Module MAU34201: Algebraic Topology I
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Section 1: Basic Results concerning Topological Spaces

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Contents

1 Results concerning Metric and Topological Spaces

1.1 Topological Spaces

A topological space (X, τ) consists of a set X which is provided with a collection τ of subsets of X, where this collection τ of subsets of X is required to satisfy appropriate axioms. The subsets of the set X that belong to the collection τ are referred to as *open sets*. The axioms which this collection τ is required to satisfy may therefore be expressed in the form of properties that the collection of open sets in any topological space must satisfy.

Definition A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set X are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

Remark If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

1.2 The Topology on a Metric Space

We now discuss metric spaces. Metric spaces are sets provided with distance functions. There are criteria, expressible through the utilization of distance functions, that determine which infinite sequences in a metric space are convergent, and which functions between metric spaces are continuous. However any metric space has a collection of open sets, determined by the distance function, that gives the metric space the structure of a topological space. The concepts of convergence and continuity that arise within the theory of topological spaces are consistent with the criteria that characterize convergence and continuity in metric space contexts using distance functions.

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality*. The elements of a metric space are usually referred to as *points* of that metric space.

An n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with respect to the *Euclidean distance function* d, defined so that

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the Euclidean distance function on $\mathbb{R}^n$.

Definition Let (X, d) be a metric space. Given a point p of X and a positive real number η, the *open ball* $B_X(p, \eta)$ of *radius* η about p in X consisting of all points of the metric space X that lie within a distance η of the given point p.

Thus, given a point p of a topological space X, and given a positive real number η, the open ball $B_X(p, \eta)$ of radius η centred on the point p is defined so that

$$B_X(p, \eta) = \{x \in X : d(x, p) < \eta\}.$$

Definition Let (X, d) be a metric space. A subset V of X is said to be an *open set* (or is said, more specifically, to be *open in* X) if and only if, given any point p of V, there exists some positive real number δ such that the open ball of radius δ centred on the point p is contained within V.

Thus a subset V of a metric space X is open in X if and only if, given any point p of V there exists some positive real number δ for which $B_X(p, \delta) \subset V$.

The empty set is considered to be an open set in any metric space. This can be justified on the grounds that, because the empty set has no points at all, it cannot contain any points for which a corresponding open ball contained in the empty set cannot be found.

Lemma 1.1 *Let X be a metric space with distance function d, and let p be a point of X. Then, for any positive real number η, the open ball $B_X(p, \eta)$ of radius η about the point p is an open set in X.*

Proof Let $q \in B_X(p, \eta)$. We must show that there exists some positive real number δ such that $B_X(q, \delta) \subset B_X(p, \eta)$. Now $d(q, p) < \eta$, and hence $\delta > 0$, where $\delta = \eta - d(q, p)$. Moreover if $x \in B_X(q, \delta)$ then

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = \eta,$$

by the Triangle Inequality, hence $x \in B_X(p, \eta)$. Thus $B_X(q, \delta) \subset B_X(p, \eta)$, showing that $B_X(p, \eta)$ is an open set, as required. ∎

Proposition 1.2 *Let X be a metric space. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open sets;

(ii) the union of any collection of open sets is itself an open set;

(iii) the intersection of any finite collection of open sets is itself an open set.

Proof The empty set is considered to be an open subset of every metric space. For, as the empty set does not contain any points, there can be no point of the empty set that is not the centre of any open ball of positive radius contained in the empty set.

The whole metric space is an open subset of itself because, given any point of the metric space, every open ball of positive radius about that point is contained within the metric space.

Let $\mathcal{C}$ be any collection of open sets in X, and let W denote the union of all the open sets belonging to $\mathcal{C}$. We must show that W is itself an open set. Let $p \in W$. Then $p \in V$ for some open set V belonging to the collection $\mathcal{C}$. Therefore there exists some positive real number δ such that $B_X(p, \delta) \subset V$. But $V \subset W$, and thus $B_X(p, \delta) \subset W$. This shows that W is open. Thus (ii) is satisfied.

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in X, and let $V = V_1 \cap V_2 \cap \cdots \cap V_k$. Let $p \in V$. Now $p \in V_j$ for all j, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(p, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of open sets.) Moreover $B_X(p, \delta) \subset B_X(p, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(p, \delta) \subset V$. This shows that the intersection V of the open sets $V_1, V_2, \ldots, V_k$ is itself open. Thus (iii) is satisfied. ∎

Any metric space may be regarded as a topological space. Indeed let X be a metric space with distance function d. We recall that a subset V of X is an *open set* if and only if, given any point v of V, there exists some positive real number δ such that

$$\{x \in X : d(x, v) < \delta\} \subset V.$$

Proposition 1.2 shows that the topological space axioms are satisfied by the collection of open sets in any metric space. We refer to this collection of open sets as the *topology* generated by the distance function d on X.

1.3 Further Examples of Topological Spaces

Example Given any set X, one can define a topology on X where every subset of X is an open set. This topology is referred to as the *discrete topology* on X.

Example Given any set X, one can define a topology on X in which the only open sets are the empty set $\emptyset$ and the whole set X.

1.4 Closed Sets

Definition Let X be a topological space. A subset F of X is said to be a *closed set* if and only if the complement $X \setminus F$ of F in X is an open set.

We recall that the complement of the union of some collection of subsets of some set X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets. The following result therefore follows reasonably directly from the definition of a topological space.

Proposition 1.3 *Let X be a topological space. Then the collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are closed sets,

(ii) the intersection of any collection of closed sets is itself a closed set,

(iii) the union of any finite collection of closed sets is itself a closed set.

1.5 Neighbourhoods of Points in Topological Spaces

Definition Let X be a topological space, let p be a point of X, and let N be a subset of X which contains the point p. Then N is said to be a *neighbourhood* of the point p if and only if there exists an open set W for which $p \in W$ and $W \subset N$.

Lemma 1.4 *Let X be a topological space. A subset V of X is open in X if and only if V is a neighbourhood of each of its points.*

Proof It follows directly from the definition of neighbourhoods that an open set V is a neighbourhood of any point belonging to V. Conversely, suppose that V is a subset of X which is a neighbourhood of each of its points. Then, given any point p of V, there exists an open set W_p such that $p \in W_p$ and $W_p \subset V$. Thus V is an open set, since it is the union of the open sets W_p as p ranges over all points of V. ∎

Let V be an open set in a topological space X, and let p be a point of X belonging to the open set V. Then V is a neighbourhood of the point p, because an open set is a neighbourhood of all of its points. Thus, given a subset V of X, and given a point p of X, asserting that the set V is both a neighbourhood of the point p and also an open set is equivalent to asserting that the set V is an open set to which the point p belongs. It is therefore appropriate to establish the following definition.

Definition Let X be a topological space, let p be a point of X and let V be a subset of X. Then the set V is said to be an *open neighbourhood* of the point p if V is an open set in X to which the point p belongs.

1.6 Interiors and Closures of Subsets of Topological Spaces

Definition Let X be a topological space and let A be a subset of X. The *interior* A° of A in X is defined to be the union of all open subsets of X that are subsets of A.

It follows directly from this definition that, given a subset A of a topological space X, and given a point p of that topological space, the point p belongs to the interior of A if and only if it belongs to some open subset V of X that is contained in the set A. Thus a point p of the topological space X belongs to the interior A^0 of the set A if and only if there exists some open set V in X for which $p \in V$ and $V \subset A$.

Let X be a topological space and let A be a subset of X. It follows from the definition of a topological space that any union of open subsets of X is itself a open subset of X. It follows that the interior of a subset A of the topological space X is an open set in X, contained in A, that contains any other open set that is also contained in A. The interior of a subset A of the topological space X is thus the largest open set that is contained within the set A.

Lemma 1.5 *Let X be a topological space, let A be a subset of X, and let p be a point of A. Then p belongs to the interior A° of the subset A if and only if this subset A is a neighbourhood of the point p.*

Proof It follows from the definition of interiors that the point p belongs to the interior of A if and only if there exists an open set V such that $p \in V$ and $V \subset A$. It then follows from the definition of neighbourhoods that this is the case if and only if the set A is a neighbourhood of the point p. ∎

Definition Let X be a topological space and let A be a subset of X. The *closure* $\overline{A}$ of A in X is defined to be the intersection of all of the closed subsets of X that contain A.

Let X be a topological space and let A be a subset of X. Then any intersection of closed subsets of X is itself a closed subset of X (see Proposition 1.3). It follows that the closure of a subset A of the topological space X is a closed set in X, containing A, that is contained in any other closed set that also contains A. The closure of a subset A of the topological space X is thus the smallest closed set that contains the set A.

Lemma 1.6 *Let X be a topological space, let A be a subset of X, and let V be an open set. Then the open set V is disjoint from the closure $\overline{A}$ of the set A if and only if it is disjoint from the set A itself. (Thus, for any open subset V of X, $V \cap \overline{A} = \emptyset$ if and only if $V \cap A = \emptyset$.)*

Proof Suppose that $V \cap \overline{A} = \emptyset$. Then $V \cap A = \emptyset$, because A is a subset of $\overline{A}$.

Conversely suppose that $V \cap A = \emptyset$. Then $A \subset X \setminus V$. Now the complement $X \setminus V$ of V is a closed set, and $\overline{A}$ is by definition the intersection of all closed sets that contain the subset A. It follows that $\overline{A} \subset X \setminus V$, and therefore $V \cap \overline{A} = \emptyset$. The result follows. ∎

Proposition 1.7 *Let X be a topological space, and let A be a subset of X. Then the complement $X \setminus \overline{A}$ of the closure $\overline{A}$ of A is equal to the interior $(X \setminus A)^\circ$ of the complement $X \setminus A$ of A. Also the complement $X \setminus A^\circ$ of the interior A° of A is equal to the closure $\overline{X \setminus A}$ of the complement of A. (Thus*

$$X \setminus \overline{A} = (X \setminus A)^\circ \quad \textit{and} \quad X \setminus A^\circ = \overline{X \setminus A}$$

for all subsets A of X.)

Proof Let $p \in X \setminus \overline{A}$, where $\overline{A}$ is the closure of the set A. Then $p \notin \overline{A}$. Now $\overline{A}$ is by definition the intersection of all closed subsets of X that contain the set A. It follows that there must exist some closed set F in X such that $A \subset F$ but $p \notin F$. Let $V = X \setminus F$. Then V is an open set, $p \in V$, and $V \subset X \setminus A$. It follows that $p \in (X \setminus A)^\circ$. We conclude from this that $X \setminus \overline{A} \subset (X \setminus A)^\circ$.

Now let $p \in (X \setminus A)^\circ$. It follows from the definition of interiors that there exists some open set V for which $p \in V$ and $V \subset X \setminus A$. Let $F = X \setminus V$. Then F is a closed set, $A \subset F$, but $p \notin F$. It now follows from the definition of closures that $p \notin \overline{A}$, and therefore $p \in X \setminus \overline{A}$. We conclude from this that $(X \setminus A)^\circ \subset X \setminus \overline{A}$. But we have previously shown that $X \setminus \overline{A} \subset (X \setminus A)^\circ$. These set inclusions together ensure that $(X \setminus A)^\circ = X \setminus \overline{A}$.

It remains to show that $X \setminus A^\circ = \overline{X \setminus A}$. Now let $B = X \setminus A$. It follows from the previous discussion, substituting the set B in place of A, that $(X \setminus B)^\circ = X \setminus \overline{B}$. Thus $A^\circ = X \setminus \overline{B}$. Taking complements, we deduce that $X \setminus A^\circ = \overline{B} = \overline{X \setminus A}$. The required result is therefore established. ∎

Proposition 1.8 *Let X be a topological space, let A be a subset of X and let p be a point of X. Then the point p belongs to the closure of the set A if and only if every neighbourhood of the point p has non-empty intersection with the set A.*

Proof First suppose that $p \notin \overline{A}$. Then $X \setminus \overline{A}$ is a neighbourhood of the point p that is disjoint from the set A.

Conversely suppose that the point p has a neighbourhood N that is disjoint from the set A. The definition of a neighbourhood of a point in a topological space ensures the existence of an open set V for which $p \in V$ and

$V \subset N$. Then $V \cap A = \emptyset$. It follows that $V \cap \overline{A} = \emptyset$, where $\overline{A}$ is the closure of A. (Lemma 1.6). Now $p \in V$. It follows that $p \notin \overline{A}$.

We have now shown that the point p belongs to the complement $X \setminus \overline{A}$ of the closure $\overline{A}$ of the set A if and only if it has a neighbourhood that is disjoint from the set A. It follows the point p belongs to the closure $\overline{A}$ of A if and only if every neighbourhood of the point p has non-empty intersection with the set A. This concludes the proof. ∎

1.7 Relationships involving Preimages of Sets

Definition Let X and Y be sets, let $\varphi\colon X \to Y$ be a function from X to Y, and let B be a subset of the set Y. The *preimage* $\varphi^{-1}(B)$ of B under the function φ is the subset of X consisting of all points p of X for which $\varphi(p) \in B$.

Thus, given a function $\varphi\colon X \to Y$ from a set X to a set Y, and given a subset B of Y, the preimage $\varphi^{-1}(B)$ of the set B under the function φ is defined so that

$$\varphi^{-1}(B) = \{p \in X : \varphi(p) \in B\}.$$

We establish some basic results concerning preimages of sets under functions between sets.

Lemma 1.9 *Let $\varphi\colon X \to Y$ be a function from a set X to a set Y, and let $\mathcal{C}$ be a collection of subsets of Y. Then the union of the preimages, under φ, of the sets in the collection $\mathcal{C}$ is the preimage of the union of those sets.*

Proof Let G denote the union of the subsets of the set Y that belong to the collection $\mathcal{C}$, and let F denote the union of the preimages, under φ, of the sets belonging to the collection $\mathcal{C}$. Then, for any point p of the set X,

$$\begin{aligned} & p \in \varphi^{-1}(G) \\ \iff & \varphi(p) \in G \\ \iff & \text{there exists } B \in \mathcal{C} \text{ for which } \varphi(p) \in B \\ \iff & \text{there exists } B \in \mathcal{C} \text{ for which } p \in \varphi^{-1}(B) \\ \iff & p \in F. \end{aligned}$$

It follows that $\varphi^{-1}(G) = F$. Thus the preimage of the union of the sets belonging to the collection $\mathcal{C}$ is the union of the preimages of those sets, as required. ∎

Lemma 1.10 *Let $\varphi\colon X \to Y$ be a function from a set X to a set Y, and let $\mathcal{C}$ be a collection of subsets of Y. Then the intersection of the preimages, under φ, of the sets in the collection $\mathcal{C}$ is the preimage of the intersection of those sets.*

Proof Let K denote the intersection of the subsets of the set Y that belong to the collection $\mathcal{C}$, and let H denote the intersection of the preimages, under φ, of the sets belonging to the collection $\mathcal{C}$. Then, for any point p of the set X,

$$\begin{aligned} & p \in \varphi^{-1}(K) \\ \iff & \varphi(p) \in K \\ \iff & \varphi(p) \in B \text{ for all } B \in \mathcal{C} \\ \iff & p \in \varphi^{-1}(B) \text{ for all } B \in \mathcal{C} \\ \iff & p \in H. \end{aligned}$$

It follows that $\varphi^{-1}(K) = H$. Thus the preimage of the intersection of the sets belonging to the collection $\mathcal{C}$ is the intersection of the preimages of those sets, as required. ∎

Lemma 1.11 *Let $\varphi\colon X \to Y$ be a function from a set X to a set Y, and let B be a subset of Y. Then $X \setminus \varphi^{-1}(B) = \varphi^{-1}(Y \setminus B)$*

Proof Let p be a point of the domain X of the function. Then

$$\begin{aligned} & p \in X \setminus \varphi^{-1}(B) \\ \iff & p \notin \varphi^{-1}(B) \\ \iff & \varphi(p) \notin B \\ \iff & \varphi(p) \in Y \setminus B \\ \iff & p \in \varphi^{-1}(Y \setminus B). \end{aligned}$$

It follows from this that $X \setminus \varphi^{-1}(B) = \varphi^{-1}(Y \setminus B)$, as required. ∎

1.8 Induced Topologies and Subspace Topologies

Lemma 1.12 *Let X be a set, let Y be a topological space, and let $\varphi\colon X \to Y$ be a function from the set X to the topological space Y. Let τ be the collection consisting of those subsets of X that are preimages, under φ, of open sets in Y. Then the collection τ of subsets of X satisfies the topological space axioms, and thus the set X, with the collection τ of open sets, is a topological space.*

Proof The empty set is the preimage of the empty set, and the whole set X is the preimage, under φ, of the whole of Y. Moreover the empty set and the whole of the topological space Y are open subsets of Y. It follows that the empty set and the whole set X belong to the collection τ of subsets of X.

Suppose that we are given a collection $\mathcal{B}$ of members of the collection τ. Then there is a corresponding collection $\mathcal{C}$ of open sets in the topological space Y determined so that the members of the collection $\mathcal{B}$ of subsets of X are preimages, under the function φ, of corresponding members of the collection $\mathcal{C}$. It follows that the union of the members of the collection $\mathcal{B}$ is the union of the preimages of the members of the collection $\mathcal{C}$, and is thus the preimage of the union of the members of the collection $\mathcal{C}$ (Lemma 1.9); it is accordingly the preimage of a union of open sets in the topological space Y, and is therefore the preimage of an open set in the topological space Y. It follows that the union of the members of the collection $\mathcal{B}$ belongs to the collection τ of subsets of X.

Also, in cases where the collection $\mathcal{B}$ is finite, the intersection of the members of the collection $\mathcal{B}$ is the intersection of the preimages of the members of the collection $\mathcal{C}$, and is thus the preimage of the intersection of the members of the collection $\mathcal{C}$ (Lemma 1.10); it is accordingly the preimage of a finite intersection of open sets in the topological space Y, and is therefore the preimage of an open set in the topological space Y. It follows that, in cases where the collection $\mathcal{B}$ is finite, the intersection of the members of the collection $\mathcal{B}$ belongs to the collection τ of subsets of X.

These results establish that the collection τ of subsets of the set X does indeed satisfy the topological space axioms, and thus the set X, with the collection τ of open sets, is a topological space. ∎

Definition Let X be a set, let Y be a topological space, and let $\varphi\colon X \to Y$ be a function from the set X to the topological space Y. The *induced topology* on X determined by the function φ is that topology whose collection τ of open sets consists of those subsets of X that are preimages, under φ, of open sets in Y.

An important special case of induced topologies arises when the functions inducing the topologies are inclusion maps. The induced topologies determined by inclusion maps are *subspace topologies*.

Definition Let X be a topological space with topology τ, and let A be a subset of the set X. The *subspace topology* on A is the topology τ_A that consists of those subsets of A that are the intersections of A with open sets in X.

Let $i\colon A \hookrightarrow X$ be the inclusion map embedding the subset A in the topological space X. Then $A \cap B = i^{-1}(B)$ for all subsets B of X. Lemma 1.12 therefore ensures that the subspace topology is indeed a topology on the set A: it is in fact the topology on the subset A induced by the inclusion map $i\colon A \hookrightarrow X$.

Lemma 1.13 *Let X be a topological space, let A be a subset of X, and let B be a subset of A. Then B is closed in A (relative to the subspace topology on A) if and only if $B = A \cap F$ for some closed subset F of X.*

Proof Suppose that $B = A \cap F$ for some closed subset F of X. Let $V = X \setminus F$. Then V is an open set in X, and

$$A \setminus B = A \setminus (A \cap F) = A \cap (X \setminus F) = A \cap V.$$

Moreover the definition of the subpace topology on A ensures that $A \cap V$ is open in A. Thus the complement $A \setminus B$ of B in A is open in A, and therefore the subset B of A is itself closed in A.

Conversely suppose that B is closed in A. Then $A \setminus B$ is open in the subspace topology on A, and therefore there exists some open set V in X such that $A \setminus B = A \cap V$. Let $F = X \setminus V$. Then F is closed in X, and

$$A \cap F = A \cap (X \setminus V) = A \setminus (A \cap V) = A \setminus (A \setminus B) = B.$$

The result follows. ∎

Lemma 1.14 *Let X be a topological space, let V be an open set in X, and let W be a subset of V. Then W is open in V if and only if W is open in X.*

Proof If W is open in X then $W = V \cap W$ and therefore W is open in V.

Conversely suppose that the set W is open in V. It then follows from the definition of subspace topologies that $W = V \cap E$ for some open set E in X. But then W is an intersection of two open sets, and is thus itself open in X. ∎

Lemma 1.15 *Let X be a topological space, let F be a closed set in X, and let G be a subset of F. Then G is closed in F if and only if G is closed in X.*

Proof If G is closed in X then $G = F \cap G$ and therefore G is closed in F.

Conversely suppose that the set G is closed in F. It then follows from Lemma 1.13 that $G = F \cap H$ for some closed set H in X. But then G is an intersection of two closed sets, and is thus itself closed in X (see Proposition 1.3). ∎

1.9 Hausdorff Spaces

Definition A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if p and q are distinct points of X then there exist open sets U and V in X such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$.

Lemma 1.16 *Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).*

Proof Let A be a subset of a Hausdorff space X and let p and q be distinct points of A. Then there exist open sets U and V in X such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$. Let $M = A \cap U$ and $N = A \cap V$. Then M and N are subsets of A that are open in the subspace topology on A. Moreover $p \in M$, $q \in N$ and $M \cap N = \emptyset$. The result follows. ∎

Lemma 1.17 *All metric spaces are Hausdorff spaces.*

Proof Let X be a metric space with distance function d, and let p and q be points of X, where $p \neq q$. Let $\varepsilon = \frac{1}{2}d(p,q)$. Then the open balls $B_X(p,\varepsilon)$ and $B_X(q,\varepsilon)$ of radius ε centred on the points p and q are open sets (see Lemma 1.1). If $B_X(p,\varepsilon) \cap B_X(q,\varepsilon)$ were non-empty then there would exist $z \in X$ satisfying $d(p,z) < \varepsilon$ and $d(z,q) < \varepsilon$. But this is impossible, since it would then follow from the Triangle Inequality that $d(p,q) < 2\varepsilon$, contrary to the choice of ε. Thus $p \in B_X(p,\varepsilon)$, $q \in B_X(q,\varepsilon)$ and $B_X(p,\varepsilon) \cap B_X(q,\varepsilon) = \emptyset$. This shows that the metric space X is a Hausdorff space. ∎

We now give an example of a topological space which is not a Hausdorff space.

Example Let X be an infinite set. The *cofinite topology* on X is defined as follows: a subset U of X is open (with respect to the cofinite topology) if and only if either $U = \emptyset$ or else $X \setminus U$ is finite. It is a straightforward exercise to verify that the topological space axioms are satisfied, so that the set X is a topological space with respect to this cofinite topology. Now the intersection of any two non-empty open sets in this topology is always non-empty. (Indeed if U and V are non-empty open sets then $U = X \setminus F$ and $V = X \setminus G$, where F and G are finite subsets of X. But then $U \cap V = X \setminus (F \cup G)$, which is non-empty, since $F \cup G$ is finite and X is infinite.) It follows immediately from this that an infinite set X is not a Hausdorff space with respect to the cofinite topology on X.

1.10 Continuous Maps between Topological Spaces

Definition A function $\varphi\colon X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if the preimage $\varphi^{-1}(V)$ of every open subset V of Y is an open set in X.

A continuous function from X to Y is often referred to as a *map* from X to Y.

Lemma 1.18 *Let X, Y and Z be topological spaces, and let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be continuous functions. Then the composition $\psi \circ \varphi\colon X \to Z$ of the functions φ and ψ is continuous.*

Proof Let V be an open set in Z. Then $\psi^{-1}(V)$ is open in Y (because ψ is continuous), and then $\varphi^{-1}(\psi^{-1}(V))$ is open in X (because φ is continuous). But $\varphi^{-1}(\psi^{-1}(V)) = (\psi \circ \varphi)^{-1}(V)$. Thus the composition function $\psi \circ \varphi$ is continuous. ∎

Lemma 1.19 *Let X and Y be topological spaces, and let $\varphi\colon X \to Y$ be a function from X to Y. The function φ is continuous if and only if $\varphi^{-1}(G)$ is closed in X for every closed subset G of Y.*

Proof Suppose first that the function $\varphi\colon X \to Y$ is continuous and that G is a closed set in Y. Then the complement $Y \setminus G$ of G in Y is an open set in Y. It follows from the continuity of the function φ that the preimage $\varphi^{-1}(Y \setminus G)$ of the complement $Y \setminus G$ of G is an open set in X. But $\varphi^{-1}(Y \setminus G) = X \setminus \varphi^{-1}(G)$. We conclude therefore that the complement $X \setminus \varphi^{-1}(G)$ of the preimage $\varphi^{-1}(G)$ of G is an open set in X, and therefore the preimage $\varphi^{-1}(G)$ of the set G is a closed set in X.

Conversely suppose that $\varphi\colon X \to Y$ is some function from X to Y with the property that the preimage $\varphi^{-1}(G)$ of every closed subset of Y is a closed set in X. We must show that the function φ is continuous. Let V be an open set in Y. Then $Y \setminus V$ is a closed set in Y, and therefore its preimage $\varphi^{-1}(Y \setminus V)$ is a closed set in X. But $\varphi^{-1}(Y \setminus V) = X \setminus \varphi^{-1}(V)$. It follows that the preimage $\varphi^{-1}(V)$ of the open set V is the complement of a closed set, and is therefore an open set in the topological space X. We have thus shown that the preimage of every open subset of Y is open in X. It follows that the function $\varphi\colon X \to Y$ is continuous, as required. ∎

1.11 Pointwise Continuity

Definition Let X and Y be topological spaces, let $\varphi\colon X \to Y$ be a function from X to Y and let p be a point of X. The function φ is said to be *continuous* *at p* if, given any open neighbourhood V in Y of the point $\varphi(p)$, the preimage $\varphi^{-1}(V)$ of V under the function φ is a neighbourhood in X of the point p.

Lemma 1.20 *Let X and Y be topological spaces, let $\varphi\colon X \to Y$ be a function from X to Y and let p be a point of X. The function φ is continuous at the point p if and only if, for all neighbourhoods N in Y of $\varphi(p)$, the preimage $\varphi^{-1}(N)$ of N is a neighbourhood in X of the point p.*

Proof If the preimage of any neighbourhood in Y of $\varphi(p)$ under the function φ is a neighbourhood in X of the point p, then, in particular, the preimage of any open neighbourhood of $\varphi(p)$ must be a neighbourhood of the point p itself, and thus the function φ is continuous at the point p.

Conversely suppose that the function φ is continuous at the point p. Let N be a neighbourhood of the point $\varphi(p)$ in Y. The definition of a neighbourhood of a point in a topological space ensures the existence of an open set V for which $p \in V$ and $V \subset N$. The continuity of the function φ at p then ensures that the preimage $\varphi^{-1}(V)$ under φ is a neighbourhood of the point p. Now $\varphi^{-1}(V) \subset \varphi^{-1}(N)$, and any superset of a neighbourhood of p is itself a neighbourhood of p. We deduce therefore that the preimage $\varphi^{-1}(N)$ under φ of the neighbourhood N of $\varphi(p)$ must be a neighbourhood of the point p, as required. ∎

Proposition 1.21 *Let X and Y be topological spaces and let $\varphi\colon X \to Y$ be a function from X to Y. Then the function φ is continuous on X if and only if it is continuous at each point of X.*

Proof Suppose that $\varphi\colon X \to Y$ is continuous on X. Let p be a point of X and let V be an open neighbourhood in Y of the point $\varphi(p)$. The continuity of φ ensures that $\varphi^{-1}(V)$ is open in X. Now an open set is a neighbourhood of each of its points. We conclude therefore that the preimage $\varphi^{-1}(V)$ of the open set V is a neighbourhood of the point p, and therefore the function $\varphi\colon X \to Y$ is continuous at the point p. Thus a continuous function is continuous at each point of its domain.

Conversely suppose that $\varphi\colon X \to Y$ is continuous at each point of X. Let V be an open set in Y. Then, the preimage of this open set V is a neighbourhood of each of its points, and is therefore open in X (see Lemma 1.4). Thus the preimage of every open set V in Y is an open set in X, and therefore the function $\varphi\colon X \to Y$ is continuous on X, as required. ∎

Lemma 1.22 *Let X and Y be topological spaces, let $\varphi\colon X \to Y$ be a function from X to Y and let p be a point of X. Then $\varphi\colon X \to Y$ is continuous at p if and only if, given any neighbourhood N of $\varphi(p)$, there exists a neighbourhood M of p for which $\varphi(M) \subset N$.*

Proof Let N be a neighbourhood of $\varphi(p)$ in Y. Suppose that there exists a neighbourhood M of p in X for which $\varphi(M) \subset N$. The definition of neighbourhoods of points in topological spaces then ensures that there exists an open set W in X for which $p \in W$ and $W \subset M$. Then $\varphi(W) \subset N$ and therefore $W \subset \varphi^{-1}(N)$. It follows that $\varphi^{-1}(N)$ is a neighbourhood of p in X, and thus the function φ is continuous at p.

Conversely suppose that the function φ is continuous at p. Let N be a neighbourhood of $\varphi(p)$ in Y, and let $M = \varphi^{-1}(N)$. Then M is a neighbourhood of p in X, because the function φ is continuous at p, and $\varphi(M) \subset N$. The result follows. ∎

Lemma 1.23 *Let X, Y and Z be topological spaces, let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be functions, and let p be a point of X. Suppose that $\varphi\colon X \to Y$ is continuous at p and that $\psi\colon Y \to Z$ is continuous at $\varphi(p)$. Then the composition $\psi \circ \varphi\colon X \to Z$ of the functions φ and ψ is continuous at p.*

Proof Let N be a neighbourhood of $\psi(\varphi(p))$ in Z. Then $\psi^{-1}(N)$ is a neighbourhood of $\varphi(p)$ in Y, because ψ is continuous at $\varphi(p)$. But then $\varphi^{-1}(\psi^{-1}(N))$ is a neighbourhood of p in X, because φ is continuous at p. But $\varphi^{-1}(\psi^{-1}(N)) = (\psi \circ \varphi)^{-1}(N)$. Thus the composition function $\psi \circ \varphi$ is continuous at p. ∎

Proposition 1.24 *Let X and Y be topological spaces and let $\varphi\colon X \to Y$ be a function from X to Y. Then $\varphi\colon X \to Y$ is continuous if and only if, given any point p of X, there exists some open set W in X such that $p \in W$ and the restriction $\varphi|W\colon W \to Y$ of the function φ to W is continuous on W.*

Proof Suppose that $\varphi\colon X \to Y$ is continuous. Let W be an open set in X, and let V be an open set in Y. Then the preimage $\varphi^{-1}(V)$ of V is open in X. Now $(\varphi|W)^{-1}(V) = \varphi^{-1}(V) \cap W$. It follows that $(\varphi|W)^{-1}(V)$ is open with respect to the subspace topology on W. Consequently the restriction $\varphi|W$ of the function φ to W is continuous on W.

We now establish the converse result. Let V be an open set in Y, and let $p \in \varphi^{-1}(V)$. Suppose that the restriction $\varphi|W\colon W \to Y$ of φ to some open neighbourhood W of the point p is continuous. Then the preimage $(\varphi|W)^{-1}(V)$ of V under the restriction function $\varphi|W$ is open with respect to the subspace topology on W. Moreover $(\varphi^{-1}|W)(V) = \varphi^{-1}(V) \cap W$. It follows from the definition of subspace topologies that there exists an open set E in X for which $\varphi^{-1}(V) \cap W = E \cap W$. Now $E \cap W$ is open in X, because the sets E and W are both open in X. Also $p \in E \cap W$ and $E \cap W \subset \varphi^{-1}(V)$. It follows that $\varphi^{-1}(V)$ is a neighbourhood of p in X. We conclude from this that φ is continuous at the point p. Thus the function φ is thus continuous at each point p of its domain. Such a function is continuous on its domain (Proposition 1.21). Accordingly the function $\varphi\colon X \to Y$ is continuous, as required. ∎

1.12 Homeomorphisms

Definition Let X and Y be topological spaces. A function $\varphi\colon X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $\varphi\colon X \to Y$ is both injective and surjective (so that the function $\varphi\colon X \to Y$ has a well-defined inverse $\varphi^{-1}\colon Y \to X$),
- the function $\varphi\colon X \to Y$ and its inverse $\varphi^{-1}\colon Y \to X$ are both continuous.

Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $\varphi\colon X \to Y$ from X to Y.

If $\varphi\colon X \to Y$ is a homeomorphism between topological spaces X and Y then φ induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being essentially identical as topological spaces.

1.13 The Pasting Lemma

We now show that, if a topological space X is the union of a finite collection of closed sets, and if a function from X to some topological space is continuous on each of these closed sets, then that function is continuous on X. The names *Pasting Lemma* and *Gluing Lemma* are both used to refer to this result.

Lemma 1.25 (Pasting Lemma) *Let X and Y be topological spaces, let $\varphi\colon X \to Y$ be a function from X to Y, and let $X = A_1 \cup A_2 \cup \cdots \cup A_k$, where $A_1, A_2, \ldots, A_k$ are closed sets in X. Suppose that the restriction of φ to the closed set A_i is continuous for $i = 1, 2, \ldots, k$. Then $\varphi\colon X \to Y$ is continuous.*

Proof Let p be a point of X, and let N be a neighbourhood of $\varphi(p)$. The continuity of the restriction of φ to each closed set A_i ensures the existence of open sets W_i for $i = 1, 2, \ldots, k$ such that $W_i \cap A_i = \emptyset$ whenever $p \notin A_i$ and $\varphi(W_i \cap A_i) \subset N$ whenever $p \in A_i$. Let

$$W = W_1 \cap W_2 \cap \cdots \cap W_k$$

Then W is an open set in X, and $p \in W$. Moreover given any point q of W, there exists some integer i between 1 and k for which $q \in A_i$ and $p \in A_i$. Indeed the point q must belong to at least one of the sets $A_1, A_2, \ldots, A_k$. But the set W, being contained in each set W_i, is disjoint from those sets A_i to which the point p does not belong. Therefore the point q must belong to some set A_i to which the point p also belongs. But then $q \in W_i \cap A_i$, and therefore $\varphi(q) \in N$. We conclude from this that the function φ is continuous at each point p of X. It follows that the function φ is continuous on X (see Proposition 1.21). ■

Alternative Proof A function $\varphi\colon X \to Y$ is continuous if and only if the preimage $\varphi^{-1}(G)$ of every closed subset G of the codomain Y is closed in the domain X (Lemma 1.19). Let G be an closed set in Y. Then $\varphi^{-1}(G) \cap A_i$ is closed in the subspace topology on A_i for $i = 1, 2, \ldots, k$, because the restriction of φ to A_i is continuous for each i. But the set A_i is closed in X, and therefore a subset of A_i is closed in A_i if and only if it is closed in X (see Lemma 1.15). Consequently $\varphi^{-1}(G) \cap A_i$ is closed in X for $i = 1, 2, \ldots, k$. Now $\varphi^{-1}(G)$ is the union of the sets $\varphi^{-1}(G) \cap A_i$ for $i = 1, 2, \ldots, k$. It follows that $\varphi^{-1}(G)$, being a finite union of closed sets, is itself closed in X. It now follows from Lemma 1.19 that $\varphi\colon X \to Y$ is continuous. ■

Example Let Y be a topological space, and let $\alpha\colon [0,1] \to Y$ and $\beta\colon [0,1] \to Y$ be continuous functions defined on the interval $[0,1]$, where $\alpha(1) = \beta(0)$. Let $\gamma\colon [0,1] \to Y$ be defined by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Now $\gamma|[0, \frac{1}{2}] = \alpha \circ \rho$ where $\rho\colon [0, \frac{1}{2}] \to [0,1]$ is the continuous function defined by $\rho(t) = 2t$ for all $t \in [0, \frac{1}{2}]$. Thus $\gamma|[0, \frac{1}{2}]$ is continuous, being a composition of two continuous functions. Similarly $\gamma|[\frac{1}{2}, 1]$ is continuous. The subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed in $[0,1]$, and $[0,1]$ is the union of these two subintervals. One applying the Pasting Lemma (Lemma 1.25), we conclude that $\gamma\colon [0,1] \to Y$ is continuous.

Example Let X be the surface of a closed cube in $\mathbb{R}^3$ and let $\varphi\colon X \to Y$ be a function mapping X into a topological space Y. The topological space X

is the union of the six square faces of the cube, and each of these faces is a closed subset of X. The Pasting Lemma Lemma 1.25 ensures that the function φ is continuous if and only if its restrictions to each of the six faces of the cube is continuous on that face.

We now present a couple of examples to show that the conclusions of the Pasting Lemma (Lemma 1.25) do not follow when the conditions stated in that lemma are relaxed.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined so that

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

and let $A_1 = \{x \in \mathbb{R} : x \leq 0\}$ and $A_2 = \{x \in \mathbb{R} : x > 0\}$. The restriction of the function f to each of the subsets A_1 and A_2 of $\mathbb{R}$ is continuous on that subset, but the function f itself is not continuous on $\mathbb{R}$. This does not contradict the Pasting Lemma because the subset A_2 of $\mathbb{R}$ is not closed in $\mathbb{R}$.

Example Let

$$X = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{Z} \text{ and } n > 0 \right\},$$

and let $f\colon X \to \mathbb{R}$ be defined so that $f(0) = 0$ and $f(1/n) = n$ for all positive integers n. For each $x \in X$, the set $\{x\}$ is a closed subset of X, and the restriction of f to each of these one-point subsets is continuous on that subset. But the function f itself is not continuous on X. This does not contradict the Pasting Lemma because the number of these one-point closed subsets of X is infinite.

1.14 Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X is said to *cover* the set A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is a collection of open sets in X that covers X.

If $\mathcal{C}$ and $\mathcal{D}$ are open covers of some topological space X then $\mathcal{D}$ is said to be a *subcover* of $\mathcal{C}$ if and only if every open set belonging to $\mathcal{D}$ also belongs to $\mathcal{C}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 1.26 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{C}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{C}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.*

Proof Given a collection $\mathcal{D}$ of subsets of A, where the members of this collection are open with respect to the subspace topology on A, there exists a corresponding collection $\mathcal{C}$ of open sets in X whose intersections with the set A are the members of the collection $\mathcal{D}$. It follows that the open cover $\mathcal{D}$ of the set A has a finite subcover if and only if some finite subcollection of the collection $\mathcal{C}$ of open sets in X covers A. The result follows. ∎

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem.* The proof of this theorem uses the *Least Upper Bound Principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 1.27 (Heine-Borel Theorem in One Dimension) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{C}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the subset of $[a, b]$ defined so that a real number τ in the interval $[a, b]$ belongs to the set S if and only if the closed bounded interval $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{C}$. Also let $u = \sup S$. Now $u \in W$ for some open set W belonging to $\mathcal{C}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some positive real number δ such that $(u-\delta, u+\delta) \subset W$. Moreover $u-\delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > u - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{C}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < u + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (u - \delta, u + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $u \in S$, and moreover $u = b$, since otherwise u would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{C}$, as required. ∎

Lemma 1.28 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{C}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{C}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{C}$ that belong to this finite subcover. It follows (applying Lemma 1.26) that A is compact, as required. ∎

Lemma 1.29 *Let $\varphi\colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $\varphi(A)$ is a compact subset of Y.*

Proof Let $\mathcal{C}$ be a collection of open sets in Y which covers $\varphi(A)$. Then A is covered by the collection of all open sets of the form $\varphi^{-1}(V)$ for some $V \in \mathcal{C}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{C}$ such that

$$A \subset \varphi^{-1}(V_1) \cup \varphi^{-1}(V_2) \cup \cdots \cup \varphi^{-1}(V_k).$$

But then $\varphi(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $\varphi(A)$ is compact. ∎

Lemma 1.30 *Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof For each positive integer j, let $V_j = \{p \in X : -j < f(p) < j\}$. Then, for each positive integer j, the subset V_j of X is the preimage under the continuous map f of the open interval $(-j, j)$, and moreover $(-j, j)$ is open in $\mathbb{R}$. It follows from the continuity of f that V_j is an open set in X for all positive integers j. Moreover the compact topological space X is covered by these open sets. It follows from the compactness of X that there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X = V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let N be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $-N < f(p) < N$ for all $p \in X$. The result follows. ∎

Proposition 1.31 *Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(p) \leq f(v)$ for all $p \in X$.*

Proof The function $f\colon X \to \mathbb{R}$ is bounded on X (Lemma 1.30). Let $m = \inf\{f(p) : p \in X\}$ and $M = \sup\{f(p) : p \in X\}$. For each positive integer j let $V_j = \{p \in X : f(p) < M - 1/j\}$. Then the set V_j is an open set in X, being the preimage of an open interval in $\mathbb{R}$ under the continuous map f. If $j_1, j_2, \ldots, j_k$ are positive integers then

$$V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k} = V_N$$

where N is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Moreover V_N is a proper subset of X, because $M - 1/N$ is not an upper bound on the values of the function f on X. It follows that X cannot covered by any finite collection of sets from the collection $(V_j : j \in \mathbb{N})$. It then follows from the compactness of X that $(V_j : j \in \mathbb{N})$ is not an open cover of X, and therefore there exists $v \in X$ for which $f(v) = M$. Applying this argument with f replaced by $-f$, we conclude that there also exists $u \in X$ for which $f(u) = m$. Then $f(u) \leq f(p) \leq f(v)$ for all $p \in X$, as required. ∎

1.15 Compact Subsets of Hausdorff Spaces

Proposition 1.32 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Let p be a point of $X \setminus K$. Then there exist open sets V and W in X such that $p \in V$, $K \subset W$ and $V \cap W = \emptyset$.*

Proof For each point $q \in K$ there exist open sets $V_{p,q}$ and $W_{p,q}$ such that $p \in V_{p,q}$, $q \in W_{p,q}$ and $V_{p,q} \cap W_{p,q} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{q_1, q_2, \ldots, q_r\}$ of points of K such that K is contained in $W_{p,q_1} \cup W_{p,q_2} \cup \cdots \cup W_{p,q_r}$, since K is compact. Define

$$V = V_{p,q_1} \cap V_{p,q_2} \cap \cdots \cap V_{p,q_r}, \qquad W = W_{p,q_1} \cup W_{p,q_2} \cup \cdots \cup W_{p,q_r}.$$

Then V and W are open sets, $p \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required. ∎

Corollary 1.33 *A compact subset of a Hausdorff topological space is closed.*

Proof Let K be a compact subset of a Hausdorff topological space X. It follows immediately from Proposition 1.32 that, for each $p \in X \setminus K$, there exists an open set V_p such that $p \in V_p$ and $V_p \cap K = \emptyset$. It follows that the complement $X \setminus K$ of K in X is a neighbourhood of each of its points, and consequently is an open set in X (see Lemma 1.4). Thus the compact set K, being the complement of an open set, is itself closed in X. ∎

Lemma 1.34 *Let $\varphi\colon X \to Y$ be a continuous function from a compact topological space X to a Hausdorff space Y. Then $\varphi(K)$ is closed in Y for every closed set K in X.*

Proof If K is a closed set in X, then K is compact (Lemma 1.28), and therefore $\varphi(K)$ is compact (Lemma 1.29). But any compact subset of a Hausdorff space is closed (Corollary 1.33). Thus $\varphi(K)$ is closed in Y, as required. ∎

Theorem 1.35 *A continuous bijection $\varphi\colon X \to Y$ from a compact topological space X to a Hausdorff space Y is a homeomorphism.*

Proof Let $\mu\colon Y \to X$ be the inverse of the bijection $\varphi\colon X \to Y$. If W is open in X then $X \setminus W$ is closed in X, and hence $\varphi(X \setminus W)$ is closed in Y (see Lemma 1.34). But

$$\varphi(X \setminus W) = \mu^{-1}(X \setminus W) = Y \setminus \mu^{-1}(W)$$

(see Lemma 1.11). It follows that $\mu^{-1}(W)$ is open in Y for every open set W in X. Therefore $\mu\colon Y \to X$ is continuous, and thus $\varphi\colon X \to Y$ is a homeomorphism. ∎

1.16 The Lebesgue Lemma and Uniform Continuity

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K with the property that $d(u, v) \leq K$ for all $u, v \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that the diameter of the set A is the least upper bound of the values of the distances between pairs of points of the set A.)

Lemma 1.36 (Lebesgue Lemma) *Let (X, d) be a compact metric space and let $\mathcal{C}$ be an open cover of X. Then there exists a positive real number δ with the following property: every subset of X whose diameter is less than δ is contained wholly within at least one of the open sets belonging to the open cover $\mathcal{C}$.*

Proof Every point of X belongs to at least one of the open sets belonging to the open cover $\mathcal{C}$. It follows from this that, for each point p of X, there exists some positive real number δ_p such that the open ball $B(p, 2\delta_p)$ of radius $2\delta_p$ centred on the point p is contained wholly within at least one of the open sets belonging to the open cover $\mathcal{C}$. But then the collection consisting of the open balls $B(p, \delta_p)$ of radius δ_p centred on the points p of X forms an open cover of the compact space X. There therefore exists a finite set $p_1, p_2, \ldots, p_k$ of points of X such that

$$B(p_1, \delta_1) \cup B(p_2, \delta_2) \cup \cdots \cup B(p_k, \delta_k) = X,$$

where $\delta_i = \delta_{p_i}$ for $i = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then, for some integer i between 1 and k, the point u belongs to $B(p_i, \delta_i)$. It then follows that $A \subset B(p_i, 2\delta_i)$, since, for each point v of A,

$$d(v, p_i) \leq d(v, u) + d(u, p_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(p_i, 2\delta_i)$ is contained wholly within at least one of the open sets belonging to the open cover $\mathcal{C}$. Thus A is contained wholly within at least one of the open sets belonging to $\mathcal{C}$, as required. ∎

Definition Let $\mathcal{C}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{C}$ is a positive real number δ with the following property: every subset of X whose diameter is less than a Lebesgue number δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{C}$.

The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Definition Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. The function φ is said to be *uniformly continuous* on X if and only if, given any positive real number ε, there exists some positive real number δ such that $d_Y(\varphi(u), \varphi(v)) < \varepsilon$ for all points u and v of X satisfying $d_X(u, v) < \delta$. (The value of δ should be independent of both u and v.)

Theorem 1.37 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $\varphi\colon X \to Y$ be a continuous function from X to Y. We must show that the function φ is uniformly continuous.

Let some positive real number ε be given. For each $q \in Y$, define

$$V_q = \{p \in X : d_Y(\varphi(p), q) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_q = \varphi^{-1}\left(B_Y(q, \tfrac{1}{2}\varepsilon)\right)$, where $B_Y(q, \tfrac{1}{2}\varepsilon)$ denotes the open ball of radius $\tfrac{1}{2}\varepsilon$ centred on the point q in Y. Now the open ball $B_Y(q, \tfrac{1}{2}\varepsilon)$ is an open set in Y, and φ is continuous. Therefore V_q is open in X for all $q \in Y$. Note that $p \in V_{\varphi(p)}$ for all $p \in X$.

Now $\{V_q : q \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 1.36) that there exists some positive real number δ such that every subset of X whose diameter is less than δ is a subset of some set V_q.

Now let u and v be points of X satisfying $d_X(u, v) < \delta$. The diameter of the set $\{u, v\}$ is $d_X(u, v)$, which is less than δ. Therefore there exists some $q \in Y$ such that $u \in V_q$ and $v \in V_q$. But then $d_Y(\varphi(u), q) < \frac{1}{2}\varepsilon$ and $d_Y(\varphi(v), q) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(\varphi(u), \varphi(v)) \leq d_Y(\varphi(u), q) + d_Y(q, \varphi(v)) < \varepsilon.$$

This shows that $\varphi\colon X \to Y$ is uniformly continuous, as required. ∎