c_3 t_{3,4})\, r_{2,k} \\ &= \frac{q_4}{t_{2,4}}\, r_{2,k} \end{aligned}$$

and thus

$$p'_k = p_k + \frac{q_4}{t_{2,4}}\, r_{2,k}$$

for $k = 1, 2, 3$.

This completes the discussion of the structure and properties of the extended simplex tableau associated with the optimization problem under discussion.

4.7 The Extended Simplex Tableau

We now consider the construction of a tableau for a linear programming problem in Dantzig standard form. Such a problem is specified by an $m \times n$ matrix A, an m-dimensional target vector $\mathbf{b} \in \mathbb{R}^m$ and an n-dimensional cost vector $\mathbf{c} \in \mathbb{R}^n$. We suppose moreover that the matrix A is of rank m. We consider procedures for solving the following linear program in Danzig standard form.

> *Determine* $\mathbf{x} \in \mathbb{R}^n$ *so as to minimize* $\mathbf{c}^T\mathbf{x}$ *subject to the constraints* $A\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$.

We denote by $A_{i,j}$ the component of the matrix A in the ith row and jth column, we denote by b_i the ith component of the target vector $\mathbf{b}$ for $i = 1, 2, \ldots, m$, and we denote by c_j the jth component of the cost vector $\mathbf{c}$ for $j = 1, 2, \ldots, n$.

We recall that a feasible solution to this problem consists of an n-dimensional vector $\mathbf{x}$ that satisfies the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ (see Subsection 4.2). A feasible solution of the linear programming problem then consists of non-negative real numbers $x_1, x_2, \ldots, x_n$ for which

$$\sum_{j=1}^{n} x_j \mathbf{a}^{(j)} = \mathbf{b}.$$

A feasible solution determined by $x_1, x_2, \ldots, x_n$ is optimal if it minimizes cost $\sum_{j=1}^{n} c_j x_j$ amongst all feasible solutions to the linear programming problem.

Let $j_1, j_2, \ldots, j_m$ be distinct integers between 1 and n that are the elements of a basis B for the linear programming problem. Then the vectors $\mathbf{a}^{(j)}$ for $j \in B$ constitute a basis of the real vector space $\mathbb{R}^m$. (see Subsection 4.4).

We denote by M_B the invertible $m \times m$ matrix whose component $(M)_{i,k}$ in the ith row and jth column satisfies $(M_B)_{i,k} = (A)_{i,j_k}$ for $i, k = 1, 2, \ldots, m$. Then the kth column of the matrix M_B is specified by the column vector $\mathbf{a}^{(j_k)}$ for $k = 1, 2, \ldots, m$, and thus the columns of the matrix M_B coincide with those columns of the matrix A that are determined by elements of the basis B.

Proposition 4.3 *Let A be an real $m \times n$ matrix of rank m with columns represented by the column vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$, let $\mathbf{b}$ be an m-dimensional column vector, and let $B = \{j_1, j_2, \ldots, j_m\}$, where $j_1, j_2, \ldots, j_m$ are integers between 1 and n for which the corresponding columns $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ of the matrix A are linearly independent. Let M_B be the invertible $m \times m$ matrix defined so that $(M_B)_{i,k} = A_{i,j_k}$ for $i, k = 1, 2, \ldots, m$. Then there are uniquely determined real numbers $t_{i,j}$ and s_i for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ for which*

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)} \quad \textit{and} \quad \mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

Moreover

$$t_{i,j} = \sum_{k=1}^{m} r_{i,k} A_{k,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k$$

for $j = 1, 2, \ldots, n$, *where* $r_{i,k} = (M_B^{-1})_{i,k}$ *for* $i, k = 1, 2, \ldots, m$.

Proof Every vector in $\mathbb{R}^m$ can be expressed as a linear combination of the basis vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$. It follows that there exist uniquely determined real numbers $t_{i,j}$ and s_i for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

Then

$$A_{i,j} = \sum_{k=1}^{n} t_{k,j} A_{i,j_k} = \sum_{k=1}^{n} (M_B)_{i,k} t_{k,j}$$

and

$$b_i = \sum_{k=1}^{m} s_k A_{i,j_k} = \sum_{k=1}^{n} (M_B)_{i,k} s_k.$$

Thus $\mathbf{a}^{(j)} = M_B \mathbf{t}^{(j)}$ and $\mathbf{b} = M_B \mathbf{s}$ for $j = 1, 2, \ldots, n$, where $\mathbf{t}^{(j)}$ and $\mathbf{s}$ denote the column vectors that satisfy $(\mathbf{t}^{(j)})_i = t_{i,j}$ and $(\mathbf{s})_i = s_i$ for $i = 1, 2, \ldots, m$. It follows that

$$\mathbf{t}^{(j)} = M_B^{-1} \mathbf{a}^{(j)} \quad \text{and} \quad \mathbf{s} = M_B^{-1} \mathbf{b}$$

for $j = 1, 2, \ldots, n$. Thus

$$t_{i,j} = (M_B^{-1} \mathbf{a}^{(j)})_i = \sum_{k=1}^{m} r_{i,k} A_{k,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and

$$s_i = (M_B^{-1} \mathbf{b})_i = \sum_{k=1}^{m} r_{i,k} b_k$$

for $i = 1, 2, \ldots, m$, where $r_{i,k} = (M_B^{-1})_{i,k}$ for $i, k = 1, 2, \ldots, m$. This completes the proof. ∎

Let A be an $m \times n$ matrix with real coefficients that is of rank m whose columns are represented by the column vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$, and let $B = \{j_1, j_2, \ldots, j_m\}$, where $j_1, j_2, \ldots, j_m$ are integers between 1 and n for which the corresponding columns $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ of the matrix A are linearly independent. Let M_B be the invertible $m \times m$ matrix defined so that $(M_B)_{i,k} = A_{i,j_k}$ for $i, k = 1, 2, \ldots, m$.

The standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$ is defined such that $(\mathbf{e}^{(k)})_i = \delta_{i,k}$ for $i, k = 1, 2, \ldots, m$, where $\delta_{i,k}$ is the *Kronecker delta*, defined such that

$$\delta_{i,k} = \begin{cases} 1 & \text{if } k = i; \\ 0 & \text{if } k \neq i. \end{cases}$$

It follows from Proposition 4.3 (with the column vector $\mathbf{b}$ of that proposition set equal to $\mathbf{e}^{(k)}$) that

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} \sum_{h=1}^{m} r_{i,h} (\mathbf{e}^{(k)})_h \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} r_{i,k} \mathbf{a}^{(j_i)},$$

where $r_{i,k}$ is the coefficient $(M_B^{-1})_{i,k}$ in the ith row and kth column of the inverse M_B^{-1} of the matrix M_B.

Let A be an $m \times n$ matrix of rank m with real coefficients, and let $\mathbf{b}$ be an m-dimensional vector, and let $\{j_1, j_2, \ldots, j_m\}$ be a subset of $\{1, 2, \ldots, n\}$ for which the corresponding columns $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ of the matrix A are linearly independent. We can then record the coefficients of the m-dimensional vectors

$$\mathbf{a}^{(1)}, \ \mathbf{a}^{(2)}, \ldots, \ \mathbf{a}^{(n)}, \ \mathbf{b}, \ \mathbf{e}^{(1)}, \ \mathbf{e}^{(2)}, \ldots, \ \mathbf{e}^{(m)}$$

with respect to the basis $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$, of $\mathbb{R}^m$ in a tableau of the following form:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\cdots$	$\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1	$r_{1,1}$	$r_{1,2}$	$\cdots$	$r_{1,m}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2	$r_{2,1}$	$r_{2,2}$	$\cdots$	$r_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m	$r_{m,1}$	$r_{m,2}$	$\cdots$	$r_{m,m}$
	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$

The definition of the quantities $t_{i,j}$ ensures that $t_{i,j_k} = \delta_{i,k}$ for $i = 1, 2, \ldots, m$, where

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

Also it follows from Proposition 4.3 that

$$t_{i,j} = \sum_{k=1}^{m} r_{i,k} A_{i,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k$$

for $i = 1, 2, \ldots, m$.

If the quantities $s_1, s_2, \ldots, s_m$ are all non-negative then they determine a basic feasible solution $\mathbf{x}$ of the linear programming problem associated with the basis B with components $x_1, x_2, \ldots, x_n$, where $x_{j_i} = s_i$ for $i = 1, 2, \ldots, m$ and $x_j = 0$ for all integers j between 1 and n that do not belong to the basis B. Indeed

$$\sum_{j=1}^{n} x_j \mathbf{a}^{(j)} = \sum_{i=1}^{m} x_{j_i} \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

The *cost* C of the basic feasible solution $\mathbf{x}$ is defined to be the value $\overline{c}^T \mathbf{x}$ of the objective function. The definition of the quantities $s_1, s_2, \ldots, s_m$ ensures that

$$C = \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} c_{j_i} s_i.$$

If the quantities $s_1, s_2, \ldots, s_n$ are not all non-negative then there is no basic feasible solution associated with the basis B.

The criterion row at the bottom of the tableau has cells to record quantities $p_1, p_2, \ldots, p_m$ associated with the vectors that constitute the standard basis $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ of $\mathbb{R}^m$. These quantities are defined so that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$$

for $k = 1, 2, \ldots, m$, where c_{j_i} is the cost associated with the basis vector $\mathbf{a}^{(j_i)}$ for $i = 1, 2, \ldots, k$, Now the quantities $r_{i,k}$ are the components of the inverse of the matrix M_B, and therefore

$$\sum_{k=1}^{m} r_{h,k} A_{k,j_i} = \delta_{h,i}$$

for $h, i = 1, 2, \ldots, m$, where

$$\delta_{h,i} = \begin{cases} 1 & \text{if } h = i; \\ 0 & \text{if } h \neq i. \end{cases}$$

It follows that

$$\sum_{k=1}^{m} p_k A_{k,j_i} = \sum_{k=1}^{m}\sum_{h=1}^{m} c_{j_h} r_{h,k} A_{k,j_i} = \sum_{h=1}^{m} c_{j_h} \left(\sum_{k=1}^{m} r_{h,k} A_{k,j_i}\right) = c_{j_i}$$

On combining the identities

$$s_i = \sum_{k=1}^{m} r_{i,k} b_k, \quad p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k} \quad \text{and} \quad C = \sum_{i=1}^{m} c_{j_i} s_i$$

derived above, we find that

$$C = \sum_{i=1}^{m} c_{j_i} s_i = \sum_{i=1}^{m}\sum_{k=1}^{m} c_{j_i} r_{i,k} b_k = \sum_{k=1}^{m} p_k b_k.$$

The tableau also has cells in the criterion row to record quantities

$$-q_1, -q_2, \ldots, -q_n,$$

where $q_1, q_2, \ldots, q_n$ are the components of the unique n-dimensional vector $\mathbf{q}$ characterized by the following properties:

- $q_{j_i} = 0$ for $i = 1, 2, \ldots, m$;
- $\mathbf{c}^T\overline{\mathbf{x}} = C + \mathbf{q}^T\overline{\mathbf{x}}$ for all $\overline{\mathbf{x}} \in \mathbb{R}^m$ satisfying the matrix equation $A\overline{\mathbf{x}} = \mathbf{b}$.

First we show that if $\mathbf{q} \in \mathbb{R}^n$ is defined such that $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$ then the vector $\mathbf{q}$ has the required properties.

The definition of $p_1, p_2, \ldots, p_k$ ensures (as noted above) that

$$\sum_{k=1}^{m} p_k A_{k,j_i} = c_{j_i}$$

for $i = 1, 2, \ldots, k$. It follows that

$$q_{j_i} = c_{j_i} - (\mathbf{p}^T A)_{j_i} = c_{j_i} - \sum_{k=1}^{m} p_k A_{k,j_i} = 0$$

for $i = 1, 2, \ldots, n$.

Also $\mathbf{p}^T\mathbf{b} = C$. It follows that if $\overline{\mathbf{x}} \in \mathbb{R}^n$ satisfies $A\overline{\mathbf{x}} = \mathbf{b}$ then

$$\mathbf{c}^T\overline{\mathbf{x}} = \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\overline{\mathbf{x}} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}} = C + \mathbf{q}^T\overline{\mathbf{x}}.$$

Thus if $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$ then the vector $\mathbf{q}$ satisfies the properties specified above.

We next show that

$$(\mathbf{p}^T A)_j = \sum_{i=1}^m c_{j_i} t_{i,j}$$

for $j = 1, 2, \ldots, n$.

Now

$$t_{i,j} = \sum_{k=1}^m r_{i,k} A_{k,j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. (see Proposition 4.3). Also the definition of p_k ensures that

$$p_k = \sum_{i=1}^m c_{j_i} r_{i,k}$$

for $k = 1, 2, \ldots, m$. These results ensure that

$$\sum_{i=1}^m c_{j_i} t_{i,j} = \sum_{i=1}^m \sum_{k=1}^m c_{j_i} r_{i,k} A_{k,j} = \sum_{k=1}^m p_k A_{k,j} = (\mathbf{p}^T A)_j.$$

It follows that

$$-q_j = \sum_{k=1}^m p_k A_{k,j} - c_j = \sum_{i=1}^m c_{j_i} t_{i,j} - c_j$$

for $j = 1, 2, \ldots, n$.

The *extended simplex tableau* associated with the basis B is obtained by entering the values of the quantities $-q_j$ (for $j = 1, 2, \ldots, n$), C and p_k (for $k = 1, 2, \ldots, m$) into the bottom row to complete the tableau described previously. The extended simplex tableau has the following structure:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$	$\cdots$	$\mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1	$r_{1,1}$	$r_{1,2}$	$\cdots$	$r_{1,m}$
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2	$r_{2,1}$	$r_{2,2}$	$\cdots$	$r_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m	$r_{m,1}$	$r_{m,2}$	$\cdots$	$r_{m,m}$
	$-q_1$	$-q_2$	$\cdots$	$-q_n$	C	p_1	p_2	$\cdots$	p_m

The extended simplex tableau can be presented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \cdots \quad \mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{p}^T A - \mathbf{c}^T$	$\mathbf{p}^T\mathbf{b}$	$\mathbf{p}^T$

Let $\mathbf{c}_B$ denote the m-dimensional vector defined so that

$$\mathbf{c}_B^T = \left(\begin{array}{cccc} c_{j_1} & c_{j_2} & \cdots & c_{j_m} \end{array} \right).$$

The identities we have verified ensure that the extended simplex tableau can therefore also be represented in block form as follows:—

	$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(n)}$	$\mathbf{b}$	$\mathbf{e}^{(1)} \quad \cdots \quad \mathbf{e}^{(m)}$
$\mathbf{a}^{(j_1)}$ $\vdots$ $\mathbf{a}^{(j_m)}$	$M_B^{-1}A$	$M_B^{-1}\mathbf{b}$	M_B^{-1}
	$\mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T$	$\mathbf{c}_B^T M_B^{-1}\mathbf{b}$	$\mathbf{c}_B^T M_B^{-1}$

Given an $m \times n$ matrix A of rank m, an m-dimensional target vector $\mathbf{b}$, and an n-dimensional cost vector $\mathbf{c}$, there exists an extended simplex tableau associated with any basis B for the linear programming problem, irrespective of whether or not there exists a basic feasible solution associated with the given basis B.

The crucial requirement that enables the construction of the tableau is that the basis B should consist of m distinct integers $j_1, j_2, \ldots, j_m$ between 1 and m for which the corresponding columns of the matrix A constitute a basis of the vector space $\mathbb{R}^m$.

A basis B is associated with a basic feasible solution of the linear programming problem if and only if the values in the column labelled by the target vector $\mathbf{b}$ and the rows labelled by $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ should be non-negative. If so, those values will include the non-zero components of the basic feasible solution associated with the basis.

If there exists a basic feasible solution associated with the basis B then that solution is optimal if and only if all the values in the criterion row in the columns labelled by $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ are all non-positive.

Versions of the Simplex Tableau Algorithm for determining a basic optimal solution to the linear programmming problem, given an initial basic feasible solution, rely on the transformation rules that determine how the values in the body of the extended simplex tableau are transformed on passing from an old basis B to an new basis B', where the new basis B' contains all but one of the members of the old basis B.

Let us refer to the rows of the extended simplex tableau labelled by the basis vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ as the *basis rows* of the tableau. The following lemma determines how elements of the basis rows of the tableau transform under changes of column bases that replace a single column of an initial basis by another column that is linearly independent of the remaining columns of that initial basis.

Lemma 4.4 *Let A be an $m \times n$ matrix of rank m with real coefficients, let $j_1, j_2, \ldots, j_m$ be distinct integers between 1 and n, let h be an integer between 1 and m, and let $j'_1, j'_2, \ldots, j'_m$ be distinct integers between 1 and n, where $j'_h \neq j_h$ and $j_i = j'_i$ for $i \neq h$. Suppose that the column vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ are linearly independent, and that the column vectors $\mathbf{a}^{(j'_1)}, \mathbf{a}^{(j'_2)}, \ldots, \mathbf{a}^{(j'_m)}$ are also linearly independent, where $\mathbf{a}^{(j)}$ denotes the jth column of the matrix A. Let $\mathbf{v}$ be an element of $\mathbb{R}^m$, let $z_1, z_2, \ldots, z_m$, $z'_1, z'_2, \ldots, z'_m$, $t_{1,j'_h}, t_{2,j'_h}, \ldots, t_{m,j'_h}$ denote the uniquely-determined real numbers for which*

$$\mathbf{v} = \sum_{i=1}^{m} z_i \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} z'_i \mathbf{a}^{(j'_i)}$$

and

$$\mathbf{a}^{(j'_h)} = \sum_{i=1}^{m} t_{i,j'_h} \mathbf{a}^{(j_i)}.$$

Then

$$z'_h = \frac{1}{t_{h,j'_h}} z_h$$

and

$$z'_i = z_i - \frac{t_{i,j'_h}}{t_{h,j'_h}} z_h \quad (i \neq h).$$

Proof Expressing the vector $\mathbf{v}$ as a linear combination of $\mathbf{a}^{(j'_h)}$ and the vectors $\mathbf{a}^{(j_i)}$ for $i \neq j$, and then substituting in the representation of $\mathbf{a}^{(j'_h)}$ as a linear combination of $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \mathbf{a}^{(j_m)}$, and using the requirement that $j'_i = j_i$ when $i \neq h$, we find that

$$\begin{aligned} \mathbf{v} &= \sum_{i=1}^{m} z'_i \mathbf{a}^{(j'_i)} \\ &= z'_h \mathbf{a}^{(j'_h)} + \sum_{\substack{1 \leq i \leq m \\ i \neq h}} z'_i \mathbf{a}^{(j_i)} \end{aligned}$$

$$= \quad z'_h t_{h,j'_h} \mathbf{a}^{(j_h)} + \sum_{\substack{1\le i\le m\\ i\ne h}} (z'_i + z'_h t_{i,j'_h}) \mathbf{a}^{(j_i)}.$$

Equating coefficients of $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots \mathbf{a}^{(j_m)}$, we deduce that

$$z_h = z'_h t_{h,j'_h}$$

and

$$z_i = z'_i + z'_h t_{i,j'_h} \qquad (1 \le i \le m \text{ and } i \ne h).$$

It follows that

$$z'_h = \frac{1}{t_{h,j'_h}} z_h$$

and

$$z'_i = z_i - \frac{t_{i,j'_h}}{t_{h,j'_h}} z_h \quad (i \ne h),$$

as required. ∎

We now apply Lemma 4.4 in order to determine how entries in the basis rows of the extended simplex tableau transform which one element of the basis is replaced by an element not belonging to the basis.

Thus we consider the manner in which the basis rows of the extended simplex tableau transform under such a change of basis. Let A be be $m \times n$ matrix of rank m and let $\mathbf{b}$ be the m-dimensional target vector that are employed in the specification of the linear programming problem. Let the old basis B consist of distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n, and let the new basis B' also consist of distinct integers $j'_1, j'_2, \ldots, j'_m$ between 1 and n. We suppose that the new basis B' is obtained from the old basis by replacing an element j_h of the old basis B by some integer j'_h between 1 and n that does not belong to the old basis. We suppose therefore that $j_i = j'_i$ when $i \ne h$, and that j'_h is some integer between 1 and n that does not belong to the basis B.

Let the coefficients $t_{i,j}$, $t'_{i,j}$, s_i, s'_i, $r_{i,k}$ and $r'_{i,k}$ be determined for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} t'_{i,j} \mathbf{a}^{(j'_i)}$$

for $j = 1, 2, \ldots, n$,

$$\mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} s'_i \mathbf{a}^{(j'_i)}$$

and

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k} \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} r'_{i,k} \mathbf{a}^{(j'_i)}$$

for $k = 1, 2, \ldots, m$.

It then follows from direct applications of Lemma 4.4 that

$$\begin{aligned}
t'_{h,j} &= \frac{1}{t_{h,j'_h}} t_{h,j}, \\
t'_{i,j} &= t_{i,j} - \frac{t_{i,j'_h}}{t_{h,j'_h}} t_{h,j} \quad (i \neq h). \\
s'_h &= \frac{1}{t_{h,j'_h}} s_h, \\
s'_i &= s_i - \frac{t_{i,j'_h}}{t_{h,j'_h}} s_h \quad (i \neq h), \\
r'_{h,k} &= \frac{1}{t_{h,j'_h}} r_{h,k}, \\
r'_{i,k} &= r_{i,k} - \frac{t_{i,j'_h}}{t_{h,j'_h}} r_{h,k} \quad (i \neq h).
\end{aligned}$$

The *pivot row* of the extended simplex tableau for this change of basis from B to B' is the row labelled by the basis vector $\mathbf{a}^{(j_h)}$ that is to be removed from the current basis. The *pivot column* of the extended simplex tableau for this change of basis is the column labelled by the vector $\mathbf{a}^{(j'_h)}$ that is to be added to the current basis. The *pivot element* for this change of basis is the element t_{h,j'_h} of the tableau located in the pivot row and pivot column of the tableau.

The identities relating the components of $\mathbf{a}^{(j)}$, $\mathbf{b}$ and $\mathbf{e}^{(k)}$ with respect to the old basis to the components of those vectors with respect to the new basis ensure that the rules for transforming the rows of the tableau other than the criterion row can be stated as follows:—

- a value recorded in the pivot row is transformed by dividing it by the pivot element;
- an value recorded in a basis row other than the pivot row is transformed by substracting from it a constant multiple of the value in the same column that is located in the pivot row, where this constant multiple is the ratio of the values in the basis row and pivot row located in the pivot column.

In order to complete the discussion of the rules for transforming the values recorded in the extended simplex tableau under a change of basis that replaces an element of the old basis by an element not in that basis, it remains to analyse the rule that determines how the elements of the criterion row are transformed under this change of basis.

First we consider the transformation of the elements of the criterion row in the columns labelled by $\mathbf{a}^{(j)}$ for $j = 1, 2, \ldots, n$. Now the coefficients $t_{i,j}$ and $t'_{i,j}$ are defined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)} = \sum_{i=1}^{m} t'_{i,j} \mathbf{a}^{(j'_i)}$$

for $j = 1, 2, \ldots, n$. Moreover

$$t'_{h,j} = \frac{1}{t_{h,j'_h}} t_{h,j}$$

and

$$t'_{i,j} = t_{i,j} - \frac{t_{i,j'_h}}{t_{h,j'_h}} t_{h,j}$$

for all integers i between 1 and m for which $i \neq h$.

Now

$$-q_j = \sum_{i=1}^{m} c_{j_i} t_{i,j} - c_j \quad \text{and} \quad -q'_j = \sum_{i=1}^{m} c_{j'_i} t'_{i,j} - c_j.$$

Therefore

$$\begin{aligned} q_j - q'_j &= \sum_{\substack{1 \leq i \leq m \\ i \neq h}} c_{j_i} (t'_{i,j} - t_{i,j}) + c_{j'_h} t'_{h,j} - c_{j_h} t_{h,j} \\ &= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) t_{h,j} \\ &= \frac{q_{j'_h}}{t_{h,j'_h}} t_{h,j} \end{aligned}$$

and thus

$$-q'_j = -q_j + \frac{q_{j'_h}}{t_{h,j'_h}} t_{h,j}$$

for $j = 1, 2, \ldots, n$.

Next we note that

$$C = \sum_{i=1}^{m} c_{j_i} s_i \quad \text{and} \quad C' = \sum_{i=1}^{m} c_{j'_i} s'_i.$$

Therefore

$$\begin{aligned}
C' - C &= \sum_{\substack{1 \le i \le m \\ i \ne h}} c_{j_i}(s'_i - s_i) + c_{j'_h} s'_h - c_{j_h} s_h \\
&= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) s_h \\
&= \frac{q_{j'_h}}{t_{h,j'_h}} s_h
\end{aligned}$$

and thus

$$C' = C + \frac{q_{j'_h}}{t_{h,j'_h}} s_h$$

for $k = 1, 2, \ldots, m$.

To complete the verification that the criterion row of the extended simplex tableau transforms according to the same rule as the other rows we note that

$$p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k} \quad \text{and} \quad p'_k = \sum_{i=1}^{m} c_{j'_i} r'_{i,k}.$$

Therefore

$$\begin{aligned}
p'_k - p_k &= \sum_{\substack{1 \le i \le m \\ i \ne h}} c_{j_i}(r'_{i,k} - r_{i,k}) + c_{j'_h} r'_{h,k} - c_{j_h} r_{h,k} \\
&= \frac{1}{t_{h,j'_h}} \left(-\sum_{i=1}^{m} c_{j_i} t_{i,j'_h} + c_{j'_h} \right) r_{h,k} = \frac{q_{j'_h}}{t_{h,j'_h}} r_{h,k}
\end{aligned}$$

and thus

$$p'_k = p_k + \frac{q_{j'_h}}{t_{h,j'_h}} r_{h,k}$$

for $k = 1, 2, \ldots, m$.

We conclude that the criterion row of the extended simplex tableau transforms under changes of basis that replace one element of the basis according to a rule analogous to that which applies to the basis rows. Indeed an element of the criterion row is transformed by subtracting from it a constant multiple of the element in the pivot row that belongs to the same column, where the multiplying factor is the ratio of the elements in the criterion row and pivot row of the pivot column.

We have now discussed how the extended simplex tableau associated with a given basis B is constructed from the constraint matrix A, target vector $\mathbf{b}$ and cost vector $\mathbf{c}$ that characterizes the linear programming problem. We have also discussed how the tableau transforms when one element of the given basis is replaced.

It remains how to replace an element of a basis associated with a non-optimal feasible solution so as to obtain a basic feasible solution of lower cost where this is possible.

We use the notation previously established. Let $j_1, j_2, \ldots, j_m$ be the elements of a basis B that is associated with some basic feasible solution of the linear programming problem. Then there are non-negative numbers $s_1, s_2, \ldots, s_m$ such that

$$\mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)},$$

where $\mathbf{a}^{(j_i)}$ is the m-dimensional vector determined by column j_i of the constraint matrix A.

Let j_0 be an integer between 1 and n that does not belong to the basis B. Then

$$\mathbf{a}^{(j_0)} - \sum_{i=1}^{m} t_{i,j_0} \mathbf{a}^{(j_i)} = \mathbf{0}.$$

and therefore

$$\lambda \mathbf{a}^{(j_0)} + \sum_{i=1}^{m} (s_i - \lambda t_{i,j_0}) \mathbf{a}^{(j_i)} = \mathbf{b}.$$

This expression representing $\mathbf{b}$ as a linear combination of the basis vectors $\mathbf{a}^{(j_0)}, \mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ determines an n-dimensional vector $\overline{\mathbf{x}}(\lambda)$ satisfying the matrix equation $A\overline{\mathbf{x}}(\lambda) = \mathbf{b}$. Let $\overline{x}_j(\lambda)$ denote the jth component of the vector $\overline{\mathbf{x}}(\lambda)$ for $j = 1, 2, \ldots, n$. Then

- $\overline{x}_{j_0}(\lambda) = \lambda$;
- $\overline{x}_{j_i}(\lambda) = s_i - \lambda t_{i,j_0}$ for $i = 1, 2, \ldots, m$;
- $\overline{x}_j = 0$ when $j \notin \{j_0, j_1, j_2, \ldots, j_m\}$.

The n-dimensional vector $\overline{\mathbf{x}}(\lambda)$ represents a feasible solution of the linear programming problem if and only if all its coefficients are non-negative. The cost is then $C + q_{j_0}\lambda$, where C is the cost of the basic feasible solution determined by the basis B.

Suppose that $q_{j_0} < 0$ and that $t_{i,j_0} \leq 0$ for $i = 1, 2, \ldots, m$. Then $\overline{\mathbf{x}}(\lambda)$ is a feasible solution with cost $C + q_{j_0}\lambda$ for all non-negative real numbers λ. In this situation there is no optimal solution to the linear programming problem, because, given any real number K, it is possible to choose λ so that $C+q_{j_0}\lambda < K$, thereby obtaining a feasible solution whose cost is less than K.

If there does exist an optimal solution to the linear programming problem then there must exist at least one integer i between 1 and m for which $t_{i,j_0} > 0$. We suppose that this is the case. Then $\overline{\mathbf{x}}(\lambda)$ is a feasible solution if and only if λ satisfies $0 \leq \lambda \leq \lambda_0$, where

$$\lambda_0 = \text{minimum}\left(\frac{s_i}{t_{i,j_0}} : t_{i,j_0} > 0\right).$$

We can then choose some integer h between 1 and n for which

$$\frac{s_h}{t_{h,j_0}} = \lambda_0.$$

Let $j'_i = j_i$ for $i \neq h$, and let $j'_h = j_0$, and let $B' = \{j'_1, j'_2, \ldots, j'_m\}$. Then $\overline{\mathbf{x}}(\lambda_0)$ is a basic feasible solution of the linear programming problem associated with the basis B'. The cost of this basic feasible solution is

$$C + \frac{s_h q_{j_0}}{t_{h,j_0}}.$$

It makes sense to select the replacement column so as to obtain the greatest cost reduction. The procedure for finding this information from the tableau can be described as follows.

We suppose that the simplex tableau for a basic feasible solution has been prepared. Examine the values in the criterion row in the columns labelled by $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$. If all those are non-positive then the basic feasible solution is optimal. If not, then consider in turn those columns $\mathbf{a}^{(j_0)}$ for which the value $-q_{j_0}$ in the criterion row is positive. For each of these columns, examine the coefficients recorded in the column in the basis rows. If these coefficients are all non-positive then there is no optimal solution to the linear programming problem. Otherwise choose h to be the value of i that minimizes the ratio $\dfrac{s_i}{t_{i,j_0}}$ amongst those values of i for which $t_{i,j_0} > 0$. The row labelled by $\mathbf{a}^{(j_h)}$ would then be the pivot row if the column $\mathbf{a}^{(j_0)}$ were used as the pivot column.

Calculate the value of the cost reduction $\dfrac{s_h(-q_{j_0})}{t_{h,j_0}}$ that would result if the column labelled by $\mathbf{a}^{(j_0)}$ were used as the pivot column. Then choose the pivot column to maximize the cost reduction amongst all columns $\mathbf{a}^{(j_0)}$ for which $-q_{j_0} > 0$. Choose the row labelled by $\mathbf{a}^{(j_h)}$, where h is determined as described above. Then apply the procedures for transforming the simplex tableau to that determined by the new basis B', where B' includes j_0 together with j_i for all integers i between 1 and m satisfying $i \neq h$.

4.8 Further Analysis of the Criterion Row Transformation Rule

We investigate further the reasons why, in a linear programming problem expressed in Dantzig standard form, the criterion row of the extended simplex tableau transforms in the same fashion under change of basis as the other rows of the tableau. Thus let A be an $m \times n$ matrix of rank m with real coefficients, where $m \leq n$, and let $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ be vectors of dimensions m and n respectively. We consider the following linear programming problem:—

> *Determine an n-dimensional vector $\mathbf{x}$ so as to minimize $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

Let $\rho\colon \mathbb{R}^m \to \mathbb{R}^{m+1}$ and $\sigma\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the embeddings of $\mathbb{R}^m$ and $\mathbb{R}^n$ in $\mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1}$ respectively defined such that

$$\rho(w_1, w_2, \ldots, w_m) = (w_1, w_2, \ldots, w_m, 0)$$

$$\sigma(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0)$$

for all $(w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$. and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Also let $\mathbf{f} \in \mathbb{R}^{m+1}$ and $\mathbf{g} \in \mathbb{R}^{n+1}$ be defined so that

$$\mathbf{f} = (0, 0, \ldots, 0, 1) \quad \text{and} \quad \mathbf{g} = (0, 0, \ldots, 0, 1).$$

Every element of $\mathbb{R}^{m+1}$ can then be expressed, uniquely, in the form $\rho(\mathbf{w}) + z\mathbf{f}$ for some $\mathbf{w} \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Similarly every element of $\mathbb{R}^{n+1}$ can then be expressed, uniquely, in the form $\sigma(\mathbf{x}) + y\mathbf{g}$ for some $\mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$. The linear transformation $A\colon \mathbb{R}^n \to \mathbb{R}^m$ determined by the constraint matrix of the linear programming problem and the cost vector $\mathbf{c} \in \mathbb{R}^m$ then together determine a linear transformation $\hat{A}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{m+1}$, where

$$\hat{A}(\sigma(\mathbf{x}) + y\mathbf{g}) = \rho(A\mathbf{x}) + (y - \mathbf{c}^T\mathbf{x})\mathbf{f}$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

This linear transformation $\hat{A}$ is specified in matrix form as follows:

$$\hat{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,n} & 0 \\ A_{2,1} & A_{2,2} & \ldots & A_{2,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{m,1} & A_{m,2} & \ldots & A_{m,n} & 0 \\ -c_1 & -c_2 & \ldots & -c_n & 1 \end{pmatrix}.$$

Let

$$\hat{\mathbf{b}} = \rho(\mathbf{b}) = (b_1, b_2, \ldots, b_m, 0)^T,$$

where $\mathbf{b}$ denotes that target vector of the linear programming problem, and let

$$\hat{\mathbf{x}} = \sigma(\mathbf{x}) + y\mathbf{g} = (x_1, x_2, \ldots, x_n, y)^T$$

for some $\mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Then $\hat{A}\hat{\mathbf{x}} = \hat{\mathbf{b}}$ if and only if $A\mathbf{x} = \mathbf{b}$ and $y = \mathbf{c}^T\mathbf{x}$.

Indeed the equation $\hat{A}\hat{\mathbf{x}} = \hat{\mathbf{b}}$, expressed in matrix notation, takes the following form:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,n} & 0 \\ A_{2,1} & A_{2,2} & \ldots & A_{2,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{m,1} & A_{m,2} & \ldots & A_{m,n} & 0 \\ -c_1 & -c_2 & \ldots & -c_n & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ 0 \end{pmatrix}$$

and this matrix equation is clearly equivalent to the two simultaneous equations $A\mathbf{x} = \mathbf{b}$ and $\mathbf{c}^T\mathbf{x} = y$. The problem of minimizing $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ is thus equivalent to the problem of minimizing y subject to the constraints $\hat{A}\hat{\mathbf{x}} = \hat{\mathbf{b}}$ and $\mathbf{x} \geq \mathbf{0}$, where $\hat{\mathbf{b}} = \rho(\mathbf{b})$ and $\hat{\mathbf{x}} = \sigma(\mathbf{x}) + y\mathbf{g}$.

Now let $\hat{\mathbf{a}}^{(1)}, \hat{\mathbf{a}}^{(2)}, \ldots \hat{\mathbf{a}}^{(n)}$ denote the first n columns of the $(m+1)\times(n+1)$ matrix $\hat{A}$. Then

$$\hat{\mathbf{a}}^{(j)} = \rho(\mathbf{a}^{(j)}) - c_j\mathbf{f}$$

for $j = 1, 2, \ldots, n$, where $\mathbf{a}^{(j)}$ denotes the jth column of the constraint matrix A.

Let $j_1, j_2, \ldots, j_m$ be a basis for the linear programming problem. Then the m-dimensional vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ are linearly independent. It then follows that the $(m+1)$-dimensional vectors

$$\rho(\mathbf{a}^{(j_1)}), \rho(\mathbf{a}^{(j_2)}), \ldots, \rho(\mathbf{a}^{(j_m)}), \mathbf{f}$$

are linearly independent, and therefore the $(m+1)$-dimensional vectors

$$\hat{\mathbf{a}}^{(j_1)}, \hat{\mathbf{a}}^{(j_2)}, \ldots, \hat{\mathbf{a}}^{(j_m)}, \mathbf{f}$$

are linearly independent, and therefore constitute a basis of $\mathbb{R}^{m+1}$.

The standard basis of $\mathbb{R}^{m+1}$ consists of the vectors $\hat{\mathbf{e}}^{(1)}, \hat{\mathbf{e}}^{(2)}, \ldots, \hat{\mathbf{e}}^{(m)}, \mathbf{f}$, where $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$ is the standard basis of $\mathbb{R}^m$ and $\hat{\mathbf{e}}^{(k)} = \rho(\mathbf{e}^{(k)})$ for $k = 1, 2, \ldots, m$. Let coefficients $r_{i,k}$ be determined for $i, k = 1, 2, \ldots, m$ so that

$$\mathbf{e}^{(k)} = \sum_{i=1}^{m} r_{i,k}\mathbf{a}^{(j_i)}$$

for $k = 1, 2, \ldots, m$. Then

$$\begin{aligned} \hat{\mathbf{e}}^{(k)} &= \rho(\mathbf{e}^{(k)}) = \sum_{i=1}^{m} r_{i,k} \rho(\mathbf{a}^{(j_i)}) = \sum_{i=1}^{m} r_{i,k} \hat{\mathbf{a}}^{(j_i)} + \sum_{i=1}^{m} c_{j_i} r_{i,k} \mathbf{f} \\ &= \sum_{i=1}^{m} r_{i,k} \hat{\mathbf{a}}^{(j_i)} + p_k \mathbf{f}, \end{aligned}$$

where $p_k = \sum_{i=1}^{m} c_{j_i} r_{i,k}$ for $k = 1, 2, \ldots, m$.

Also let $s_1, s_2, \ldots, s_k$ be the components of the target vector $\mathbf{b}$ with respect to the basis $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ of $\mathbb{R}^m$, so that

$$\mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)},$$

Then

$$\begin{aligned} \hat{\mathbf{b}} &= \rho(\mathbf{b}) = \sum_{i=1}^{m} s_i \rho(\mathbf{a}^{(j_i)}) = \sum_{i=1}^{m} s_i \hat{\mathbf{a}}^{(j_i)} + \sum_{i=1}^{m} c_{j_i} s_i \mathbf{f} \\ &= \sum_{i=1}^{m} s_i \hat{\mathbf{a}}^{(j_i)} + C\mathbf{f}, \end{aligned}$$

where $C = \sum_{i=1}^{m} c_{j_i} s_i$.

Next let coefficients $t_{i,j}$ be determined so that

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j} \mathbf{a}^{(j_i)}$$

for $j = 1, 2, \ldots, n$. Then

$$\begin{aligned} \hat{\mathbf{a}}^{(j)} &= \rho(\mathbf{a}^{(j)}) - c_j \mathbf{f} = \sum_{i=1}^{m} t_{i,j} \rho(\mathbf{a}^{(j_i)}) - c_j \mathbf{f} \\ &= \sum_{i=1}^{m} t_{i,j} \hat{\mathbf{a}}^{(j_i)} + \left(\sum_{i=1}^{m} c_{j_i} t_{i,j} - c_j \right) \mathbf{f} \\ &= \sum_{i=1}^{m} r_{i,k} \hat{\mathbf{a}}^{(j_i)} - q_j \mathbf{f}, \end{aligned}$$

where $q_j = c_j - \sum_{i=1}^{m} c_{j_i} t_{i,j}$ for $j = 1, 2, \ldots, n$.

These identities show that the coefficients $t_{i,j}$ and $-q_j$ in the column of the extended simplex tableau labelled by the vector $\mathbf{a}^{(j)}$ are the coefficients of $\hat{\mathbf{a}}^{(j)}$ with respect to the basis

$$\hat{\mathbf{a}}^{(j_1)}, \hat{\mathbf{a}}^{(j_2)}, \ldots, \hat{\mathbf{a}}^{(j_m)}, \mathbf{f}$$

of $\mathbb{R}^{m+1}$ for $j = 1, 2, \ldots, n$. Similarly the coefficients s_i and C in the column of the extended simplex tableau labelled by the target vector $\mathbf{b}$ are the coefficients of $\hat{\mathbf{b}}$ with respect to the same basis of $\mathbb{R}^{m+1}$. Also the coefficients $r_{i,k}$ and p_k in the column of the extended simplex tableau labelled by the standard basis vector $\mathbf{e}^{(k)}$ are the coefficients of $\hat{\mathbf{e}}^{(k)}$ with respect to the above basis of $\mathbb{R}^{m+1}$.

The results just described ensure that the criterion row of the extended simplex tableau transforms according to the same rules as the rows above it under change of basis.

4.9 The Simplex Tableau Algorithm

In describing the Simplex Tableau Algorithm, we adopt notation previously introduced. Thus we are concerned with the solution of a linear programming problem in Dantzig standard form, specified by positive integers m and n, an $m \times n$ constraint matrix A of rank m, a target vector $\mathbf{b} \in \mathbb{R}^m$ and a cost vector $\mathbf{c} \in \mathbb{R}^n$. The optimization problem requires us to find a vector $\mathbf{x} \in \mathbb{R}^n$ that minimizes $\mathbf{c}^T\mathbf{x}$ amongst all vectors $\mathbf{x} \in \mathbb{R}^n$ that satisfy the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

We denote by $A_{i,j}$ the coefficient in the ith row and jth column of the matrix A, we denote the ith component of the target vector $\mathbf{b}$ by b_i and we denote the jth component of the cost vector $\mathbf{c}$ by c_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

As usual, we define vectors $\mathbf{a}^{(j)} \in \mathbb{R}^m$ for $j = 1, 2, \ldots, n$ such that $(\mathbf{a}^{(j)})_i = A_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Distinct integers $j_1, j_2, \ldots, j_m$ between 1 and n determine a basis B, where

$$B = \{j_1, j_2, \ldots, j_m\},$$

if and only if the corresponding vectors $\mathbf{a}^{(j_1)}, \mathbf{a}^{(j_2)}, \ldots, \mathbf{a}^{(j_m)}$ constitute a basis of $\mathbb{R}^m$. Given such a basis B we let M_B denote the invertible $m \times m$ matrix defined such that $(M_B)_{i,k} = A_{i,j_k}$ for all integers i and k between 1 and m.

We let $t_{i,j} = (M_B^{-1}A)_{i,j}$ and $s_i = (M_B^{-1}\mathbf{b})_i$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then

$$\mathbf{a}^{(j)} = \sum_{i=1}^{m} t_{i,j}\mathbf{a}^{(j_i)}$$

for $j = 1, 2, \ldots, n$, and

$$\mathbf{b} = \sum_{i=1}^{m} s_i \mathbf{a}^{(j_i)}.$$

A basis B determines an associated basic feasible solution if and only if $s_i \geq 0$ for $i = 1, 2, \ldots, m$. We suppose in what follows that the basis B determines a basic feasible solution.

Let

$$C = \sum_{i=1}^{m} c_{j_i} s_i.$$

Then C is the cost of the basic feasible solution associated with the basis B.

Let

$$-q_j = \sum_{i=1}^{m} c_{j_i} t_{i,j} - c_j.$$

Then $q_j = 0$ for all $j \in \{j_1, j_2, \ldots, j_m\}$. Also the cost of any feasible solution $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ of the linear programming problem is

$$C + \sum_{j=1}^{n} q_j \overline{x}_j.$$

The *simplex tableau* associated with the basis B is that portion of the extended simplex tableau that omits the columns labelled by $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \ldots, \mathbf{e}^{(m)}$. The simplex table has the following structure:

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\cdots$	$\mathbf{a}^{(n)}$	$\mathbf{b}$
$\mathbf{a}^{(j_1)}$	$t_{1,1}$	$t_{1,2}$	$\cdots$	$t_{1,n}$	s_1
$\mathbf{a}^{(j_2)}$	$t_{2,1}$	$t_{2,2}$	$\cdots$	$t_{2,n}$	s_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\mathbf{a}^{(j_m)}$	$t_{m,1}$	$t_{m,2}$	$\cdots$	$t_{m,n}$	s_m
	$-q_1$	$-q_2$	$\cdots$	$-q_n$	C

Let $\mathbf{c}_B$ denote the m-dimensional vector defined such that

$$\mathbf{c}_B^T = \begin{pmatrix} c_{j_1} & c_{j_2} & \cdots & c_{j_m} \end{pmatrix}.$$

Then the simplex tableau can be presented in block form as follows:—

<table>
<tr><td></td><td>$\mathbf{a}^{(1)} \quad \cdots \quad \mathbf{a}^{(n)}$</td><td>$\mathbf{b}$</td></tr>
<tr><td>$\mathbf{a}^{(j_1)}$
$\vdots$
$\mathbf{a}^{(j_m)}$</td><td>$M_B^{-1} A$</td><td>$M_B^{-1} \mathbf{b}$</td></tr>
<tr><td></td><td>$\mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T$</td><td>$\mathbf{c}_B^T M_B^{-1} \mathbf{b}$</td></tr>
</table>

Example We consider again the following linear programming problem:—

minimize

$$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$

subject to the following constraints:
$5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11$;
$4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6$;
$x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5$.

We are given the following initial basic feasible solution $(1, 2, 0, 0, 0)$. We need to determine whether this initial basic feasible solution is optimal and, if not, how to improve it till we obtain an optimal solution.

The constraints require that x_1, x_2, x_3, x_4, x_5 be non-negative real numbers satisfying the matrix equation

$$\begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Thus we are required to find a (column) vector $\mathbf{x}$ with components x_1, x_2, x_3, x_4 and x_5 that maximizes $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix},$$

and

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix}.$$

Our initial basis B satisfies $B = \{j_1, j_2\}$, where $j_1 = 1$ and $j_2 = 2$. The first two columns of the matrix A provide the corresponding invertible 2×2 matrix M_B. Thus

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

Inverting this matrix, we find that

$$M_B^{-1} = -\frac{1}{7} \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}.$$

For each integer j between 1 and 5, let $\mathbf{a}^{(j)}$ denote the m-dimensional vector whose ith component is $A_{i,j}$ for $i = 1, 2$. Then

$$\mathbf{a}^{(j)} = \sum_{i=1}^{2} t_{i,j}\mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{2} s_i\mathbf{a}^{(j_i)},$$

where $t_{i,j} = (M_B^{-1}A)_{i,j}$ and $s_i = (M_B^{-1}\mathbf{b})_i$ for $j = 1, 2, 3, 4, 5$ and $i = 1, 2$.

Calculating $M_B^{-1}A$ we find that

$$M_B^{-1}A = \begin{pmatrix} 1 & 0 & \frac{5}{7} & \frac{17}{7} & \frac{9}{7} \\ 0 & 1 & \frac{1}{7} & -\frac{12}{7} & -\frac{8}{7} \end{pmatrix}.$$

Also

$$M_B^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The coefficients of these matrices determine the values of $t_{i,j}$ and s_i to be entered into the appropriate cells of the simplex tableau.

The basis rows of the simplex tableau corresponding to the basis $\{1, 2\}$ are thus as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$
$\mathbf{a}^{(1)}$	1	0	$\frac{5}{7}$	$\frac{17}{7}$	$\frac{9}{7}$	1
$\mathbf{a}^{(2)}$	0	1	$\frac{1}{7}$	$-\frac{12}{7}$	$-\frac{8}{7}$	2
	·	·	·	·	·	·

Now the cost C of the current feasible solution satisfies the equation

$$C = \sum_{i=1}^{2} c_{j_i} s_i = c_1 s_1 + c_2 s_2,$$

where $c_1 = 3$, $c_2 = 4$, $s_1 = 1$ and $s_2 = 2$. It follows that $C = 11$.

To complete the simplex tableau, we need to compute $-q_j$ for $j = 1, 2, 3, 4, 5$, where

$$-q_j = \sum_{i=1}^{2} c_{j_i} t_{i,j} - c_j.$$

Let $\mathbf{c}_B$ denote the 2-dimensional vector whose ith component is (c_{j_i}). Then $\mathbf{c}_B = (3, 4)$. Let $\mathbf{q}$ denote the 5-dimensional vector whose jth component is q_j for $j = 1, 2, 3, 4, 5$. Then

$$-\mathbf{q}^T = \mathbf{c}_B^T M_B^{-1} A - \mathbf{c}^T.$$

It follows that

$$\begin{aligned} -\mathbf{q}^T &= \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{5}{7} & \frac{17}{7} & \frac{9}{7} \\ 0 & 1 & \frac{1}{7} & -\frac{12}{7} & -\frac{8}{7} \end{pmatrix} \\ &\quad - \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \frac{5}{7} & -\frac{60}{7} & -\frac{40}{7} \end{pmatrix}. \end{aligned}$$

The simplex tableau corresponding to basis $\{1, 2\}$ is therefore completed as follows:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$
$\mathbf{a}^{(1)}$	1	0	$\frac{5}{7}$	$\frac{17}{7}$	$\frac{9}{7}$	1
$\mathbf{a}^{(2)}$	0	1	$\frac{1}{7}$	$-\frac{12}{7}$	$-\frac{8}{7}$	2
	0	0	$\frac{5}{7}$	$-\frac{60}{7}$	$-\frac{40}{7}$	11

The values of $-q_j$ for $j = 1, 2, 3, 4, 5$ are not all non-positive ensures that the initial basic feasible solution is not optimal. Indeed the cost of a feasible solution $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5)$ is

$$11 - \tfrac{5}{7}\overline{x}_3 + \tfrac{60}{7}\overline{x}_4 + \tfrac{40}{7}\overline{x}_5.$$

Thus a feasible solution with $\overline{x}_3 > 0$ and $\overline{x}_4 = \overline{x}_5 = 0$ will have lower cost than the initial feasible basic solution. We therefore implement a change of basis whose pivot column is that labelled by the vector $\mathbf{a}^{(3)}$.

We must determine which row to use as the pivot row. We need to determine the value of i that minimizes the ratio $\dfrac{s_i}{t_{i,3}}$, subject to the requirement that $t_{i,3} > 0$. This ratio has the value $\frac{7}{5}$ when $i = 1$ and 14 when $i = 2$. Therefore the pivot row is the row labelled by $\mathbf{a}^{(1)}$. The pivot element $t_{1,3}$ then has the value $\frac{5}{7}$.

The simplex tableau corresponding to basis $\{2, 3\}$ is then obtained by subtracting the pivot row multiplied by $\frac{1}{5}$ from the row labelled by $\mathbf{a}^{(2)}$, subtracting the pivot row from the criterion row, and finally dividing all values in the pivot row by the pivot element $\frac{5}{7}$.

The simplex tableau for the basis $\{2, 3\}$ is thus the following:—

	$\mathbf{a}^{(1)}$	$\mathbf{a}^{(2)}$	$\mathbf{a}^{(3)}$	$\mathbf{a}^{(4)}$	$\mathbf{a}^{(5)}$	$\mathbf{b}$
$\mathbf{a}^{(3)}$	$\frac{7}{5}$	0	1	$\frac{17}{5}$	$\frac{9}{5}$	$\frac{7}{5}$
$\mathbf{a}^{(2)}$	$-\frac{1}{5}$	1	0	$-\frac{11}{5}$	$-\frac{7}{5}$	$\frac{9}{5}$
	-1	0	0	-11	-7	10

All the values in the criterion row to the left of the new cost are non-positive. It follows that we have found a basic optimal solution to the linear programming problem. The values recorded in the column labelled by $\mathbf{b}$ show that this basic optimal solution is

$$(0, \tfrac{9}{5}, \tfrac{7}{5}, 0, 0).$$

4.10 The Revised Simplex Algorithm

The Simplex Tableau Algorithm restricts attention to the columns to the left of the extended simplex tableau. The Revised Simplex Algorithm proceeds by maintaining the columns to the right of the extended simplex tableau, calculating values in the columns to the left of that tableau only as required.

We show how the Revised Simplex Algorithm is implemented by applying it to the example used to demonstrate the implementation of the Simplex Algorithm.

Example We apply the Revised Simplex Algorithm to determine a basic optimal solution to the the following linear programming problem:—

> *minimize*
> $$3x_1 + 4x_2 + 2x_3 + 9x_4 + 5x_5$$
> *subject to the following constraints:*
> $5x_1 + 3x_2 + 4x_3 + 7x_4 + 3x_5 = 11$*;*
> $4x_1 + x_2 + 3x_3 + 8x_4 + 4x_5 = 6$*;*
> $x_j \geq 0$ *for* $j = 1, 2, 3, 4, 5$.

We are given the following initial basic feasible solution $(1, 2, 0, 0, 0)$. We need to determine whether this initial basic feasible solution is optimal and, if not, how to improve it till we obtain an optimal solution.

The constraints require that x_1, x_2, x_3, x_4, x_5 be non-negative real numbers satisfying the matrix equation

$$\begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Thus we are required to find a (column) vector $\mathbf{x}$ with components x_1, x_2, x_3, x_4 and x_5 that maximizes $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 5 & 3 & 4 & 7 & 3 \\ 4 & 1 & 3 & 8 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 6 \end{pmatrix},$$

and

$$\mathbf{c}^T = \begin{pmatrix} 3 & 4 & 2 & 9 & 5 \end{pmatrix}.$$

Our initial basis B satisfies $B = \{j_1, j_2\}$, where $j_1 = 1$ and $j_2 = 2$. The first two columns of the matrix A provide the corresponding invertible 2×2 matrix M_B. Thus

$$M_B = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}.$$

Inverting this matrix, we find that

$$M_B^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix}.$$

For each integer j between 1 and 5, let $\mathbf{a}^((j)$ denote the m-dimensional vector whose ith component is $A_{i,j}$ for $i = 1, 2$. Then

$$\mathbf{a}^{(j)} = \sum_{i=1}^{2} t_{i,j}\mathbf{a}^{(j_i)} \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{2} s_i \mathbf{a}^{(j_i)},$$

where $t_{i,j} = (M_B^{-1}A)_{i,j}$ and $s_i = (M_B^{-1}\mathbf{b})_i$ for $j = 1, 2, 3, 4, 5$ and $i = 1, 2$.

Let $r_{i,k} = (M_B^{-1})_{i,k}$ for $i = 1, 2$ and $k = 1, 2$, and let

$$\begin{aligned} C &= c_{j_1}s_1 + c_{j_2}s_2 = c_1 s_1 + c_2 s_2 = 11 \\ p_1 &= c_{j_1}r_{1,1} + c_{j_2}r_{2,1} = c_1 r_{1,1} + c_2 r_{2,1} = \tfrac{13}{7} \\ p_2 &= c_{j_1}r_{1,2} + c_{j_2}r_{2,2} = c_1 r_{1,2} + c_2 r_{2,2} = -\tfrac{11}{7} \end{aligned}$$

The values of s_i, $r_{i,k}$, C and p_k are inserted into the following tableau, which consists of the columns to the right of the extended simplex tableau:—

	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{a}^{(1)}$	1	$-\frac{1}{7}$	$\frac{3}{7}$
$\mathbf{a}^{(2)}$	2	$\frac{4}{7}$	$-\frac{5}{7}$
	11	$\frac{13}{7}$	$-\frac{11}{7}$

To proceed with the algorithm, one computes values $-q_j$ for $j \notin B$ using the formula

$$-q_j = p_1 A_{1,j} + p_2 A_{2,j} - c_j,$$

seeking a value of j for which $-q_j > 0$. Were all the values $-q_j$ are non-positive (i.e., if all the q_j are non-negative), then the initial solution would be optimal. Computing $-q_j$ for $j = 5, 4, 3$, we find that

$$\begin{aligned} -q_5 &= \tfrac{13}{7} \times 3 - \tfrac{11}{7} \times 4 - 5 = -\tfrac{40}{7} \\ -q_4 &= = \tfrac{13}{7} \times 7 - \tfrac{11}{7} \times 8 - 9 = -\tfrac{60}{7} \\ -q_3 &= = \tfrac{13}{7} \times 4 - \tfrac{11}{7} \times 3 - 2 = \tfrac{5}{7} \end{aligned}$$

The inequality $q_3 > 0$ shows that the initial basic feasible solution is not optimal, and we should seek to change basis so as to include the vector $\mathbf{a}^{(3)}$. Let

$$\begin{aligned} t_{1,3} &= r_{1,1}A_{1,3} + r_{1,2}A_{2,3} = -\tfrac{1}{7} \times 4 + \tfrac{3}{7} \times 3 = \tfrac{5}{7} \\ t_{2,3} &= r_{2,1}A_{1,3} + r_{2,2}A_{2,3} = \tfrac{4}{7} \times 4 - \tfrac{5}{7} \times 3 = \tfrac{1}{7} \end{aligned}$$

Then

$$\mathbf{a}^{(3)} = t_{1,3}\mathbf{a}^{(j_1)} + t_{2,3}\mathbf{a}^{(j_2)} = \tfrac{5}{7}\mathbf{a}^{(1)} + \tfrac{1}{7}\mathbf{a}^{(2)}.$$

We introduce a column representing the vector $\mathbf{a}^{(3)}$ into the tableau to serve as a pivot column. The resultant tableau is as follows:—

	$\mathbf{a}^{(3)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{a}^{(1)}$	$\frac{5}{7}$	1	$-\frac{1}{7}$	$\frac{3}{7}$
$\mathbf{a}^{(2)}$	$\frac{1}{7}$	2	$\frac{4}{7}$	$-\frac{5}{7}$
	$\frac{5}{7}$	11	$\frac{13}{7}$	$-\frac{11}{7}$

To determine a pivot row we must pick the row index i so as to minimize the ratio $\dfrac{s_i}{t_{i,3}}$, subject to the requirement that $t_{i,3} > 0$. In the context of this example, we should pick $i = 1$. Accordingly the row labelled by the vector $\mathbf{a}^{(1)}$ is the pivot row. To implement the change of basis we must subtract from the second row the values above them in the pivot row, multiplied by $\frac{1}{5}$; we must subtract the values in the pivot row from the values below them in the criterion row, and we must divide the values in the pivot row itself by the pivot element $\frac{5}{7}$.

The resultant tableau corresponding to the basis $2, 3$ is then as follows:—

	$\mathbf{a}^{(3)}$	$\mathbf{b}$	$\mathbf{e}^{(1)}$	$\mathbf{e}^{(2)}$
$\mathbf{a}^{(3)}$	1	$\frac{7}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$
$\mathbf{a}^{(2)}$	0	$\frac{9}{5}$	$\frac{3}{5}$	$-\frac{4}{7}$
	0	10	2	-2

A straightforward computation then shows that if

$$\mathbf{p}^T = \begin{pmatrix} 2 & -2 \end{pmatrix}$$

then

$$\mathbf{p}^T A - \mathbf{c}^T = \begin{pmatrix} -1 & 0 & 0 & -11 & -7 \end{pmatrix}.$$

The components of this row vector are all non-positive. It follows that the basis $\{2, 3\}$ determines a basic optimal solution

$$(0, \tfrac{9}{5}, \tfrac{7}{5}, 0, 0).$$

4.11 Finding Initial Basic Solutions

Suppose that we are given a linear programming problem in Dantzig standard form, specified by positive integers m and n, an $m \times n$ matrix A of rank m, an m-dimensional target vector $\mathbf{b} \in \mathbb{R}^m$ and an n-dimensional cost vector $\mathbf{c} \in \mathbb{R}^n$. The problem requires us to find an n-dimensional vector $\mathbf{x}$ that minimizes the objective function $\mathbf{c}^T\mathbf{x}$ subject to the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Now, in the event that the column vector $\mathbf{b}$ has negative coefficients, the relevant rows of the constraint matrix A and target vector $\mathbf{b}$ can be multiplied by -1 to yield an equivalent problem in which the coefficients of the target vector are all non-negative. Therefore we may assume, without loss of generality, that $\mathbf{b} \geq \mathbf{0}$.

The Simplex Tableau Algorithm and the Revised Simplex Algorithm provided methods for passing from an initial basic feasible solution to a basic optimal solution, provided that such a basic optimal solution exists. However, we need first to find an initial basic feasible solution for this linear programming problem.

One can find such an initial basic feasible solution by solving an auxiliary linear programming problem. This auxiliary problem requires us to find n-dimensional vectors $\mathbf{x}$ and $\mathbf{z}$ that minimize the objective function $\sum_{j=1}^{n} (\mathbf{z})_j$ subject to the constraints $A\mathbf{x} + \mathbf{z} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$.

This auxiliary linear programming problem is itself in Dantzig standard form. Moreover it has an initial basic feasible solution specified by the simultaneous equations $\mathbf{x} = \mathbf{0}$ and $\mathbf{z} = \mathbf{b}$. The objective function of a feasible solution is always non-negative. Applications of algorithms based on the Simplex Method should identify a basic optimal solution $(\mathbf{x}, \mathbf{z})$ for this problem. If the cost $\sum_{j=1}^{n} (\mathbf{z})_j$ of this basic optimal solution is equal to zero then $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. If the cost of the basic optimal solution is positive then the problem does not have any basic feasible solutions.

The process of solving a linear programming problem in Dantzig standard form thus typically consists of two *phases*. The *Phase I* calculation aims to solve the auxiliary linear programming problem of seeking n-dimensional vectors $\mathbf{x}$ and $\mathbf{z}$ that minimize $\sum_{i=1}^{n} (\mathbf{z})_j$ subject to the constraints $A\mathbf{x} + \mathbf{z} = \mathbf{b}$, $\mathbf{x} \geq 0$ and $\mathbf{z} \geq 0$. If the optimal solution $(\mathbf{x}, \mathbf{z})$ of the auxiliary problem satisfies $\mathbf{z} \neq \mathbf{0}$ then there is no initial basic solution of the original linear programming problem. But if $\mathbf{z} = \mathbf{0}$ then $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and thus the Phase I calculation has identified an initial basic feasible solution of the original linear programmming problem. The *Phase II* calculation is the process of successively changing bases to lower the cost of the corresponding basic feasible solutions until either a basic optimal solution has been found or else it has been demonstated that no such basic optimal solution exists.