MAU34201—Algebraic Topology I
School of Mathematics, Trinity College
Michaelmas Term 2022
Section 7: Winding Numbers of Loops in the Plane

David R. Wilkins

7. Winding Numbers of Loops in the Plane

7.1. Lifts of Angle Functions along Plane Curves

Proposition 7.1

For each real number θ, let Ω_θ be the open set in $\mathbb{R}^2$ that is the complement of the ray

$$\{(t\cos\theta, t\sin\theta) : t \in \mathbb{R} \text{ and } t \leq 0\}.$$

Then, for each real number θ, there exists a corresponding continuous function $\omega_\theta \colon \Omega_\theta \to \mathbb{R}$ characterized by the properties that $\theta - \pi < \omega_\theta(x, y) < \theta + \pi$,

$$x = \sqrt{x^2 + y^2}\cos\omega_\theta(x, y) \quad \text{and} \quad y = \sqrt{x^2 + y^2}\sin\omega_\theta(x, y)$$

for all $(x, y) \in \Omega_\theta$.

Proof

For each real number θ, the open subset Ω_θ of the plane $\mathbb{R}^2$ is the union of the three open sets $V_{\theta-\frac{1}{2}\pi}$, V_θ and $V_{\theta+\frac{1}{2}\pi}$, where, for each real number θ,

$$V_\theta = \{(x, y) \in \mathbb{R}^n : x \cos\theta + y \sin\theta > 0\}.$$

The open set V_θ then consists of those points (x, y) of $\mathbb{R}^2$ distinct from $(0, 0)$ that are such that the displacement vector from the origin to the point in question makes an angle with the vector $(\cos\theta, \sin\theta)$ that is an acute angle.

Let θ be a real number, and let $(x, y) \in V_\theta$. Then $(-\sin\theta, \cos\theta)$ is not a scalar multiple of (x, y), and therefore

$$|y \cos\theta - x \sin\theta| < \sqrt{x^2 + y^2}.$$

Indeed the left hand side of this inequality is the absolute value of the scalar product, in $\mathbb{R}^2$, of the vectors $(-\sin\theta, \cos\theta)$ and (x, y), this scalar product is equal to the length $\sqrt{x^2 + y^2}$ of the vector (x, y) multiplied by the cosine of the angle between the two vectors, and this cosine lies strictly between -1 and 1, because the vectors are not scalar multiples of one another when $(x, y) \in V_\theta$.

Let

$$\hat{\omega}_\theta(x, y) = \theta + \arcsin\left(\frac{y\,\cos\theta - x\,\sin\theta}{\sqrt{x^2 + y^2}}\right)$$

for all $x, y \in V_\theta$, where $\arcsin\colon [-1, 1] \to [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ is the inverse of the restriction of the sine function to the closed interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. The function arcsin is continuous. Also

$$\left|\arcsin\left(\frac{y\,\cos\theta - x\,\sin\theta}{\sqrt{x^2 + y^2}}\right)\right| < \tfrac{1}{2}\pi,$$

because

$$|y\cos\theta - x\,\sin\theta| < \sqrt{x^2 + y^2}$$

as previously noted.

Consequently $\theta - \frac{1}{2}\pi < \hat{\omega}_\theta(x, y) < \theta + \frac{1}{2}\pi$ for all $(x, y) \in V_\theta$.

Now, given any point (x, y) of V_θ, there exists a unique real number ψ satisfying $\theta - \frac{1}{2}\pi < \psi < \theta + \frac{1}{2}\pi$ for which $x = \sqrt{x^2 + y^2} \cos\psi$ and $y = \sqrt{x^2 + y^2} \sin\psi$. Then

$$\begin{aligned} y \cos\theta - x \sin\theta &= \sqrt{x^2 + y^2} (\sin\psi \cos\theta - \cos\psi \sin\theta) \\ &= \sqrt{x^2 + y^2} \sin(\psi - \theta). \end{aligned}$$

It follows that $\hat{\omega}_\theta(x, y) = \psi$, and thus

$$x = \sqrt{x^2 + y^2} \cos\hat{\omega}_\theta(x, y) \quad \text{and} \quad y = \sqrt{x^2 + y^2} \sin\hat{\omega}_\theta(x, y).$$

Now $\hat{\omega}_{\theta+\frac{1}{2}\pi}(x, y) = \hat{\omega}_\theta(x, y)$ for all $(x, y) \in V_{\theta+\frac{1}{2}\pi} \cap V_\theta$. Indeed, for any point (x, y) in the set $V_{\theta+\frac{1}{2}\pi} \cap V_\theta$, the values of $\hat{\omega}_{\theta+\frac{1}{2}\pi}(x, y)$ and $\hat{\omega}_\theta(x, y)$ differ by an amount whose absolute value is less than $\frac{3}{2}\pi$ and are mapped to the same value under both the sine and cosine functions, and must therefore be equal to one another. Similarly $\hat{\omega}_{\theta-\frac{1}{2}\pi}(x, y) = \hat{\omega}_\theta(x, y)$ for all $(x, y) \in V_{\theta-\frac{1}{2}\pi} \cap V_\theta$. There is thus a well-defined continuous function $\omega_\theta \colon \Omega_\theta \to \mathbb{R}$ defined such that

$$\omega_\theta(x, y) = \begin{cases} \hat{\omega}_{\theta-\frac{1}{2}\pi}(x, y) & \text{if } (x, y) \in V_{\theta-\frac{1}{2}\pi}; \\ \hat{\omega}_\theta(x, y) & \text{if } (x, y) \in V_\theta; \\ \hat{\omega}_{\theta+\frac{1}{2}\pi}(x, y) & \text{if } (x, y) \in V_{\theta+\frac{1}{2}\pi}. \end{cases}$$

This function $\omega_\theta \colon \Omega_\theta \to \mathbb{R}$ has all the required properties. ∎

Proposition 7.2

Let $\gamma\colon [0,1] \to \mathbb{R}^2$ *be a path in the plane* $\mathbb{R}^2$ *that does not pass through the origin* $(0,0)$*. Then there exists a continuous function* $\hat{\gamma}\colon [0,1] \to \mathbb{R}$ *with the property that*

$$\gamma(t) = (|\gamma(t)| \cos\hat{\gamma}(t), |\gamma(t)| \sin\hat{\gamma}(t))$$

for all $t \in [0,1]$*.*

Proof

For each real number θ, let Ω_θ be the open set in $\mathbb{R}^2$ that is the complement of the ray

$$\{(t\cos\theta, t\sin\theta) : t \in \mathbb{R} \text{ and } t \leq 0\}.$$

Then, for each real number θ, there exists a corresponding continuous function $\omega_\theta \colon \Omega_\theta \to \mathbb{R}$ characterized by the properties that $\theta - \pi < \omega_\theta(x, y) < \theta + \pi$,

$$x = \sqrt{x^2 + y^2}\,\cos\omega_\theta(x, y) \quad \text{and} \quad y = \sqrt{x^2 + y^2}\,\sin\omega_\theta(x, y)$$

for all $(x, y) \in \Omega_\theta$. (see Proposition 7.1.)

The open sets Ω_θ that result as θ ranges over the set of all real numbers cover the complement $\mathbb{R}^2 \setminus \{(0,0)\}$ of the origin in the plane $\mathbb{R}^2$. The preimages of these open sets under the continuous function γ then cover the closed unit interval $[0,1]$. The closed unit interval is a compact metric space. It follows, on applying the Lebesgue Lemma (Lemma 1.36), that there exists a positive real number δ with the property that, given any subinterval of the closed unit interval whose length is less than δ, there exists some real number θ that is such as to ensure that the subinterval is mapped by the continuous function γ into the open set Ω_θ.

Accordingly let n be a positive integer large enough to ensure that $1/n < \delta$, and let $u_j = j/n$ for all integers j between 0 and n. There then exist real numbers $\theta_1, \theta_2, \ldots, \theta_n$ chosen so that $\gamma([u_{j-1}, u_j]) \subset \Omega_{\theta_j}$ for $j = 1, 2, \ldots, n$. Then, for $j = 1, 2, \ldots, n$, let $\eta_j(t) = \omega_{\theta_j}(\gamma(t))$ for all real numbers t satisfying $u_{j-1} \leq t \leq u_j$. Then

$$\gamma(t) = (|\gamma(t)| \cos \eta_j(t), \, |\gamma(t)| \sin \eta_j(t))$$

for all $t \in [u_{j-1}, u_j]$.

Now
$$\frac{\eta_{j+1}(u_j) - \eta_j(u_j)}{2\pi}$$
is an integer for all integers j between 1 and $n-1$. Accordingly let $m_1 = 0$ and let integers $m_2, m_3, \ldots, m_n$ be successively determined so that
$$\eta_j(u_j) + 2\pi m_j = \eta_{j+1}(u_j) + 2\pi m_{j+1}$$
for $j = 1, 2, \ldots, n-1$. Then let $\hat{\gamma}\colon [0,1] \to \mathbb{R}$ be the function defined so that
$$\hat{\gamma}(t) = \eta_j(t) + 2\pi m_j$$
for $j = 1, 2, \ldots, n$ and for all real numbers t satisfying $u_{j-1} \leq t \leq u_j$. Then the function $\hat{\gamma}\colon [0,1] \to \mathbb{R}$ is the required continuous function with the property that
$$\gamma(t) = (|\gamma(t)| \cos \hat{\gamma}(t),\ |\gamma(t)| \sin \hat{\gamma}(t))$$
for all $t \in [0,1]$. ∎

7.2. The Winding Number of a Loop around a Point

Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a path in the plane $\mathbb{R}^2$. We say that a point $\mathbf{p}$ of $\mathbb{R}^2$ *does not lie on* the path γ if $\mathbf{p}$ does not belong to the range $\gamma([0,1])$ of the continuous function representing the path.

Now let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a loop in the plane $\mathbb{R}^2$ and let $\mathbf{p}$ be a point of $\mathbb{R}^2$ that does not lie on the loop γ. Then there exists a continuous function $\theta\colon [0,1] \to \mathbb{R}$ with the property that

$$\gamma(t) - \mathbf{p} = |\gamma(t) - \mathbf{p}| \,(\cos\theta(t), \sin\theta(t))$$

for all $t \in [0,1]$. (see Proposition 7.2). Moreover

$$\frac{\theta(1) - \theta(0)}{2\pi}$$

is an integer because the function $\gamma\colon [0,1] \to \mathbb{R}^2$ representing the loop has the property that $\gamma(1) = \gamma(0)$.

Now let $\varphi\colon [0,1] \to \mathbb{R}^2$ be any continuous function with the property that

$$\gamma(t) - \mathbf{p} = |\gamma(t) - \mathbf{p}|\,(\cos\varphi(t), \sin\varphi(t))$$

for all $t \in [0,1]$. Then

$$\frac{\theta(t) - \varphi(t)}{2\pi}$$

is a continuous function of the real number t, as t ranges over the closed unit interval $[0,1]$. Moreover the values of this continuous function are all integers. Such a function must be a constant function. It follows that

$$\theta(1) - \varphi(1) = \theta(0) - \varphi(0),$$

and therefore

$$\theta(1) - \theta(0) = \varphi(1) - \varphi(0).$$

We are therefore justified in making the following definition.

Definition

Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a loop in the plane $\mathbb{R}^2$ and let $\mathbf{p}$ be a point of $\mathbb{R}^2$ that does not lie on the loop γ. The *winding number* of the loop γ about the point $\mathbf{p}$ is the unique integer $n(\gamma, \mathbf{p})$, determined by the loop γ and the point $\mathbf{p}$, that is characterized by the property that

$$2\pi n(\gamma, \mathbf{p}) = \theta(1) - \theta(0)$$

for any continuous function $\theta\colon [0,1] \to \mathbb{R}$ with the property that

$$\gamma(t) - \mathbf{p} = |\gamma(t) - \mathbf{p}| \left(\cos\theta(t), \sin\theta(t)\right)$$

for all $t \in [0,1]$.

7.3. The Dog-Walking Lemma

Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a path in the plane $\mathbb{R}^2$. Then the range $\gamma([0,1])$ of the continuous function representing the path is a compact subset of the plane, because the closed unit interval is a compact set, and because continuous functions map compact sets to compact sets. It follows that the range $\gamma([0,1])$ of the function γ is a closed set. We denote this closed set by $[\gamma]$.

Lemma 7.3 (Dog-Walking Lemma)

Let $\gamma_1\colon [0,1] \to \mathbb{R}^2$ and $\gamma_2\colon [0,1] \to \mathbb{R}^2$ be loops in the plane, and let $\mathbf{p}$ be a point of the plane $\mathbb{R}^2$ that does not lie on γ_1. Suppose that $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t) - \mathbf{p}|$ for all $t \in [0,1]$. Then the winding numbers of the loops γ_1 and γ_2 about the point $\mathbf{p}$ are equal to one another.

Proof

Note that the inequality requiring that $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t) - \mathbf{p}|$ for all $t \in [0,1]$ ensures that the point $\mathbf{p}$ does not lie on the path γ_2. Now it follows from Proposition 7.2 that there exist continuous functions $\hat{\gamma}_1 \colon [0,1] \to \mathbb{R}$ and $\hat{\gamma}_2 \colon [0,1] \to \mathbb{R}$ with the properties that

$$\begin{aligned} \gamma_1(t) - \mathbf{p} &= (|\gamma_1(t) - \mathbf{p}| \cos \hat{\gamma}_1(t), \, |\gamma_1(t) - \mathbf{p}| \sin \hat{\gamma}_1(t)), \\ \gamma_2(t) - \mathbf{p} &= (|\gamma_2(t) - \mathbf{p}| \cos \hat{\gamma}_2(t), \, |\gamma_2(t) - \mathbf{p}| \sin \hat{\gamma}_2(t)) \end{aligned}$$

for all $t \in [0,1]$.

The hypotheses of the lemma ensure that, for each real number t satisfying $0 \leq t \leq 1$, the point $\gamma_2(t)$ belongs to the open disk of radius $|\gamma_1(t) - \mathbf{p}|$ centred on the point $\gamma_1(t)$. Now the displacement vector from the point $\mathbf{p}$ to any point of this open disk makes an acute angle with the displacement vector from $\mathbf{p}$ to the centre of the disk. Consequently the angle between the displacement vectors from the point $\mathbf{p}$ to the points $\gamma_1(t)$ and $\gamma_2(t)$ is an acute angle.

This geometric fact can be verified algebraically as follows. The hypotheses of the lemma require that

$$|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t) - \mathbf{p}|$$

for all real numbers t satisfying $0 \leq t \leq 1$. Taking scalar products of vectors in $\mathbb{R}^2$, and applying Schwarz's Inequality we find that

$$\begin{aligned}
&(\gamma_1(t) - \mathbf{p}) \,.\, (\gamma_2(t) - \mathbf{p}) \\
&\qquad = |\gamma_1(t) - \mathbf{p}|^2 - (\gamma_1(t) - \mathbf{p}) \,.\, (\gamma_1(t) - \gamma_2(t)) \\
&\qquad \geq |\gamma_1(t) - \mathbf{p}|^2 - |\gamma_1(t) - \mathbf{p}|\,|\gamma_2(t) - \gamma_1(t)| \\
&\qquad > 0.
\end{aligned}$$

Consequently the angle between the displacement vectors $\gamma_1(t) - \mathbf{p}$ and $\gamma_2(t) - \mathbf{p}$ is indeed an acute angle.

It follows that, for each real number t between 0 and 1, there is a corresponding uniquely-determined integer m_t such that

$$2\pi m_t - \tfrac{1}{2}\pi < \hat{\gamma}_2(t) - \hat{\gamma}_1(t) < 2\pi m_t + \tfrac{1}{2}\pi.$$

The continuity of the functions $\hat{\gamma}_1$ and $\hat{\gamma}_2$ ensures that, given any real number t satisfying $0 \leq t \leq 1$ there exists some positive real number δ such that

$$2\pi m_t - \tfrac{1}{2}\pi < \hat{\gamma}_2(u) - \hat{\gamma}_1(u) < 2\pi m_t + \tfrac{1}{2}\pi.$$

for all real numbers u satisfying the inequalities $0 \leq u \leq 1$ and $t - \delta < u < t + \delta$. It follows that $m_u = m_t$ for all real numbers u satisfying these inequalities. Thus the function that sends each real number t in the closed unit interval to the integer m_t is a continuous integer-valued function on the closed unit interval, and is thus a constant function on that interval.

Consequently there exists some integer m, independent of t, with the property that

$$2\pi m - \tfrac{1}{2}\pi < \hat{\gamma}_2(t) - \hat{\gamma}_1(t) < 2\pi m + \tfrac{1}{2}\pi.$$

for all $t \in [0, 1]$.

Now $\hat{\gamma}_1(1)$ and $\hat{\gamma}_1(0)$ differ by an integer multiple of 2π, because $\gamma_1(1) = \gamma_1(0)$. Similarly $\hat{\gamma}_2(1)$ and $\hat{\gamma}_2(0)$ differ by an integer multiple of 2π, because $\gamma_2(1) = \gamma_2(0)$. It follows that the real numbers $\hat{\gamma}_2(1) - \hat{\gamma}_1(1)$ and $\hat{\gamma}_2(0) - \hat{\gamma}_1(0)$ differ by an integer multiple of 2π. But both these numbers differ from the constant $2\pi m$ by an amount whose absolute value is less than $\frac{1}{2}\pi$. It follows that

$$\hat{\gamma}_2(1) - \hat{\gamma}_1(1) = \hat{\gamma}_2(0) - \hat{\gamma}_1(0).$$

Rearranging this equality we find that

$$\hat{\gamma}_2(1) - \hat{\gamma}_2(0) = \hat{\gamma}_1(1) - \hat{\gamma}_1(0).$$

Now the winding numbers of the loops γ_1 and γ_2 about the point $\mathbf{p}$ respectively are the integers that result on dividing by 2π the quantities $\hat{\gamma}_1(1) - \hat{\gamma}_1(0)$ and $\hat{\gamma}_2(1) - \hat{\gamma}_2(0)$ respectively. Consequently the winding numbers of the loops γ_1 and γ_2 about the point $\mathbf{p}$ are equal to one another, as required. ∎

Remark

Imagine that you are exercising a dog in a park. You walk along a path close to the perimeter of the park that remains at all times at at least 200 metres from an oak tree in the centre of the park. Your dog runs around in your vicinity, but remains at all times within 100 metres of you. In order to leave the park you and your dog return to the point at which you entered the park. The Dog-Walking Lemma then ensures that the number of times that your dog went around the oak tree in the centre of the park is equal to the number of times that you yourself went around that tree.

Lemma 7.4

Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a loop in the plane and let W be the set $\mathbb{R}^2 \setminus [\gamma]$ of all points of the plane that do not lie on the loop γ. Then the function that sends each point of W to the winding number of the loop γ about that point is a continuous function on W.

Proof

Let $\mathbf{p} \in W$. Now the range $[\gamma]$ of the function representing the loop γ is a compact subset of the plane. It is therefore a closed subset of the plane. Consequently there exists some positive real number δ small enough to ensure that $|\gamma(t) - \mathbf{p}| \geq \delta > 0$ for all $t \in [0,1]$. Let $\mathbf{q}$ be a point of the plane $\mathbb{R}^2$ satisfying $|\mathbf{q} - \mathbf{p}| < \delta$, and let $\eta\colon [0,1] \to \mathbb{R}^2$ be the loop in the plane defined such that $\eta(t) = \gamma(t) + \mathbf{p} - \mathbf{q}$ for all $t \in [0,1]$. Then $\gamma(t) - \mathbf{q} = \eta(t) - \mathbf{p}$ for all $t \in [0,1]$, and therefore the winding number of the loop γ about the point $\mathbf{q}$ is equal to the winding number of the loop η about the point $\mathbf{p}$.

Also $|\eta(t) - \gamma(t)| < |\gamma(t) - \mathbf{p}|$ for all $t \in [0, 1]$. It follows from the Dog-Walking Lemma (Lemma 7.3) that the winding number of the loop γ about the point $\mathbf{q}$ is equal to the winding number of the loop η about the point $\mathbf{p}$, and is therefore equal to the winding number of the loop γ about the point $\mathbf{p}$. This shows that the function sending each point $\mathbf{p}$ to the winding number of the loop γ about that point is a continuous function on the set W, as required. $\blacksquare$

Lemma 7.5

Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a loop in the plane, and let R be a positive real number with the property that $|\gamma(t)| < R$ for all $t \in [0,1]$. Then the winding number of γ about a point $\mathbf{p}$ of the plane is zero if that point $\mathbf{p}$ satisfies $|\mathbf{p}| \geq R$.

Proof

Let $\gamma_0\colon [0,1] \to \mathbb{R}^2$ be the constant path defined by $\gamma_0(t) = 0$ for all $[0,1]$. If $|\mathbf{p}| \geq R$ then

$$|\gamma(t) - \gamma_0(t)| = |\gamma(t)| < |\mathbf{p}| = |\gamma_0(t) - \mathbf{p}|.$$

It follows from the Dog-Walking Lemma (Lemma 7.3) that the winding numbers of the loops γ and γ_0 about the point $\mathbf{p}$ are equal to one another, and thus the winding number of the loop γ about the point $\mathbf{p}$ is equal to zero, as required. ∎

7.4. The Homotopy Invariance of Winding Numbers

Proposition 7.6

For each $\tau \in [0,1]$, let $\gamma_\tau\colon [0,1] \to \mathbb{R}^2$ be a loop in the plane. Also let $\mathbf{p}$ be a point of $\mathbb{R}^2$ that does not lie on any of the loops γ_τ. Suppose that the function $H\colon [0,1] \times [0,1] \to \mathbb{R}^2$ is continuous, where $H(t,\tau) = \gamma_\tau(t)$ for all $t \in [0,1]$ and $\tau \in [0,1]$. Then the winding numbers of the loops γ_0 and γ_1 about the point $\mathbf{p}$ are equal to one another.

Proof

The closed unit square $[0,1] \times [0,1]$ is a closed bounded subset of $\mathbb{R}^2$, and is therefore compact. It follows that the continuous function on the closed unit square $[0,1] \times [0,1]$ that sends a point (t,τ) of the square to $|H(t,\tau) - \mathbf{p}|^{-1}$ is a bounded function on $[0,1] \times [0,1]$ (see, for example, Lemma 1.30). Therefore there exists some positive real number ε such that $|H(t,\tau) - \mathbf{p}| \geq \varepsilon > 0$ for all $t \in [0,1]$ and $\tau \in [0,1]$.

Now any continuous vector-valued function on a a closed bounded subset of a Euclidean space is uniformly continuous. (This follows, for example, on combining the results of Theorem 2.9 and Theorem 1.37.) Therefore there exists some positive real number δ such that $|H(t,\tau) - H(t,\tau')| < \varepsilon$ for all $t \in [0,1]$ and for all $\tau, \tau' \in [0,1]$ satisfying $|\tau - \tau'| < \delta$. Let $\tau_0, \tau_1, \ldots, \tau_m$ be real numbers chosen such that $0 = \tau_0 < \tau_1 < \ldots < \tau_m = 1$ and $|\tau_j - \tau_{j-1}| < \delta$ for $j = 1, 2, \ldots, m$.

Then

$$\begin{array}{rcl} |\gamma_{\tau_j}(t) - \gamma_{\tau_{j-1}}(t)| & = & |H(t, \tau_j) - H(t, \tau_{j-1})| \\ & < & \varepsilon \leq |H(t, \tau_{j-1}) - \mathbf{p}| = |\gamma_{\tau_{j-1}}(t) - \mathbf{p}| \end{array}$$

for all $t \in [0, 1]$, and for each integer j between 1 and m. It therefore follows from the Dog-Walking Lemma (Lemma 7.3) that the winding numbers of the loops $\gamma_{\tau_{j-1}}$ and γ_{τ_j} about the point $\mathbf{p}$ are equal to one another for each integer j between 1 and m. Consequently the winding numbers of the loops γ_0 and γ_1 about the point $\mathbf{p}$ are equal to one another, as required. $\blacksquare$

7.5. Winding Numbers of Loops in the Complex Plane

Definition

Let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop in the complex plane, and let w be a complex number that does not lie on the loop γ. The *winding number* of γ about w is defined to be the unique integer $n(\gamma, w)$ with the property that

$$\hat{\gamma}(1) - \hat{\gamma}(0) = 2\pi n(\gamma, w)$$

for any continuous real-valued function $\hat{\gamma}\colon [0,1] \to \mathbb{R}$ that is determined so as to ensure that

$$\gamma(t) - w = |\gamma(t) - w| \left(\cos \hat{\gamma}(t) + i \sin \hat{\gamma}(t)\right)$$

for all $t \in [0,1]$, where $i = \sqrt{-1}$.

Lemma 7.7

Let c be a non-zero complex number, let m be a positive integer, and let P be the polynomial function defined such that $P(z) = cz^m$ for all complex numbers z. Also let R be a positive real number, and let $\sigma\colon [0,1] \to \mathbb{C}$ be the loop in the complex plane defined so that

$$\sigma(t) = R\cos 2\pi t + iR\sin 2\pi t$$

for all $t \in [0,1]$, where $i = \sqrt{-1}$. Then the winding number $n(P \circ \sigma, 0)$ of the loop that sends each real number t satisfying $0 \leq t \leq 1$ to $P(\sigma(t))$ satisfies $n(P \circ \sigma, 0) = m$.

Proof
Let η denote the loop $P \circ \sigma$. Then $\eta(t) = P(\sigma(t))$ for all $t \in [0, 1]$. Also let θ_0 be a real number for which

$$c = |c|(\cos\theta_0 + i\sin\theta_0).$$

Now De Moivre's Theorem ensures that

$$(\cos 2\pi t + i\sin 2\pi t)^m = \cos 2m\pi t + i\sin 2m\pi t$$

for all $t \in [0, 1]$. Consequently

$$\begin{aligned}\eta(t) &= |c|R^m(\cos\theta_0 + i\sin\theta_0)(\cos 2m\pi t + i\sin 2m\pi t)\\ &= |c|R^m(\cos\hat{\eta}(t) + i\sin\hat{\eta}(t))\end{aligned}$$

for all $t \in [0, 1]$, where $\hat{\eta}(t) = \theta_0 + 2m\pi t$. It follows that $2\pi n(\eta, 0) = \hat{\eta}(1) - \hat{\eta}(0) = 2m\pi$. The result follows. ∎

The Dog-Walking Lemma (Lemma 7.3), when reformulated for loops in the complex plane, may be restated as follows.

Lemma 7.8

Let $\gamma_1 \colon [0,1] \to \mathbb{C}$ and $\gamma_2 \colon [0,1] \to \mathbb{C}$ be loops in the complex plane, and let w be a complex number that does not lie on γ_1. Suppose that $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t) - w|$ for all $t \in [0,1]$. Then the winding numbers of the loops γ_1 and γ_2 about the complex number w are equal to one another.

7.6. The Fundamental Theorem of Algebra

Theorem 7.9 (The Fundamental Theorem of Algebra)

Any non-constant polynomial with complex coefficients has at least one root in the complex plane.

Proof

We shall prove that any polynomial with complex coefficients that is non-zero throughout the complex plane must be a constant polynomial.

Let $P(z) = c_0 + c_1 z + \cdots + c_m z^m$, where $c_1, c_2, \ldots, c_m$ are complex numbers and $c_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = c_m z^m$ and $Q(z) = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1}$. Let

$$R = \frac{|c_0| + |c_1| + \cdots + |c_m|}{|c_m|}.$$

If $|z| \geq R$ then $|z| \geq 1$, and therefore

$$\begin{aligned}
\left|\frac{Q(z)}{P_m(z)}\right| &= \frac{1}{|c_m z|}\left|\frac{c_0}{z^{m-1}} + \frac{c_1}{z^{m-2}} + \cdots + c_{m-1}\right| \\
&\leq \frac{1}{|c_m|\,|z|}\left(\left|\frac{c_0}{z^{m-1}}\right| + \left|\frac{c_1}{z^{m-2}}\right| + \cdots + |c_{m-1}|\right) \\
&\leq \frac{1}{|c_m|\,|z|}(|c_0| + |c_1| + \cdots + |c_{m-1}|) < \frac{R}{|z|} \leq 1.
\end{aligned}$$

It follows that $|P(z) - P_m(z)| < |P_m(z)|$ for all complex numbers z satisfying $|z| \geq R$.

For each real number τ satisfying $0 \leq \tau \leq 1$ let $\gamma_\tau \colon [0,1] \to \mathbb{C}$ be the loop in the complex plane defined so that

$$\gamma_\tau(t) = P(R\tau(\cos 2\pi t + i \sin(2\pi t))$$

for all $t \in [0,1]$. Also let $\eta \colon [0,1] \to \mathbb{C}$ be the loop in the complex plane defined so that

$$\eta(t) = P_m(R(\cos 2\pi t + i \sin(2\pi t)) = R^m c_m(\cos 2m\pi t + i \sin(2m\pi t))$$

for all $t \in [0,1]$. Then $n(\eta, 0) = m$ (see Lemma 7.7).

Now $|\gamma_1(t) - \eta(t)| < |\eta(t)|$ for all $t \in [0,1]$. It therefore follows from the Dog-Walking Lemma (Lemma 7.8) that $n(\gamma_1, 0) = n(\eta, 0) = m$.

Now if the polynomial P is everywhere non-zero then the loops γ_0 and γ_1 have the same winding number about zero. (This follows directly on applying Proposition 7.6.) But γ_0 is the constant loop defined so that $\gamma_0(t) = P(0)$ for all $t \in [0, 1]$, and consequently the winding number of the loop γ_0 about zero is equal to zero. It follows that if the polynomial P is everywhere non-zero then the winding number of the loop γ_1 about zero is also equal to zero. But we have already shown that this winding number is equal to the degree m of the polynomial P. It follows that if the polynomial P is everywhere non-zero, then it must be a constant polynomial. Consequently every non-constant polynomial with complex coefficients must have at least one root in the complex plane. This completes the proof of the Fundamental Theorem of Algebra. ∎