# Module MAU34802: The Theory of Linear Programming
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# Section 3: The Transportation Problem

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## Contents

# 3 The Transportation Problem

## 3.1 The General Transportation Problem

The Transportation Problem can be expressed in the following form. Some commodity is supplied by $m$ suppliers and is transported from those suppliers to $n$ recipients. The $i$th supplier can supply at most $s_i$ units of the commodity, and the $j$th recipient requires at least $d_j$ units of the commodity. The cost of transporting a unit of the commodity from the $i$th supplier to the $j$th recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the $i$th supplier to the $j$th recipient.

The Transportation Problem can then be presented as follows:

> *determine* $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *so as minimize* $\sum_{i,j} c_{i,j} x_{i,j}$ *subject to the constraints* $x_{i,j} \geq 0$ *for all* $i$ *and* $j$, $\sum_{j=1}^{n} x_{i,j} \leq s_i$ *and* $\sum_{i=1}^{m} x_{i,j} \geq d_j$, *where* $s_i \geq 0$ *for all* $i$, $d_j \geq 0$ *for all* $i$, *and* $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.

The quantities $s_1, s_2, \ldots, s_m$ representing the quantities of the transported commodity supplied by the suppliers are the components of an $m$-dimensional vector $(s_1, s_2, \ldots, s_m)$. We refer to this vector as the *supply vector* for the transportation problem.

The quantities $d_1, d_2, \ldots, d_n$ representing the quantities of the transported commodity demanded by the recipients are the components of an $n$-dimensional vector $(d_1, d_2, \ldots, d_n)$. We refer to this vector as the *demand vector* for the transportation problem.

The quantities $c_{i,j}$ that represent the cost of transporting the commodity from the $i$th supplier to the $j$th recipient are the components of an $m \times n$ matrix. We refer to this matrix as the *cost matrix* for the transportation problem.

## 3.2 Transportation Problems where Supply equals Demand

Consider a transportation problem with $m$ suppliers and $n$ recipients. The following proposition shows that a solution to the transportation problem can only exist if total supply of the relevant commodity exceeds total demand for that commodity.

**Proposition 3.1** *Let $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$ be non-negative real numbers. Suppose that there exist non-negative real numbers $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ that satisfy the inequalities*

$$\sum_{j=1}^{n} x_{i,j} \leq s_i \quad \textit{and} \quad \sum_{i=1}^{m} x_{i,j} \geq d_j.$$

*Then*

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} s_i.$$

*Moreover if it is the case that*

$$\sum_{j=1}^{n} d_j = \sum_{i=1}^{m} s_i.$$

*then*

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \textit{for } i = 1, 2, \ldots, m$$

*and*

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \textit{for } j = 1, 2, \ldots, n.$$

**Proof** The inequalities satisfied by the non-negative real numbers $x_{i,j}$ ensure that

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \leq \sum_{i=1}^{m} s_i.$$

Thus the total supply must equal or exceed the total demand.

Now $s_i - \sum\limits_{j=1}^{n} x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$. It follows that if $s_i > \sum_{j=1}^{n} x_{i,j}$ for at least one value of $i$ then $\sum\limits_{i=1}^{m} s_i > \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j}$. Similarly $\sum\limits_{i=1}^{m} x_{i,j} - d_j \geq 0$

for $j = 1, 2, \ldots, n$. It follows that if it is the case that $\sum_{i=1}^{m} x_{i,j} > d_j$ for at least one value of $j$ then $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} > \sum_{j=1}^{n} d_j$.

It follows that if total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n,$$

as required. ∎

We analyse the Transportation Problem in the case where total supply equals total demand. The optimization problem in this case can then be stated as follows:—

> *determine $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so as minimize $\sum_{i,j} c_{i,j} x_{i,j}$ subject to the constraints $x_{i,j} \geq 0$ for all $i$ and $j$, $\sum_{j=1}^{n} x_{i,j} = s_i$ and $\sum_{i=1}^{m} x_{i,j} = d_j$, where $s_i \geq 0$ and $d_j \geq 0$ for all $i$ and $j$, and $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.*

**Definition** A *feasible* solution to a transportation problem (with equality of total supply and total demand) is represented by real numbers $x_{i,j}$, where

- $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;
- $\sum_{j=1}^{n} x_{i,j} = s_i$ for $= 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

**Definition** A feasible solution $(x_{i,j})$ of a transportation problem is said to be *optimal* if it minimizes cost amongst all feasible solutions of that transportation problem.

## 3.3 Bases for the Transportation Problem

**Definition** Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where $m$ and $n$ are positive integers. Then a subset $B$ of $I \times J$ is said to be a *basis* for the transportation problem with $m$ suppliers and $n$ recipients if, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j$, there exists a unique $m \times n$ matrix $X$ with real coefficients satisfying the following properties:—

(i) $\sum_{j=1}^{n} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$;

(ii) $\sum_{i=1}^{m} (X)_{i,j} = (\mathbf{z})_j$ for $j = 1, 2, \ldots, n$;

(iii) $(X)_{i,j} = 0$ unless $(i, j) \in B$.

**Lemma 3.2** *Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where $m$ and $n$ are positive integers. and let*

$$B = \{(i, j) \in I \times J : i = m \textit{ or } j = n\}.$$

*Then $B$ is a basis for a transportation problem with $m$ suppliers and $n$ recipients.*

**Proof** The result can readily be verified when $m = 1$ or $n = 1$. We therefore restrict attention to cases where $m > 1$ and $n > 1$.

Let

$$B = \{(i, j) \in I \times J : i = m \text{ or } j = n\},$$

where $m > 1$ and $n > 1$. Then, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ that satisfy $\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j$, there exists a unique $m \times n$ matrix $X$ with real coefficients with all the following properties:

(i) $\sum_{j=1}^{n} (X)_{i,j} = y_i$ for $i = 1, 2, \ldots, m$;

(ii) $\sum_{i=1}^{m} (X)_{i,j} = z_j$ for $j = 1, 2, \ldots, n$;

(iii) $(X)_{i,j} = 0$ unless $(i, j) \in B$.

This matrix $X$ has coefficients as follows: $X_{i,j} = 0$ if $i < m$ and $j < n$; $X_{i,n} = y_i$ for $i < m$; $X_{m,j} = z_j$ for $j < n$; $X_{m,n} = w$, where

$$w = y_m - \sum_{j=1}^{n-1} z_j = z_n - \sum_{i=1}^{m-1} y_i.$$

This matrix $X$ is thus of the form

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & y_1 \\ 0 & 0 & \dots & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & y_{m-1} \\ z_1 & z_2 & \dots & z_{n-1} & w \end{pmatrix},$$

where

$$w = y_m - \sum_{j=1}^{n-1} z_j = z_n - \sum_{i=1}^{m-1} y_i.$$

It follows from the definition of bases for transportation problems that the subset $B$ of $I \times J$ is a basis for a transportation problem with $m$ suppliers and $n$ recipients. This completes the proof. ∎

We now introduce some notation for use in discussion of the theory of transportation problems.

For each integer $i$ between 1 and $m$, let $\mathbf{e}^{(i)}$ denote the $m$-dimensional vector whose $i$th component is equal to 1 and whose other components are zero. For each integer $j$ between 1 and $n$, let $\hat{\mathbf{e}}^{(j)}$ denote the $n$-dimensional vector whose $j$th component is equal to 1 and whose other components are zero. Thus

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \quad \text{and} \quad (\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases}$$

Moreover $\mathbf{y} = \sum_{i=1}^{m} (\mathbf{y})_i \mathbf{e}^{(i)}$ for all $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} = \sum_{j=1}^{n} (\mathbf{z})_j \hat{\mathbf{e}}^{(j)}$ for all $\mathbf{z} \in \mathbb{R}^n$. Also, for each ordered pair $(i, j)$ of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E^{(i,j)}$ denote the $m \times n$ matrix that has a single non-zero coefficient equal to 1 located in the $i$th row and $j$th column of the matrix. Thus

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Moreover

$$X = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} E^{(i,j)}$$

for all $m \times n$ matrices $X$ with real coefficients.

We let $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ be the linear transformations defined such that $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Then $\rho(E^{(i,j)}) = \mathbf{e}^{(i)}$ for $i = 1, 2, \ldots, m$ and $\sigma(E^{(i,j)}) = \hat{\mathbf{e}}^{(j)}$ for $j = 1, 2, \ldots, n$.

A feasible solution of the transportation problem with given supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix $C$ is represented by an $m \times n$ matrix $X$ satisfying the following three conditions:—

- The coefficients of $X$ are all non-negative;
- $\rho(X) = \mathbf{s}$;
- $\sigma(X) = \mathbf{d}$.

The cost functional $f\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$ is defined so that

$$f(X) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} (X)_{i,j} = \operatorname{trace}(C^T X)$$

for all $X \in M_{m,n}(\mathbb{R})$, where $C$ is the cost matrix and $c_{i,j} = (C)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

A feasible solution $\hat{X}$ of the Transportation problem is optimal if and only if $f(\hat{X}) \leq f(X)$ for all feasible solutions $X$ of that problem.

**Lemma 3.3** *Let $X$ be an $m \times n$ matrix, let $\rho(X) \in \mathbb{R}^m$ and $\sigma(X) \in \mathbb{R}^n$ be defined so that $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$, and let*

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

*Then $(\rho(X), \sigma(X)) \in W$.*

**Proof** Summing the components of the vectors $\rho(X)$ and $\sigma(X)$, we find that

$$\sum_{i=1}^{m}(\rho(X))_i = \sum_{i=1}^{m}\sum_{j=1}^{n}(X)_{i,j} = \sum_{j=1}^{n}(\sigma(X))_j.$$

Thus $(\rho(X), \sigma(X)) \in W$, as required. ∎

Given a subset $K$ of $I \times J$, where $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, we denote by $M_K$ the vector subspace of the space $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real coefficients defined such that

$$M_K = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in K\}.$$

The definition of bases for transportation problems then ensures that a subset $B$ of $I \times J$ is a basis for a transportation problem with $m$ suppliers and $n$ recipients if and only if the linear transformation $\theta_B \colon M_B \to W$ is an isomorphism of vector spaces, where

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m}(\mathbf{y})_i = \sum_{j=1}^{n}(\mathbf{z})_j \right\},$$

and $\theta_B(X) = (\rho(X), \sigma(X))$ for all $X \in M_B$, where $(\rho(X))_i = \sum_{j=1}^{n}(X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m}(X)_{i,j}$ for $j = 1, 2, \ldots, n$.

**Proposition 3.4** *A basis for a transportation problem with $m$ suppliers and $n$ recipients has $m + n - 1$ elements.*

**Proof** Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ and, for all $(i,j) \in I \times J$, let $E^{(i,j)}$ denote the $m \times n$ matrix defined so that

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Let $B$ be a basis for the transportation problem with $m$ suppliers and $n$ recipients. Then the $m \times n$ matrices $E^{(i,j)}$ for which $(i,j) \in B$ constitute a basis of the vector space $M_B$ where

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in B\}.$$

It follows that the dimension of the vector space $M_B$ is equal to the number of elements in the basis $B$.

Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\},$$

and let $\theta_B \colon M_B \to W$ be defined so that $\theta_B(X) = (\rho(X), \sigma(X))$ for all $X \in M_B$, where $\rho(X)_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$, and $\sigma(X)_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Now the definition of bases for transportation problems ensures that $\theta \colon M_B \to W$ is an isomorphism of vector spaces. Therefore $\dim M_B = \dim W$. It follows that any two bases for a transportation problem with $m$ suppliers and $n$ recipients have the same number of elements.

Lemma 3.2 showed that

$$\{(i, j) \in I \times J : i = m \text{ or } j = n\}$$

is a basis for a transportation problem with $m$ suppliers and $n$ recipients. This basis has $m + n - 1$ elements. It follows that $\dim W = m + n - 1$, and therefore every basis for a transportation problem with $m$ suppliers and $n$ recipients has $m + n - 1$ elements, as required. ∎

**Proposition 3.5** *Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where $m$ and $n$ are positive integers, and let $K$ be a subset of $I \times J$. Suppose that, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j$, there exists an $m \times n$ matrix $X$ with real coefficients belonging to $M_K$ with the following properties:*

*(i)* $\sum_{j=1}^{n} (X)_{i,j} = y_i$ *for* $i = 1, 2, \ldots, m$;

*(ii)* $\sum_{i=1}^{m} (X)_{i,j} = z_j$ *for* $j = 1, 2, \ldots, n$;

*(iii)* $(X)_{i,j} = 0$ *unless* $(i, j) \in K$.

*Then there exists a basis $B$ for the transportation problem satisfying $B \subset K$.*

**Proof** First we define bases for the vector spaces involved in the proof. For each integer $i$ between 1 and $m$, let $\mathbf{e}^{(i)} \in \mathbb{R}^m$ be defined such that

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

For each integer $j$ between 1 and $n$, let $\hat{\mathbf{e}}^{(j)} \in \mathbb{R}^n$ be defined such that

$$(\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases}$$

For each ordered pair $(i, j)$ of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E^{(i,j)} \in M_n(\mathbb{R})$ be defined such that

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Let $M_K$ denote the vector subspace of the space $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real coefficients defined such that

$$M_K = \{X \in M_{m.n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in K\},$$

let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^m (\mathbf{y})_i = \sum_{j=1}^n (\mathbf{z})_j \right\},$$

and let $\theta_K \colon M_K \to W$ be the linear transformation defined so that $\theta_K(X) = (\rho(X), \sigma(X))$ for all $X \in M_{m,n}(\mathbb{R})$, where $\rho(X)_i = \sum_{j=1}^n (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^m (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Then

$$X = \sum_{(i,j) \in K} (X)_{i,j} E^{(i,j)}$$

for all $X \in M_K$, and therefore

$$\theta_K(X) = \sum_{(i,j) \in K} (X)_{i,j} \theta(E^{(i,j)}) = \sum_{(i,j) \in K} (X)_{i,j} (\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$$

for all $X \in M_K$. The conditions of the proposition ensure that that the ordered pairs $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ of basis vectors for which $(i, j)$ belongs to $K$ span the vector space $W$. It then follows from standard linear algebra that there exists a subset $B$ of $K$ such that those ordered pairs $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i, j)$ belongs to $B$ constitute a basis for the vector space $W$ (see Corollary 2.3).

Thus, given any ordered pair $(\mathbf{y}, \mathbf{z})$ of vectors belonging to $W$, there exist uniquely determined real numbers $x_{i,j}$ for all $(i, j) \in B$ such that

$$(\mathbf{y}, \mathbf{z}) = \sum_{(i,j) \in B} x_{i,j} (\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)}).$$

Let $X \in M_B$ be the $m \times n$ matrix defined such that $(X)_{i,j} = x_{i,j}$ for all $(i,j) \in B$ and $(X)_{i,j} = 0$ for all $(i,j) \in (I \times J) \setminus B$. Then $X$ is the unique $m \times n$ matrix with the properties that $\rho(X) = \mathbf{y}$, $\sigma(X) = \mathbf{z}$ and $X_{(i,j)} = 0$ unless $(i,j) \in B$. It follows that the subset $B$ of $K$ is the required basis for the transportation problem. ∎

**Proposition 3.6** *Let $m$ and $n$ be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let $K$ be a subset of $I \times J$. Suppose that there is no basis $B$ of the transportation problem for which $K \subset B$. Then there exists a non-zero $m \times n$ matrix $Y$ with real coefficients which satisfies the following conditions:*

- $\sum_{j=1}^{n} (Y)_{i,j} = 0$ *for* $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} (Y)_{i,j} = 0$ *for* $j = 1, 2, \ldots, n$;
- $(Y)_{i,j} = 0$ *when* $(i,j) \notin K$.

**Proof** For each integer $i$ between 1 and $m$, let $\mathbf{e}^{(i)} \in \mathbb{R}^m$ be defined such that

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

For each integer $j$ between 1 and $n$, let $\hat{\mathbf{e}}^{(j)} \in \mathbb{R}^n$ be defined such that

$$(\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases},$$

and let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Now follows from Proposition 2.2 that if the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i,j) \in K$ were linearly independent then there would exist a subset $B$ of $I \times J$ satisfying $K \subset B$ such that the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i,j) \in B$ would constitute a basis of $W$. It would then follow that, given any ordered pair $(\mathbf{y}, \mathbf{z})$ of vectors belonging to $W$, there would exist a unique $m \times n$ matrix $X$ with real coefficients with the properties that $\sum_{j=1}^{m} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{n} (X)_{i,j} = (\mathbf{z})_i$ for $j = 1, 2, \ldots, n$, and $(X)_{i,j} = 0$ unless $(i,j) \in B$. The subset $B$ of $I \times J$ would thus be a basis for the transportation problem. But the subset $K$ is not contained in any basis for the Transportation Problem. It follows that the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which

$(i, j) \in K$ must be linearly dependent. Therefore there exists a non-zero $m \times n$ matrix $Y$ with real coefficients such that $(Y)_{i,j} = 0$ when $(i, j) \notin K$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} (\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)}) = (\mathbf{0}, \mathbf{0}).$$

But then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \mathbf{e}^{(i)} = \mathbf{0} \quad \text{and} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \hat{\mathbf{e}}^{(j)} = \mathbf{0},$$

and therefore

$$\sum_{j=1}^{n} (Y)_{i,j} = 0 \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} (Y)_{i,j} = 0 \quad \text{for } j = 1, 2, \ldots, n.$$

Also $(Y)_{i,j} = 0$ unless $(i, j) \in K$. The result follows. ∎

## 3.4 Basic Feasible Solutions of Transportation Problems

Consider the transportation problem with $m$ suppliers and $n$ recipients, where the $i$th supplier can provide at most $s_i$ units of some given commodity, where $s_i \geq 0$, and the $j$th recipient requires at least $d_j$ units of that commodity, where $d_j \geq 0$. We suppose also that total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

The cost of transporting the commodity from the $i$th supplier to the $j$th recipient is $c_{i,j}$.

**Definition** A feasible solution $(x_{i,j})$ of a transportation problem is said to be *basic* if there exists a basis $B$ for that transportation problem such that $x_{i,j} = 0$ whenever $(i, j) \notin B$.

**Example** Consider a transportation problem where $m = n = 2$, $s_1 = 8$, $s_2 = 3$, $d_1 = 2$, $d_2 = 9$, $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

A feasible solution takes the form of a $2 \times 2$ matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

with non-negative components which satisfies the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}.$$

A basic feasible solution will have at least one component equal to zero. There are four matrices with at least one zero component which satisfy the required equations. They are the following:—

$$\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 8 & 0 \\ -6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 9 \\ 3 & 0 \end{pmatrix}.$$

The first and third of these matrices have non-negative components. These two matrices represent basic feasible solutions to the problem, and moreover they are the only basic feasible solutions.

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

The cost of the basic feasible solution $\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}$ is

$$8c_{1,2} + 2c_{2,1} + c_{2,2} = 24 + 8 + 1 = 33.$$

The cost of the basic feasible solution $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is

$$2c_{1,1} + 6c_{1,2} + 3c_{2,2} = 4 + 18 + 3 = 25.$$

Now any $2 \times 2$ matrix $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ satisfying the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}$$

must be of the form

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$$

for some real number $\lambda$.

But the matrix $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ has non-negative components if and only if $0 \leq \lambda \leq 2$. It follows that the set of feasible solutions of this instance of the transportation problem is

$$\left\{ \begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix} : \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 2 \right\}.$$

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$. Therefore, for each real number $\lambda$ satisfying $0 \leq \lambda \leq 2$, the cost $f(\lambda)$ of the feasible solution $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ is given by

$$f(\lambda) = 2\lambda + 3(8-\lambda) + 4(2-\lambda) + (1+\lambda) = 33 - 4\lambda.$$

Cost is minimized when $\lambda = 2$, and thus $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is the optimal solution of this transportation problem. The cost of this optimal solution is 25.

**Proposition 3.7** *Given any feasible solution of a transportation problem, there exists a basic feasible solution with whose cost does not exceed that of the given solution.*

**Proof** Let $m$ and $n$ be positive integers, and let let the $m \times n$ matrix $X$ represent a feasible solution of a transportation problem with supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix $C$, where $C$ is an $m \times n$ matrix with real coefficients. Then $s_i \geq 0$ for $i = 1, 2, \ldots, m$ and $d_j \geq 0$ for $j = 1, 2, \ldots, n$, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m), \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

Also $x_{i,j} \geq 0$ for all $i$ and $j$, $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. The cost of the feasible solution $X$ is then $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the coefficient in the $i$th row and $j$th column of the cost matrix $C$.

If the feasible solution $X$ is itself basic then there is nothing to prove. Suppose therefore that $X$ is not a basic solution. We show that there then exists a feasible solution $\overline{X}$ with fewer non-zero components than the given feasible solution.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let

$$K = \{(i, j) \in I \times J : x_{i,j} > 0\}.$$

Because $X$ is not a basic solution to the Transportation Problem, there does not exist any basis $B$ for the transportation problem satisfying $K \subset B$. It therefore follows from Proposition 3.6 that there exists a non-zero $m \times n$ matrix $Y$ whose coefficients $y_{i,j}$ satisfy the following conditions:—

- $\sum_{j=1}^{n} y_{i,j} = 0$ for $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} y_{i,j} = 0$ for $j = 1, 2, \ldots, n$;
- $y_{i,j} = 0$ when $(i, j) \notin K$.

We can assume without loss of generality that $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} y_{i,j} \geq 0$, where the quantities $c_{i,j}$ are the coefficients of the cost matrix $C$, because otherwise we can replace $Y$ with $-Y$.

Let $Z_\lambda = X - \lambda Y$ for all real numbers $\lambda$, and let $z_{i,j}(\lambda)$ denote the coefficient $(Z_\lambda)_{i,j}$ in the $i$th row and $j$th column of the matrix $Z_\lambda$. Then $z_{i,j}(\lambda) = x_{i,j} - \lambda y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Moreover

- $\sum_{j=1}^{n} z_{i,j}(\lambda) = s_i$;
- $\sum_{i=1}^{m} z_{i,j}(\lambda) = d_j$;
- $z_{i,j}(\lambda) = 0$ whenever $(i, j) \notin K$;
- $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} z_{i,j}(\lambda) \leq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$ whenever $\lambda \geq 0$.

Now the matrix $Y$ is a non-zero matrix whose rows and columns all sum to zero. It follows that at least one of its coefficients must be strictly positive. Thus there exists at least one ordered pair $(i, j)$ belonging to the set $K$ for which $y_{i,j} > 0$. Let

$$\lambda_0 = \text{minimum} \left\{ \frac{x_{i,j}}{y_{i,j}} : (i, j) \in K \text{ and } y_{i,j} > 0 \right\}.$$

Then $\lambda_0 > 0$. Moreover if $0 \leq \lambda < \lambda_0$ then $x_{i,j} - \lambda y_{i,j} > 0$ for all $(i,j) \in K$, and if $\lambda > \lambda_0$ then there exists at least one element $(i_0, j_0)$ of $K$ for which $x_{i_0,j_0} - \lambda y_{i_0,j_0} < 0$. It follows that $x_{i,j} - \lambda_0 y_{i,j} \geq 0$ for all $(i,j) \in K$, and $x_{i_0,j_0} - \lambda_0 y_{i_0,j_0} = 0$.

Thus $Z_{\lambda_0}$ is a feasible solution of the given transportation problem whose cost does not exceed that of the given feasible solution $X$. Moreover $Z_{\lambda_0}$ has fewer non-zero components than the given feasible solution $X$.

If $Z_{\lambda_0}$ is itself a basic feasible solution, then we have found the required basic feasible solution whose cost does not exceed that of the given feasible solution. Otherwise we can iterate the process until we arrive at the required basic feasible solution whose cost does not exceed that of the given feasible solution. ∎

A transportation problem has only finitely many basic feasible solutions. Indeed there are only finitely many bases for the problem, and any basis is associated with at most one basic feasible solution. Therefore there exists a basic feasible solution whose cost does not exceed the cost of any other basic feasible solution. It then follows from Proposition 3.7 that the cost of this basic feasible solution cannot exceed the cost of any other feasible solution of the given transportation problem. This basic feasible solution is thus a basic optimal solution of the Transportation Problem.

The transportation problem determined by the supply vector, demand vector and cost matrix has only finitely many basic feasible solutions, because there are only finitely many bases for the problem, and each basis can determine at most one basic feasible solution. Nevertheless the number of basic feasible solutions may be quite large.

But it can be shown that a transportation problem always has a basic optimal solution. It can be found using an algorithm that implements the Simplex Method devised by George B. Dantzig in the 1940s. This algorithm involves passing from one basis to another, lowering the cost at each stage, until one eventually finds a basis that can be shown to determine a basic optimal solution of the transportation problem.

## 3.5 The Northwest Corner Method

**Example** We discuss in detail how to find an initial basic feasible solution of a transportation problem with 4 suppliers and 5 recipients, using a method known as the *Northwest Corner Method.* This method does not make use of cost information.

The course of the calculation is determined by the supply vector $\mathbf{s}$ and the demand vector $\mathbf{d}$, where

$$\mathbf{s} = (9, 11, 4, 5), \quad \mathbf{d} = (6, 7, 5, 3, 8).$$

We need to fill in the entries in a tableau of the form

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | · | · | · | · | · | 9 |
| 2 | · | · | · | · | · | 11 |
| 3 | · | · | · | · | · | 4 |
| 4 | · | · | · | · | · | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

In the tableau just presented the labels on the left hand side identify the suppliers, the labels at the top identify the recipients, the numbers on the right hand side list the number of units that the relevant supplier must provide, and the numbers at the bottom identify the number of units that the relevant recipient must obtain. Number in the bottom right hand corner gives the common value of the total supply and the total demand.

The values in the individual cells must be non-zero, the rows must sum to the value on the left, and the columns must sum to the value on the bottom. The Northwest Corner Method is applied recursively. At each stage the undetermined cell in at the top left (the northwest corner) is given the maximum possible value allowable with the constraints. The remainder of either the first row or the first column must then be completed with zeros. This leads to a reduced tableau to be determined with either one fewer row or else one fewer column. One continues in this fashion, as exemplified in the solution of this particular problem, until the entire tableau has been completed.

The method will also determine a basis associated with the basic feasible solution determined by the Northwest Corner Method. This basis lists the cells that play the role of northwest corner at each stage of the method. At the first stage, the northwest corner cell is associated with supplier 1 and recipient 1. This cell is assigned a value equal to the mimimum of the corresponding column and row sums. Thus, this example, the northwest corner cell, is given the value 6, which is the desired column sum. The remaining cells in that row are given the value 0.

The tableau then takes the following form:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | 6 | · | · | · | · | 9 |
| 2 | 0 | · | · | · | · | 11 |
| 3 | 0 | · | · | · | · | 4 |
| 4 | 0 | · | · | · | · | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

The ordered pair $(1, 1)$ commences the list of elements making up the associated basis.

At the second stage, one applies the Northwest Corner Method to the following reduced tableau:—

| $x_{i,j}$ | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|
| 1 | · | · | · | · | 3 |
| 2 | · | · | · | · | 11 |
| 3 | · | · | · | · | 4 |
| 4 | · | · | · | · | 5 |
| $d_j$ | 7 | 5 | 3 | 8 | 23 |

The required value for the first row sum of the reduced tableau has been reduced to reflect the fact that the values in the remaining undetermined cells of the first row must sum to the value 3.

The value 3 is then assigned to the northwest corner cell of the reduced tableau (as 3 is the maximum possible value for this cell subject to the constraints on row and column sums). The reduced tableau therefore takes the following form after the second stage:—

| $x_{i,j}$ | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|
| 1 | 3 | 0 | 0 | 0 | 3 |
| 2 | · | · | · | · | 11 |
| 3 | · | · | · | · | 4 |
| 4 | · | · | · | · | 5 |
| $d_j$ | 7 | 5 | 3 | 8 | 23 |

The main tableau at the completion of the second stage then stands as follows:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | 6 | 3 | 0 | 0 | 0 | 9 |
| 2 | 0 | · | · | · | · | 11 |
| 3 | 0 | · | · | · | · | 4 |
| 4 | 0 | · | · | · | · | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

The list of ordered pairs representing the basis elements determined at the second stage then stands as follows:—

Basis: $(1,1),(1,2),\ldots$.

The reduced tableau for the third stage then stands as follows:—

| $x_{i,j}$ | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|
| 2 | · | · | · | · | 11 |
| 3 | · | · | · | · | 4 |
| 4 | · | · | · | · | 5 |
| $d_j$ | 4 | 5 | 3 | 8 | 20 |

Accordingly the northwest corner of the reduced tableau should be assigned the value 4, and the remaining elements of the first column should be assigned the value 0.

The reduced tableau at the completion of the third stage stands as follows:—

| $x_{i,j}$ | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|
| 2 | 4 | · | · | · | 11 |
| 3 | 0 | · | · | · | 4 |
| 4 | 0 | · | · | · | 5 |
| $d_j$ | 4 | 5 | 3 | 8 | 20 |

The main tableau and list of basis elements at the completion of the third stage then stand as follows:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | 6 | 3 | 0 | 0 | 0 | 9 |
| 2 | 0 | 4 | · | · | · | 11 |
| 3 | 0 | 0 | · | · | · | 4 |
| 4 | 0 | 0 | · | · | · | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

Basis: $(1,1), (1,2), (2,2), \ldots$.

The reduced tableau at the completion of the fourth stage is as follows:—

| $x_{i,j}$ | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|
| 2 | 5 | · | · | 7 |
| 3 | 0 | · | · | 4 |
| 4 | 0 | · | · | 5 |
| $d_j$ | 5 | 3 | 8 | 16 |

The main tableau and list of basis elements at the completion of the fourth stage then stand as follows:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | 6 | 3 | 0 | 0 | 0 | 9 |
| 2 | 0 | 4 | 5 | · | · | 11 |
| 3 | 0 | 0 | 0 | · | · | 4 |
| 4 | 0 | 0 | 0 | · | · | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

Basis: $(1,1), (1,2), (2,2), (2,3), \ldots$.

At the fifth stage the sum of the undetermined cells for the 2nd supplier must sum to 2. Therefore the main tableau and list of basis elements at the completion of the fifth stage then stand as follows:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | 6 | 3 | 0 | 0 | 0 | 9 |
| 2 | 0 | 4 | 5 | 2 | 0 | 11 |
| 3 | 0 | 0 | 0 | · | · | 4 |
| 4 | 0 | 0 | 0 | · | · | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

Basis: $(1,1), (1,2), (2,2), (2,3), (2,4), \ldots$.

At the sixth stage the sum of the undetermined cells for the 4th recipient must sum to 1. Therefore the main tableau and list of basis elements at the completion of the sixth stage then stand as follows:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | 6 | 3 | 0 | 0 | 0 | 9 |
| 2 | 0 | 4 | 5 | 2 | 0 | 11 |
| 3 | 0 | 0 | 0 | 1 | · | 4 |
| 4 | 0 | 0 | 0 | 0 | · | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),\ldots$.

Two further stages suffice to complete the tableau. Moreover, at the completion of the eighth and final stage the main tableau and list of basis elements stand as follows:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | $s_i$ |
|---|---|---|---|---|---|---|
| 1 | 6 | 3 | 0 | 0 | 0 | 9 |
| 2 | 0 | 4 | 5 | 2 | 0 | 11 |
| 3 | 0 | 0 | 0 | 1 | 3 | 4 |
| 4 | 0 | 0 | 0 | 0 | 5 | 5 |
| $d_j$ | 6 | 7 | 5 | 3 | 8 | 29 |

Basis: $(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)$.

We now check that we have indeed obtained a basis $B$, where

$$B = \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5),(4,5)\}.$$

If $B$ is indeed a basis, then arbitrary values $s_1, s_2, s_3, s_4$ and $d_1, d_2, d_3, d_4, d_5$ should determine corresponding values of $x_{i,j}$ for $(i,j) \in B$, as indicated in the following tableau:—

| $x_{i,j}$ | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| 1 | $x_{1,1}$ | $x_{1,2}$ | | | | $s_1$ |
| 2 | | $x_{2,2}$ | $x_{2,3}$ | $x_{2,4}$ | | $s_2$ |
| 3 | | | | $x_{3,4}$ | $x_{3,5}$ | $s_3$ |
| 4 | | | | | $x_{4,5}$ | $s_4$ |
| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | |

Now analysis of the Northwest Corner Method shows that, when successive elements of the set $B$ are ordered by the stage of the method at which they are determined. Then the value of $x_{i',j'}$ for a given ordered pair $(i', j') \in B$ is determined by the values of the row sums $s_i$, the column sums $d_j$, together with the values $x_{i,j}$ for the ordered pairs $(i, j)$ in the set $B$ determined at earlier stages of the method.

In the specific numerical example that we have just considered, we find that the values of $x_{i,j}$ for ordered pairs $(i, j)$ in the set $B$, where

$$B = \{(1, 1), (1, 2), (2, 2), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5)\},$$

are determined by solving, successively, the following equations:—

$$x_{1,1} = d_1, \quad x_{1,2} = s_1 - x_{1,1}, \quad x_{2,2} = d_2 - x_{1,2},$$

$$x_{2,3} = d_3, \quad x_{2,4} = s_2 - x_{2,3} - x_{2,2}, \quad x_{3,4} = d_4 - x_{2,4},$$

$$x_{3,5} = s_3 - x_{3,4}, \quad x_{4,5} = d_5 - x_{3,5},$$

It follows that the values of $x_{i,j}$ for $(i, j) \in B$ are indeed determined by $s_1, s_2, s_3, s_4$ and $d_1, d_2, d_3, d_4, d_5$.

Indeed we find that

$$\begin{aligned}
x_{1,1} &= d_1, \\
x_{1,2} &= s_1 - d_1, \\
x_{2,2} &= d_2 - s_1 + d_1, \\
x_{2,3} &= d_3, \\
x_{2,4} &= s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{3,4} &= d_4 - s_2 + d_3 + d_2 - s_1 + d_1, \\
x_{3,5} &= s_3 - d_4 + s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{4,5} &= d_5 - s_3 + d_4 - s_2 + d_3 + d_2 - s_1 + d_1.
\end{aligned}$$

Note that, in this specific example, the values of $x_{i,j}$ for ordered pairs $(i, j)$ in the basis $B$ are expressed as sums of terms of the form $\pm s_i$ and $\pm d_j$. Moreover the summands $s_i$ all have the same sign, the summands $d_j$ all have the same sign, and the sign of the terms $s_i$ is opposite to the sign of the terms $d_j$. Thus, for example

$$x_{4,5} = (d_1 + d_2 + d_3 + d_4 + d_5) - (s_1 + s_2 + s_3).$$

This pattern is in fact a manifestation of a general result applicable to all instances of the Transportation Problem.

**Remark** The basic feasible solution produced by applying the Northwest Corner Method is just one amongst many basic feasible solutions. There are many others. Some of these may be obtained on applying the Northwest Corner Method after reordering the rows and columns (thus renumbering the suppliers and recipients).

It would take significant work to calculate all basic feasible solutions and then calculate the cost associated with each one.

## 3.6 The Minimum Cost Method for finding Basic Feasible Solutions

We discuss another method for finding an initial basic feasible solution of a transportation problem. This method is similar to the Northwest Corner Method, but takes account of the transport costs encoded in the cost matrix. The method is known as the *Minimum Cost Method*, on account of the method of selecting the cell of the tableau to be filled in at each stage in the application of the algorithm. The initial basic feasible solution obtained by this method is not necessarily optimal.

**Example** Let $c_{i,j}$ be the coefficient in the $i$th row and $j$th column of the cost matrix $C$, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

and let

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$

$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We seek to find non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$ subject to the following constraints:

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad \text{for} \quad i = 1, 2, 3, 4,$$

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad \text{for} \quad j = 1, 2, 3,$$

and $x_{i,j} \geq 0$ for all $i$ and $j$.

For this problem the supply vector is $(13, 8, 11, 13)$ and the demand vector is $(19, 12, 14)$. The components of both the supply vector and the demand vector add up to 45.

In order to start the process of finding an initial basic solution for this problems, we set up a tableau that records the row sums (or supplies), the column sums (or demands) and the costs $c_{i,j}$ for the given problem, whilst leaving cells to be filled in with the values of the non-negative real numbers $x_{i,j}$ that will specify the initial basic feasible solution. The resultant tableau is structured as follows:—

| $c_{i,j} \searrow x_{i,j}$ | 1 | 2 | 3 | $s_i$ |
|---|---|---|---|---|
| 1 | 8 ? | 4 ? | 16 ? | 13 |
| 2 | 3 ? | 7 ? | 2 ? | 8 |
| 3 | 13 ? | 8 ? | 6 ? | 11 |
| 4 | 5 ? | 7 ? | 8 ? | 13 |
| $d_j$ | 19 | 12 | 14 | 45 |

We apply the minimum cost method to find an initial basic solution.

The cell with lowest cost is the cell $(2, 3)$. We assign to this cell the maximum value possible, which is the minimum of $s_2$, which is 8, and $d_3$, which is 14. Thus we set $x_{2,3} = 8$. This forces $x_{2,1} = 0$ and $x_{2,2} = 0$. The pair $(2, 3)$ is added to the current basis. At the completion of the first stage the tableau is structured as follows:—

| $c_{i,j} \searrow x_{i,j}$ | 1 | 2 | 3 | $s_i$ |
|---|---|---|---|---|
| 1 | 8 ? | 4 ? | 16 ? | 13 |
| 2 | 3 0 | 7 0 | 2 • 8 | 8 |
| 3 | 13 ? | 8 ? | 6 ? | 11 |
| 4 | 5 ? | 7 ? | 8 ? | 13 |
| $d_j$ | 19 | 12 | 14 | 45 |

We enter a • symbol into the tableau in the relevant cell to indicate that $(1, 2)$ will be belong to the basis constructed by this method.

The next undetermined cell of lowest cost is $(1,2)$. We assign to this cell the minimum of $s_1$, which is 13, and $d_2 - x_{2,2}$, which is 12. Thus we set $x_{1,2} = 12$. This forces $x_{3,2} = 0$ and $x_{4,2} = 0$. The pair $(1,2)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

| $c_{i,j} \searrow x_{i,j}$ | 1 | 2 | 3 | $s_i$ |
|---|---|---|---|---|
| 1 | 8 ? | 4 • 12 | 16 ? | 13 |
| 2 | 3 0 | 7 0 | 2 • 8 | 8 |
| 3 | 13 ? | 8 0 | 6 ? | 11 |
| 4 | 5 ? | 7 0 | 8 ? | 13 |
| $d_j$ | 19 | 12 | 14 | 45 |

The next undetermined cell of lowest cost is $(4,1)$. We assign to this cell the minimum of $s_4 - x_{4,2}$, which is 13, and $d_1 - x_{2,1}$, which is 19. Thus we set $x_{4,1} = 13$. This forces $x_{4,3} = 0$. The pair $(4,1)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

| $c_{i,j} \searrow x_{i,j}$ | 1 | 2 | 3 | $s_i$ |
|---|---|---|---|---|
| 1 | 8 ? | 4 • 12 | 16 ? | 13 |
| 2 | 3 0 | 7 0 | 2 • 8 | 8 |
| 3 | 13 ? | 8 0 | 6 ? | 11 |
| 4 | 5 • 13 | 7 0 | 8 0 | 13 |
| $d_j$ | 19 | 12 | 14 | 45 |

The next undetermined cell of lowest cost is $(3,3)$. We assign to this cell the minimum of $s_3 - x_{3,2}$, which is 11, and $d_3 - x_{2,3} - x_{4,3}$, which is 6 $(= 14 - 8)$. Thus we set $x_{3,3} = 6$. This forces $x_{1,3} = 0$. The pair $(3,3)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

| $c_{i,j} \searrow x_{i,j}$ | 1 | 2 | 3 | $s_i$ |
|---|---|---|---|---|
| 1 | 8 | 4 $\bullet$ | 16 | |
| | ? | 12 | 0 | 13 |
| 2 | 3 | 7 | 2 $\bullet$ | |
| | 0 | 0 | 8 | 8 |
| 3 | 13 | 8 | 6 $\bullet$ | |
| | ? | 0 | 6 | 11 |
| 4 | 5 $\bullet$ | 7 | 8 | |
| | 13 | 0 | 0 | 13 |
| $d_j$ | 19 | 12 | 14 | 45 |

The next undetermined cell of lowest cost is $(1, 1)$. We assign to this cell the minimum of $s_1 - x_{1,2} - x_{1,3}$, which is 1, and $d_1 - x_{2,1} - x_{4,1}$, which is 6. Thus we set $x_{1,1} = 1$. The pair $(1, 1)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

| $c_{i,j} \searrow x_{i,j}$ | 1 | 2 | 3 | $s_i$ |
|---|---|---|---|---|
| 1 | 8 $\bullet$ | 4 $\bullet$ | 16 | |
| | 1 | 12 | 0 | 13 |
| 2 | 3 | 7 | 2 $\bullet$ | |
| | 0 | 0 | 8 | 8 |
| 3 | 13 | 8 | 6 $\bullet$ | |
| | ? | 0 | 6 | 11 |
| 4 | 5 $\bullet$ | 7 | 8 | |
| | 13 | 0 | 0 | 13 |
| $d_j$ | 19 | 12 | 14 | 45 |

The final undetermined cell is $(3, 1)$. We assign to this cell the common value of $s_3 - x_{3,2} - x_{3,3}$ and $d_1 - x_{1,1} - x_{2,1} - x_{4,1}$, which is 5. Thus we set $x_{3,1} = 5$. The pair $(3, 1)$ is added to the current basis. At the completion of this final stage the tableau is structured as follows:—

| $c_{i,j} \searrow x_{i,j}$ | 1 | | 2 | | 3 | | $s_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 8 $\bullet$ | 1 | 4 $\bullet$ | 12 | 16 | 0 | 13 |
| 2 | 3 | 0 | 7 | 0 | 2 $\bullet$ | 8 | 8 |
| 3 | 13 $\bullet$ | 5 | 8 | 0 | 6 $\bullet$ | 6 | 11 |
| 4 | 5 $\bullet$ | 13 | 7 | 0 | 8 | 0 | 13 |
| $d_j$ | | 19 | | 12 | | 14 | 45 |

The initial basis is thus $B$ where

$$B = \{(1,1),\ \ (1,2),\ \ (2,3),\ \ (3,1),\ \ (3,3),\ \ (4,1)\}.$$

The basic feasible solution is represented by the $6 \times 5$ matrix $X$, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

The cost of this initial feasible basic solution is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 2 \times 8 + 13 \times 5 + 6 \times 6 \\ & \qquad + 5 \times 13 \\ = {} & 8 + 48 + 16 + 65 + 36 + 65 \\ = {} & 238. \end{aligned}$$

## 3.7 Effectiveness of the Minimum Cost Method

We now discuss the reasons why the Minimum Cost Method yields a feasible solution to a transportation problem that is a basic feasible solution.

Consider a transportation problem with $m$ suppliers and $n$ recipients, determined by a supply vector $\mathbf{s}$, a demand vector $\mathbf{d}$ and a cost matrix $C$, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m), \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

and where $\mathbf{d} \in \mathbb{R}^n$ and cost matrix $C$, We denote by $c_{i,j}$ the coefficient in the $i$th row and $j$th column of the matrix $C$.

The Minimum Cost Method determines a feasible solution to this transportation problem. A feasible solution is represented by an $m \times n$ matrix $X$ whose coefficients $x_{i,j}$ satisfy the following conditions: $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$; $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$; $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. We must show that there exists a basis $B$ such that the feasible solution determined by the Minimum Cost Method satisfies $x_{i,j} = 0$ when $(i, j) \notin B$.

In applying the Minimum Cost Method, we begin by locating a coefficient of the cost matrix which does not exceed the other coefficients of this matrix. Renumbering the suppliers and recipients, if necessary, we may assume, without loss of generality, that $c_{i,j} \geq c_{m,n}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The feasible solution with coefficients $x_{i,j}$ that results from application of the Minimum Cost Method then conforms to a structure specified in at least one of the two cases that are described immediately below:—

- in *Case I*, the following conditions are satisfied: $d_n \leq s_m$; $x_{m,n} = d_n$; $x_{i,n} = 0$ when $1 \leq i < n$; $\sum_{j=1}^{n-1} x_{i,j} = s_i$ for $1 \leq i < m$; $\sum_{j=1}^{n-1} x_{m,j} = s_m - d_n$; $\sum_{i=1}^{m} x_{i,j} = d_j$ for $1 \leq j < n$; and the coefficients $x_{i,j}$ with $1 \leq i \leq m$ and $1 \leq j < n$ constitute a solution of the relevant transportation problem arising from application of the Minimum Cost Method.

- in *Case II*, the following conditions are satisfied: $s_m \leq d_n$; $x_{m,n} = s_m$; $x_{m,j} = 0$ when $1 \leq j < n$; $\sum_{i=1}^{m-1} x_{i,j} = d_j$ for $1 \leq j < n$; $\sum_{i=1}^{m-1} x_{i,n} = d_n - s_m$; $\sum_{j=1}^{n} x_{i,j} = s_i$ for $1 \leq i < m$; and the coefficients $x_{i,j}$ with $1 \leq i < m$ and $1 \leq j \leq n$ constitute a solution of the relevant transportation problem arising from application of the Minimum Cost Method.

The recursive nature of the Minimum Cost Method therefore enables us to prove that the Minimum Cost Method yields a basic feasible solution by induction on $m+n$, where $m$ is the number of suppliers and $n$ is the number of recipients. The Minimum Cost Method clearly yields a basic feasible solution in the trivial case where $m = n = 1$. We suppose therefore as our inductive hypothesis that the feasible solution determined by application of the Minimum Cost Method is a basic feasible solution in those cases where adding the number of suppliers to the number of recipients results in a number less than $m + n$.

In particular, we may assume that, in applying the Minimum Cost Method to the given problem with $m$ suppliers and $n$ recipients the matrices $X'$ and $X''$ that result from application of the Minimum Cost Method to a smaller transportation problem as specified in the descriptions of *Case I* and *Case II* above.

Let us now restrict attention to *Case I*. In this case the reduced transportation is a transportation problem with $m$ suppliers and $n-1$ recipients. The inductive hypothesis guarantees that the feasible solution that results from application of the Minimum Cost Method is a basic solution. Therefore there exists a basis $B'$ for this reduced problem with $n+m-2$ elements, Moreover if $1 \leq i \leq m$, $1 \leq j \leq n-1$ and if $x_{i,j} \neq 0$ then $(i,j) \in B'$. The elements of the basis $B'$ take the form of ordered pairs $(i,j)$, where $i$ is some integer between 1 and $m$ and $j$ is some integer between 1 and $n-1$. Let

$$B = B' \cup \{(m,n)\}.$$

We claim that $B$ is a basis for a transportation problem with $m$ suppliers and $n$ recipients.

Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. We must show that there exist unique real numbers $z_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that $\sum_{j=1}^{n} z_{i,j} = a_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} z_{i,j} = b_j$ for $j = 1, 2, \ldots, n$, and $z_{i,j} = 0$ unless $(i,j) \in B$.

In particular these equations require that $\sum_{i=1}^{m} z_{i,n} = b_n$. But $m$ is the only value of $i$ for which $(i,n) \in B$. It follows that the coefficients $z_{i,j}$ of any basic solution determined by the basis $B$ must satisfy $z_{i,n} = 0$ for $i < m$ and $z_{m,n} = b_n$.

It then follows that, in *Case I*, if the coefficients $z_{i,j}$ satisfy the equations $\sum_{j=1}^{n} z_{i,j} = a_i$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} z_{i,j} = b_j$ for $1 \leq j \leq n$, and if $z_{i,j} = 0$ unless $(i,j) \in B$, then these coefficients must satisfy the following conditions:—

(i) $z_{m,n} = b_n$;

(ii) $z_{i,n} = 0$ when $1 \leq i < m$;

(iii) $\sum_{j=1}^{n-1} z_{m,j} = a_m - b_n$

(iv) $\sum_{j=1}^{n-1} z_{i,j} = a_i$ when $1 \leq i < m$;

(v) $\sum_{i=1}^{m} z_{i,j} = b_j$ when $1 \leq j < n$.

(vi) if $j < n$ and $z_{i,j} \neq 0$ then $(i, j) \in B'$.

Now $B'$ is a basis for a transportation problem with $m$ suppliers and $n-1$ recipients. It follows that there exist unique real numbers $z_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j < n$ that satisfy conditions (iii), (iv), (v) and (vi) above. It follows from this that if the numbers $z_{i,n}$ are determined in accordance with conditions (i) and (ii) above then the numbers $z_{i,j}$ are the unique real numbers that solve the equations $\sum_{j=1}^{n} z_{i,j} = a_i$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} z_{i,j} = b_j$ for $1 \leq j \leq n$, and that also satisfy $z_{i,j} = 0$ whenever $(i, j) \notin B$.

We conclude that, when the Minimum Cost Method proceeds so as to produce a feasible solution to a transportation problem with $m$ suppliers and $n$ recipients that conforms to the conditions specified in *Case I* above, then that feasible solution is a basic feasible solution with associated basis $B$. A similar argument applies when the feasible solution conforms to the conditions specified in *Case II* above. The feasible solution produced by the Minimum Cost Method conforms to conditions specified in one or other of these two cases. We conclude therefore that the Minimum Cost Method always determines a basic feasible solution to a transportation problem.

## 3.8 Formal Description of the Minimum Cost Method

We describe the *Minimum Cost Method* for finding an initial basic feasible solution to a transportation problem.

Consider a transportation problem specified by positive integers $m$ and $n$ and non-negative real numbers $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$, where $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, n\}$. A feasible solution consists of an array of non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in J$ with the property that $\sum_{j \in J} x_{i,j} = s_i$ for all $i \in I$ and $\sum_{i \in I} x_{i,j} = d_j$ for all $j \in J$. The objective of the problem is to find a feasible solution that minimizes cost, where the cost of a feasible solution $(x_{i,j} : i \in I \text{ and} j \in J)$ is $\sum_{i \in I} \sum_{j \in J} c_{i,j} x_{i,j}$.

In applying the Minimum Cost Method to find an initial basic solution to the Transportation we apply an algorithm that corresponds to the determination of elements $(i_1, j_1), (i_2, j_2), \ldots, (i_{m+n-1}, j_{m+n-1})$ of $I \times J$ and of subsets $I_0, I_1, \ldots, I_{m+n-1}$ of $I$ and $J_0, J_1, \ldots, J_{m+n-1}$ of $J$ such that $I_0 = I$,

$J_0 = J$, and such that, for each integer $k$ between 1 and $m+n-1$, exactly one of the following two conditions is satisfied:—

(i) $i_k \notin I_k$, $j_k \in J_k$, $I_{k-1} = I_k \cup \{i_k\}$ and $J_{k-1} = J_k$;

(ii) $i_k \in I_k$, $j_k \notin J_k$, $I_{k-1} = I_k$ and $J_{k-1} = J_k \cup \{j_k\}$;

Indeed let $I_0 = I$, $J_0 = J$ and $B_0 = \{0\}$. The Minimum Cost Method algorithm is accomplished in $m+n-1$ stages.

Let $k$ be an integer satisfying $1 \leq k \leq m+n-1$ and that subsets $I_{k-1}$ of $I$, $J_{k-1}$ of $J$ and $B_{k-1}$ of $I \times J$ have been determined in accordance with the rules that apply at previous stages of the Minimum Cost algorithm. Suppose also that non-negative real numbers $x_{i,j}$ have been determined for all ordered pairs $(i,j)$ in $I \times J$ that satisfy either $i \notin I_{k-1}$ or $j \notin J_{k-1}$ so as to satisfy the following conditions:—

- $\sum\limits_{j \in J \setminus J_{k-1}} x_{i,j} \leq s_i$ whenever $i \in I_{k-1}$;
- $\sum\limits_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_{k-1}$;
- $\sum\limits_{i \in I \setminus I_{k-1}} x_{i,j} \leq d_j$ whenever $j \in J_{k-1}$;
- $\sum\limits_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_{k-1}$.

The Minimum Cost Method specifies that one should choose $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ so that

$$c_{i_k,j_k} \leq c_{i,j} \quad \text{for all } (i,j) \in I_{k-1} \times J_{k-1},$$

and set $B_k = B_{k-1} \cup \{(i_k, j_k)\}$. Having chosen $(i_k, j_k)$, the non-negative real number $x_{i_k,j_k}$ is then determined so that

$$x_{i_k,j_k} = \min \left( s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j},\ d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k} \right).$$

The subsets $I_k$ and $J_k$ of $I$ and $J$ respectively are then determined, along with appropriate values of $x_{i,j}$, according to the following rules:—

(i) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} < d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $I_k = I_{k-1} \setminus \{i_k\}$ and $J_k = J_{k-1}$, and we also let $x_{i_k,j} = 0$ for all $j \in J_{k-1} \setminus \{j_k\}$;

(ii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} > d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $J_k = J_{k-1} \setminus \{j_k\}$ and $I_k = I_{k-1}$, and we also let $x_{i,j_k} = 0$ for all $i \in I_{k-1} \setminus \{i_k\}$;

(iii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} = d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we determine $I_k$ and $J_k$ and the corresponding values of $x_{i,j}$ either in accordance with the specification in rule (i) above or else in accordance with the specification in rule (ii) above.

These rules ensure that the real numbers $x_{i,j}$ determined at this stage are all non-negative, and that the following conditions are satisfied at the conclusion of the $k$th stage of the Minimum Cost Method algorithm:—

- $\sum\limits_{j \in J \setminus J_k} x_{i,j} \leq s_i$ whenever $i \in I_k$;
- $\sum\limits_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_k$;
- $\sum\limits_{i \in I \setminus I_k} x_{i,j} \leq d_j$ whenever $j \in J_k$;
- $\sum\limits_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_k$.

At the completion of the final stage (for which $k = m + n - 1$) we have determined a subset $B$ of $I \times J$, where $B = B_{m+n-1}$, together with non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in I$ that constitute a feasible solution to the given transportation problem.

## 3.9 Formal Description of the Northwest Corner Method

The *Northwest Corner Method* for finding a basic feasible solution proceeds according to the stages of the *Minimum Cost Method* above, differing only from that method in the choice of the ordered pair $(i_k, j_k)$ at the $k$th stage of the method. In the Minimum Cost Method, the ordered pair $(i_k, j_k)$ is chosen such that $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ and

$$c_{i_k,j_k} \leq c_{i,j} \quad \text{for all } (i,j) \in I_{k-1} \times J_{k-1}$$

(where the sets $I_{k-1}$, $J_{k-1}$ are determined as in the specification of the Minimum Cost Method). In applying the Northwest Corner Method, costs associated with ordered pairs $(i, j)$ in $I \times J$ are not taken into account. Instead $(i_k, j_k)$ is chosen so that $i_k$ is the minimum of the integers in $I_{k-1}$ and $j_k$ is the minimum of the integers in $J_{k-1}$. Otherwise the specification of the Northwest Corner Method corresponds to that of the Minimum Cost Method, and results in a basic feasible solution of the given transportation problem.

## 3.10 A Method for finding Basic Optimal Solutions

We continue with the study of the optimization problem introduced in the discussion of the minimum cost method.

**Example** We seek to determine non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the coefficient in the $i$th row and $j$th column of the cost matrix $C$, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

subject to the constraints

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad (i = 1, 2, 3, 4)$$

and

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad (j = 1, 2, 3),$$

where

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$
$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We have found an initial basic feasible solution by the Minimum Cost Method. This solution satisfies $x_{i,j} = (X)_{i,j}$ for all $i$ and $j$, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

We next determine whether this initial basic feasible solution is an optimal solution, and, if not, how to adjust the basis to obtain a solution of lower cost.

We determine $u_1, u_2, u_3, u_4$ and $v_1, v_2, v_3$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$, where $B$ is the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all $i$ and $j$.

We therefore complete the following tableau:—

| $c_{i,j} \searrow q_{i,j}$ | 1 | | 2 | | 3 | | $u_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 8 | $\bullet$ | 4 | $\bullet$ | 16 | | 0 |
| | | 0 | | 0 | | ? | |
| 2 | 3 | | 7 | | 2 | $\bullet$ | ? |
| | | ? | | ? | | 0 | |
| 3 | 13 | $\bullet$ | 8 | | 6 | $\bullet$ | ? |
| | | 0 | | ? | | 0 | |
| 4 | 5 | $\bullet$ | 7 | | 8 | | ? |
| | | 0 | | ? | | ? | |
| $v_j$ | ? | | ? | | ? | | |

Now $u_1 = 0$, $(1, 1) \in B$ and $(1, 2) \in B$ force $v_1 = 8$ and $v_2 = 4$. After entering these values the tableau stands as follows:

| $c_{i,j} \searrow q_{i,j}$ | 1 | | 2 | | 3 | | $u_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 8 | $\bullet$ | 4 | $\bullet$ | 16 | | 0 |
| | | 0 | | 0 | | ? | |
| 2 | 3 | | 7 | | 2 | $\bullet$ | ? |
| | | ? | | ? | | 0 | |
| 3 | 13 | $\bullet$ | 8 | | 6 | $\bullet$ | ? |
| | | 0 | | ? | | 0 | |
| 4 | 5 | $\bullet$ | 7 | | 8 | | ? |
| | | 0 | | ? | | ? | |
| $v_j$ | 8 | | 4 | | ? | | |

Then $v_1 = 8$, $(3, 1) \in B$ and $(4, 1) \in B$ force $u_3 = -5$ and $u_4 = 3$. After entering these values the tableau stands as follows:

| $c_{i,j} \searrow q_{i,j}$ | 1 | 2 | 3 | $u_i$ |
|---|---|---|---|---|
| 1 | 8 • | 4 • | 16 | 0 |
| | 0 | 0 | ? | |
| 2 | 3 | 7 | 2 • | ? |
| | ? | ? | 0 | |
| 3 | 13 • | 8 | 6 • | $-5$ |
| | 0 | ? | 0 | |
| 4 | 5 • | 7 | 8 | 3 |
| | 0 | ? | ? | |
| $v_j$ | 8 | 4 | ? | |

Then $u_3 = -5$ and $(3,3) \in B$ force $v_3 = 1$. After entering this value the tableau stands as follows:

| $c_{i,j} \searrow q_{i,j}$ | 1 | 2 | 3 | $u_i$ |
|---|---|---|---|---|
| 1 | 8 • | 4 • | 16 | 0 |
| | 0 | 0 | ? | |
| 2 | 3 | 7 | 2 • | ? |
| | ? | ? | 0 | |
| 3 | 13 • | 8 | 6 • | $-5$ |
| | 0 | ? | 0 | |
| 4 | 5 • | 7 | 8 | 3 |
| | 0 | ? | ? | |
| $v_j$ | 8 | 4 | 1 | |

Then $v_3 = 1$ and $(2,3) \in B$ force $u_2 = -1$.
After entering the numbers $u_i$ and $v_j$, the tableau is as follows:—

| $c_{i,j} \searrow q_{i,j}$ | 1 | 2 | 3 | $u_i$ |
|---|---|---|---|---|
| 1 | 8 • | 4 • | 16 | 0 |
| | 0 | 0 | ? | |
| 2 | 3 | 7 | 2 • | $-1$ |
| | ? | ? | 0 | |
| 3 | 13 • | 8 | 6 • | $-5$ |
| | 0 | ? | 0 | |
| 4 | 5 • | 7 | 8 | 3 |
| | 0 | ? | ? | |
| $v_j$ | 8 | 4 | 1 | |

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 15$, $q_{2,1} = -6$, $q_{2,2} = 2$, $q_{3,2} = -1$, $q_{4,2} = 6$ and $q_{4,3} = 10$.

The completed tableau is as follows:—

| $c_{i,j} \searrow q_{i,j}$ | 1 | 2 | 3 | $u_i$ |
|---|---|---|---|---|
| 1 | 8 $\bullet$ | 4 $\bullet$ | 16 | 0 |
| | 0 | 0 | 15 | |
| 2 | 3 | 7 | 2 $\bullet$ | $-1$ |
| | $-6$ | 2 | 0 | |
| 3 | 13 $\bullet$ | 8 | 6 $\bullet$ | $-5$ |
| | 0 | $-1$ | 0 | |
| 4 | 5 $\bullet$ | 7 | 8 | 3 |
| | 0 | 6 | 10 | |
| $v_j$ | 8 | 4 | 1 | |

The initial basic feasible solution is not optimal because some of the quantities $q_{i,j}$ are negative. To see this, suppose that the numbers $\overline{x}_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ constitute a feasible solution to the given problem. Then $\sum_{j=1}^{3} \overline{x}_{i,j} = s_i$ for $i = 1, 2, 3$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1, 2, 3, 4$. It follows that

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{3} c_{i,j}\overline{x}_{i,j} &= \sum_{i=1}^{4}\sum_{j=1}^{3}(v_j - u_i + q_{i,j})\overline{x}_{i,j} \\ &= \sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i + \sum_{i=1}^{4}\sum_{j=1}^{3} q_{i,j}\overline{x}_{i,j}.\end{aligned}$$

Applying this identity to the initial basic feasible solution, we find that $\sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i = 238$, given that 238 is the cost of the initial basic feasible solution. Thus the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 238 + 15\overline{x}_{1,3} - 6\overline{x}_{2,1} + 2\overline{x}_{2,2} - \overline{x}_{3,2} + 6\overline{x}_{4,2} + 10\overline{x}_{4,3}.$$

One could construct feasible solutions with $\overline{x}_{2,1} < 0$ and $\overline{x}_{i,j} = 0$ for $(i, j) \notin B \cup \{(2, 1)\}$, and the cost of such feasible solutions would be lower than that of the initial basic solution. We therefore seek to bring $(2, 1)$ into the basis, removing some other element of the basis to ensure that the new basis corresponds to a feasible basic solution.

The procedure for achieving this requires us to determine a $4 \times 3$ matrix $Y$ satisfying the following conditions:—

- $y_{2,1} = 1$;
- $y_{i,j} = 0$ when $(i, j) \notin B \cup \{(2, 1)\}$;

- all rows and columns of the matrix $Y$ sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix $Y$ that correspond to cells in the current basis (marked with the $\bullet$ symbol), so that all rows sum to zero and all columns sum to zero:—

| $y_{i,j}$ | 1 | 2 | 3 | |
|---|---|---|---|---|
| 1 | ? $\bullet$ | ? $\bullet$ | | 0 |
| 2 | 1 $\circ$ | | ? $\bullet$ | 0 |
| 3 | ? $\bullet$ | | ? $\bullet$ | 0 |
| 4 | ? $\bullet$ | | | 0 |
| | 0 | 0 | 0 | 0 |

The constraints that $y_{2,1} = 1$, $y_{i,j} = 0$ when $(i,j) \notin B$ and the constraints requiring the rows and columns to sum to zero determine the values of $y_{i,j}$ for all $y_{i,j} \in B$. These values are recorded in the following tableau:—

| $y_{i,j}$ | 1 | 2 | 3 | |
|---|---|---|---|---|
| 1 | 0 $\bullet$ | 0 $\bullet$ | | 0 |
| 2 | 1 $\circ$ | | $-1$ $\bullet$ | 0 |
| 3 | $-1$ $\bullet$ | | 1 $\bullet$ | 0 |
| 4 | 0 $\bullet$ | | | 0 |
| | 0 | 0 | 0 | 0 |

We now determine those values of $\lambda$ for which $X+\lambda Y$ is a feasible solution, where

$$X + \lambda Y = \begin{pmatrix} 1 & 12 & 0 \\ \lambda & 0 & 8-\lambda \\ 5-\lambda & 0 & 6+\lambda \\ 13 & 0 & 0 \end{pmatrix}.$$

In order to drive down the cost as far as possible, we should make $\lambda$ as large as possible, subject to the requirement that all the coefficients of the above matrix should be non-negative numbers.

Accordingly we take $\lambda = 5$. Our new basic feasible solution $X$ is then as follows:—

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$

We regard $X$ as the current feasible basic solution.

The cost of the current feasible basic solution $X$ is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 3 \times 5 + 2 \times 3 + 6 \times 11 \\ & \qquad + 5 \times 13 \\ = \;& 8 + 48 + 15 + 6 + 66 + 65 \\ = \;& 208. \end{aligned}$$

The cost has gone down by 30, as one would expect (the reduction in the cost being $-\lambda q_{2,1}$ where $\lambda = 5$ and $q_{2,1} = -6$).

The current basic feasible solution $X$ is associated with the basis $B$ where

$$B = \{(1,1),\ (1,2),\ (2,1),\ (2,3),\ (3,3),\ (4,1)\}.$$

We now determine, for the current basis $B$ values $u_1, u_2, u_3, u_4$ and $v_1, v_2, v_3$ such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all $i$ and $j$.

We therefore complete the following tableau:—

| $c_{i,j} \searrow q_{i,j}$ | 1 | 2 | 3 | $u_i$ |
|---|---|---|---|---|
| 1 | 8 $\bullet$ | 4 $\bullet$ | 16 | 0 |
| | 0 | 0 | ? | |
| 2 | 3 $\bullet$ | 7 | 2 $\bullet$ | ? |
| | 0 | ? | 0 | |
| 3 | 13 | 8 | 6 $\bullet$ | ? |
| | ? | ? | 0 | |
| 4 | 5 $\bullet$ | 7 | 8 | ? |
| | 0 | ? | ? | |
| $v_j$ | ? | ? | ? | |

Now $u_1 = 0$, $(1,1) \in B$ and $(1,2) \in B$ force $v_1 = 8$ and $v_2 = 4$.

Then $v_1 = 8$, $(2,1) \in B$ and $(4,1) \in B$ force $u_2 = 5$ and $u_4 = 3$.

Then $u_2 = 5$ and $(3,3) \in B$ force $v_3 = 7$.

Then $v_3 = 7$ and $(3,3) \in B$ force $u_3 = 1$.

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 9$, $q_{2,2} = 8$, $q_{3,1} = 6$, $q_{3,2} = 5$, $q_{4,2} = 6$ and $q_{4,3} = 4$.

The completed tableau is as follows:—

| $c_{i,j} \searrow q_{i,j}$ | 1 | 2 | 3 | $u_i$ |
|---|---|---|---|---|
| 1 | 8 • | 4 • | 16 | 0 |
| | 0 | 0 | 9 | |
| 2 | 3 • | 7 | 2 • | 5 |
| | 0 | 8 | 0 | |
| 3 | 13 | 8 | 6 • | 1 |
| | 6 | 5 | 0 | |
| 4 | 5 • | 7 | 8 | 3 |
| | 0 | 6 | 4 | |
| $v_j$ | 8 | 4 | 7 | |

All numbers $q_{i,j}$ are non-negative for the current feasible basic solution. This solution is therefore optimal. Indeed, arguing as before we find that the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 208 + 9\overline{x}_{1,3} + 8\overline{x}_{2,2} + 6\overline{x}_{3,1} + 5\overline{x}_{3,2} + 6\overline{x}_{4,2} + 4\overline{x}_{4,3}.$$

We conclude that $X$ is an basic optimal solution, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$

## 3.11 Formal Analysis of the Solution of the Transportation Problem

We now describe in general terms the method for solving a transportation problem in which total supply equals total demand.

We suppose that an initial basic feasible solution has been obtained. We apply an iterative method (based on the general Simplex Method for the solution of linear programming problems) that will test a basic feasible solution for optimality and, in the event that the feasible solution is shown not to be optimal, establishes information that (with the exception of certain 'degenerate' cases of the transportation problem) enables one to find a basic feasible solution with lower cost. Iterating this procedure a finite number of times, one should arrive at a basic feasible solution that is optimal for the given transportation problem.

We suppose that the given instance of the Transportation Problem involves $m$ suppliers and $n$ recipients. The required supplies are specified by non-negative real numbers $s_1, s_2, \ldots, s_m$, and the required demands are specified by non-negative real numbers $d_1, d_2, \ldots, d_n$. We further suppose that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. A *feasible solution* is represented by non-negative real numbers $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. A subset $B$ of $I \times J$ is a *basis* for the transportation problem if and only if, given any real numbers $y_1, y_2, \ldots, y_m$ and $z_1, z_2, \ldots, z_n$, where $\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j$, there exist uniquely determined real numbers $\overline{x}_{i,j}$ for $i \in I$ and $j \in J$ such that $\sum_{j=1}^{n} \overline{x}_{i,j} = y_i$ for $i \in I$, $\sum_{i=1}^{m} \overline{x}_{i,j} = z_j$ for $j \in J$, where $\overline{x}_{i,j} = 0$ whenever $(i, j) \notin B$.

A feasible solution $(x_{i,j})$ is said to be a basic feasible solution associated with the basis $B$ if and only if $x_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i, j) \notin B$.

Let $x_{i,j}$ be a non-negative real number for each $i \in I$ and $j \in J$. Suppose that $(x_{i,j})$ is a basic feasible solution to the transportation problem associated with basis $B$, where $B \subset I \times J$.

The cost associated with a feasible solution $(x_{i,j}$ is given by $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$, where the constants $c_{i,j}$ are real numbers for all $i \in I$ and $j \in J$. A feasible solution for a transportation problem is an optimal solution if and only if it minimizes cost amongst all feasible solutions to the problem.

In order to test for optimality of a basic feasible solution $(x_{i,j})$ associated with a basis $B$, we determine real numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ with the property that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. (Proposition 3.10 below guarantees that, given any basis $B$, it is always possible to find the required quantities $u_i$ and $v_j$.) Having calculated these quantities $u_i$ and $v_j$ we determine the values of $q_{i,j}$, where $q_{i,j} = c_{i,j} - v_j + u_i$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $(i, j) \in B$.

We claim that a basic feasible solution $(x_{i,j})$ associated with the basis $B$ is optimal if and only if $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$. This is a consequence of the identity established in the following proposition.

**Proposition 3.8** *Let $x_{i,j}$, $c_{i,j}$ and $q_{i,j}$ be real numbers defined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be real numbers.*

*Suppose that*

$$c_{i,j} = v_j - u_i + q_{i,j}$$

*for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j},$$

*where* $s_i = \sum_{j=1}^{n} x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $d_j = \sum_{i=1}^{m} x_{i,j}$ *for* $j = 1, 2, \ldots, n$.

**Proof** The definitions of the relevant quantities ensure that

$$\begin{aligned}
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} &= \sum_{i=1}^{m} \sum_{j=1}^{n} (v_j - u_i + q_{i,j}) x_{i,j} \\
&= \sum_{j=1}^{n} \left( v_j \sum_{i=1}^{m} x_{i,j} \right) - \sum_{i=1}^{m} \left( u_i \sum_{j=1}^{n} x_{i,j} \right) \\
&\qquad + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j} \\
&= \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j},
\end{aligned}$$

as required. ∎

**Corollary 3.9** *Let* $m$ *and* $n$ *be integers, and let* $I = \{1, 2, \ldots, m\}$ *and* $J = \{1, 2, \ldots, n\}$. *Let* $x_{i,j}$ *and* $c_{i,j}$ *be real numbers defined for all* $i \in I$ *and* $j \in I$, *and let* $u_1, u_2, \ldots, u_m$ *and* $v_1, v_2, \ldots, v_n$ *be real numbers. Suppose that* $c_{i,j} = v_j - u_i$ *for all* $(i, j) \in I \times J$ *for which* $x_{i,j} \neq 0$. *Then*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

*where* $s_i = \sum_{j=1}^{n} x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $d_j = \sum_{i=1}^{m} x_{i,j}$ *for* $j = 1, 2, \ldots, n$.

**Proof** Let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $x_{i,j} \neq 0$. It follows from this that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j} = 0.$$

It then follows from Proposition 3.8 that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \sum_{i=1}^{m}\sum_{j=1}^{n}(v_j - u_i + q_{i,j})x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

as required. ∎

Let $m$ and $n$ be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let the subset $B$ of $I \times J$ be a basis for a transportation problem with $m$ suppliers and $n$ recipients. Let the cost of a feasible solution $(\overline{x}_{i,j})$ be $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j}$. Now $\sum_{j=1}^{n} \overline{x}_{i,j} = s_i$ and $\sum_{i=1}^{m} \overline{x}_{i,j} = d_j$, where the quantities $s_i$ and $d_j$ are determined by the specification of the problem and are the same for all feasible solutions of the problem. Let quantities $u_i$ for $i \in I$ and $v_j$ for $j \in J$ be determined such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$, and let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ for all $(i,j) \in B$.

It follows from Proposition 3.8 that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} = \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j}.$$

Now if the quantities $x_{i,j}$ for $i \in I$ and $j \in J$ constitute a basic feasible solution associated with the basis $B$ then $x_{i,j} = 0$ whenever $(i,j) \notin B$. It follows that $\sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j} = 0$, and therefore

$$\sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i = C,$$

where

$$C = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j}.$$

The cost $\overline{C}$ of the feasible solution $(\overline{x}_{i,j})$ then satisfies the equation

$$\overline{C} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} = C + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j}.$$

If $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$, then the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ is bounded below by the cost of the basic feasible solution $(x_{i,j})$. It follows that, in this case, the basic feasible solution $(x_{i,j})$ is optimal.

Suppose that $(i_0, j_0)$ is an element of $I \times J$ for which $q_{i_0,j_0} < 0$. Then $(i_0, j_0) \notin B$. There is no basis for the transportation problem that includes the set $B \cup \{(i_0, j_0)\}$. A straightforward application of Proposition 3.6 establishes the existence of quantities $y_{i,j}$ for $i \in I$ and $j \in J$ such that $y_{i_0,j_0} = 1$ and $y_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i, j) \notin B \cup \{(i_0, j_0)\}$.

Let the $m \times n$ matrices $X$ and $Y$ be defined so that $(X)_{i,j} = x_{i,j}$ and $(Y)_{i,j} = y_{i,j}$ for all $i \in I$ and $j \in J$. Suppose that $x_{i,j} > 0$ for all $(i, j) \in B$. Then the components of $X$ in the basis positions are strictly positive. It follows that, if $\lambda$ is positive but sufficiently small, then the components of the matrix $X + \lambda Y$ in the basis positions are also strictly positive, and therefore the components of the matrix $X + \lambda Y$ are non-negative for all sufficiently small non-negative values of $\lambda$. There will then exist a maximum value $\lambda_0$ that is an upper bound on the values of $\lambda$ for which all components of the matrix $X + \lambda Y$ are non-negative. It is then a straightforward exercise in linear algebra to verify that $X + \lambda_0 Y$ is another basic feasible solution associated with a basis that includes $(i_0, j_0)$ together with all but one of the elements of the basis $B$.

Moreover the cost of this new basic feasible solution is $C + \lambda_0 q_{i_0,j_0}$, where $C$ is the cost of the basic feasible solution represented by the matrix $X$. Thus if $q_{i_0,j_0} < 0$ then the cost of the new basic feasible solution is lower than that of the basic feasible solution $X$ from which it was derived.

Suppose that, for all basic feasible solutions of the given Transportation problem, the coefficients of the matrix specifying the basic feasible solution are strictly positive at the basis positions. Then a finite number of iterations of the procedure discussed above with result in an basic optimal solution of the given transportation problem. Such problems are said to be *non-degenerate*.

However if it turns out that a basic feasible solution $(x_{i,j})$ associated with a basis $B$ satisfies $x_{i,j} = 0$ for some $(i, j) \in B$, then we are in a *degenerate* case of the transportation problem. The theory of degenerate cases of linear programming problems is discussed in detail in textbooks that discuss the details of linear programming algorithms.

We now establish the proposition that guarantees that, given any basis $B$, there exist quantities $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ such that the costs $c_{i,j}$ associated with the given transportation problem satisfy $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. This result is an essential component of the method described here for testing basic feasible solutions to determine whether or not they are optimal.

**Proposition 3.10** *Let $m$ and $n$ be integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let $B$ be a subset of $I \times J$ that is a basis for the transporta-*

*tion problem with $m$ suppliers and $n$ recipients. For each $(i,j) \in B$ let $c_{i,j}$ be a corresponding real number. Then there exist real numbers $u_i$ for $i \in I$ and $v_j$ for $j \in J$ such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. Moreover if $\overline{u}_i$ and $\overline{v}_j$ are real numbers for $i \in I$ and $j \in J$ that satisfy the equations $c_{i,j} = \overline{v}_j - \overline{u}_i$ for all $(i,j) \in B$, then there exists some real number $k$ such that $\overline{u}_i = u_i + k$ for all $i \in I$ and $\overline{v}_j = v_j + k$ for all $j \in J$.*

**Proof** Let

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in B\}.$$

It follows from the definition of bases for transportation problems that there exist unique $m \times n$ matrices $S_1, S_2, \ldots, S_m$ belonging to $M_B$, where $S_1$ is the zero matrix, and where, for each integer $i$ satisfying $1 < i \leq m$, the matrix $S_k$ has the properties that

$$\sum_{\ell=1}^{n} (S_i)_{k,\ell} = \begin{cases} 1 & \text{if } k = 1, \\ -1 & \text{if } k = i, \\ 0 & \text{if } k \in I \setminus \{1, i\}, \end{cases}$$

and

$$\sum_{k=1}^{m} (S_i)_{k,\ell} = 0 \text{ for all } \ell \in J.$$

Also there exist unique $m \times n$ matrices $T_1, T_2, \ldots, T_m$ belonging to $M_B$ where, for each integer $j$ satisfying $1 \leq j \leq n$, the matrix $T_j$ has the properties that

$$\sum_{j=1}^{n} (T_j)_{k,l} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \in I \setminus \{1\}, \end{cases}$$

and

$$\sum_{i=1}^{m} (T_j)_{k,\ell} = \begin{cases} 1 & \text{if } \ell = j, \\ 0 & \text{if } \ell \in J \setminus \{j\}, \end{cases}$$

Let

$$u_i = \sum_{k=1}^{n} \sum_{\ell=1}^{n} c_{k,\ell} (S_i)_{k,\ell}$$

for $i = 1, 2, \ldots, m$ and

$$v_j = \sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (T_j)_{k,\ell}.$$

for $j = 1, 2, \ldots, n$. We claim the that numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ have the required properties.

Let $X$ be an $m \times n$ matrix belonging to $M_B$, and let

$$y_i = \sum_{j=1}^{n} (X)_{i,j} \quad \text{for all } i \in I$$

and

$$z_j = \sum_{i=1}^{m} (X)_{i,j} \quad \text{for all } j \in J,$$

and let

$$\overline{X} = \sum_{\ell=1}^{n} z_\ell T_\ell - \sum_{k=1}^{m} y_k S_k.$$

Then

$$\sum_{i=1}^{m} (\overline{X})_{i,j} = z_j \quad \text{for all } j \in J.$$

and

$$\sum_{j=1}^{n} (\overline{X})_{i,j} = y_i \quad \text{for all } i \in I \setminus \{1\},$$

Moreover

$$\sum_{j=1}^{n} (\overline{X})_{1,j} = \sum_{\ell=1}^{n} z_\ell - \sum_{k=2}^{m} y_k = y_1,$$

because $\sum\limits_{i=1}^{m} y_i = \sum\limits_{j=1}^{n} z_j$.

But the definition of bases for transportation problems ensures that $X$ is the unique $m \times n$ matrix belonging to $M_B$ with the properties that $\sum\limits_{j=1}^{n} (X)_{i,j} = y_i$ for all $i \in I$ and $\sum\limits_{i=1}^{m} (X)_{i,j} = z_j$ for all $j \in J$. It follows that

$$X = \overline{X} = \sum_{j=1}^{n} z_j T_j - \sum_{i=1}^{m} y_i S_i,$$

and therefore

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (X)_{k,\ell} = \sum_{j=1}^{n} z_j v_j - \sum_{i=1}^{m} y_i u_i.$$

Let $(i, j) \in B$. Then $E^{(i,j)} \in M_B$, where

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

It follows from the result just obtained that

$$c_{i,j} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (E^{(i,j)})_{k,\ell} = v_j - u_i.$$

We have thus shown that, given any basis $B$ for the transportation problem with $m$ suppliers and $n$ recipients, there exist real numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ with the required property that

$$c_{i,j} = v_j - u_i \quad \text{for all } (i, j) \in B..$$

Now let $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m$ and $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n$ be real numbers with the property that

$$c_{i,j} = \overline{v}_j - \overline{u}_i \quad \text{for all } (i, j) \in B..$$

Then $b_j - a_i = 0$ for all $(i, j) \in B$, where $a_i = \overline{u}_i - u_i$ for $i = 1, 2, \ldots, m$ and $b_j = \overline{v}_j - v_j$ for $j = 1, 2, \ldots, n$, and therefore

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(E^{i,j})_{k,\ell} = 0$$

for all $(i, j) \in B$. Now the $m \times n$ matrices $E^{(i,j)}$ for which $(i, j) \in B$ constitute a basis of the vector space $M_B$. It follows that

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(X)_{k,\ell} = 0$$

for all $X \in M_B$.

In particular

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(S_i)_{k,\ell} = 0$$

for $i = 2, 3, \ldots, m$, and

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(T_j)_{k,\ell} = 0$$

for $j = 1, 2, \ldots, n$.

But it follows from the definitions of the matrices $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_n$ that

$$\begin{aligned}
\sum_{k=1}^{m}\sum_{\ell=1}^{n} b_\ell (S_i)_{k,\ell} &= \sum_{\ell=1}^{n}\left(b_\ell \sum_{k=1}^{m}(S_i)_{k,\ell}\right) = 0,\\
\sum_{k=1}^{m}\sum_{\ell=1}^{n} a_k (S_i)_{k,\ell} &= \sum_{k=1}^{m}\left(a_k \sum_{\ell=1}^{n}(S_i)_{k,\ell}\right) = a_1 - a_i
\end{aligned}$$

for $i = 2, 3, \ldots, m$, and

$$\begin{aligned}
\sum_{k=1}^{m}\sum_{\ell=1}^{n} b_\ell (T_j)_{k,\ell} &= \sum_{\ell=1}^{n}\left(b_\ell \sum_{k=1}^{m}(T_j)_{k,\ell}\right) = b_j,\\
\sum_{k=1}^{m}\sum_{\ell=1}^{n} a_k (S_i)_{k,\ell} &= \sum_{k=1}^{m}\left(a_k \sum_{\ell=1}^{n}(S_i)_{k,\ell}\right) = a_1
\end{aligned}$$

for $j = 1, 2, \ldots, n$.

It follows that $a_i - a_1 = 0$ for $i = 2, \ldots, n$ and $b_j - a_1 = 0$ for $j = 1, 2, \ldots, n$. Thus if $k = a_1$ then $\overline{u}_i = u_i + a_i = u_i + k$ for $i = 1, 2, \ldots, m$ and $\overline{v}_j = v_j + b_j = v_j + k$ for $j = 1, 2, \ldots, n$, as required. ∎