Module MAU34201: Algebraic Topology I
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Section 3: Connected, Path-Connected and Simply Connected Spaces

D. R. Wilkins

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3 Connected, Path-Connected and Simply Connected Spaces

3.1 Connected Topological Spaces

Definition A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

Lemma 3.1 *A topological space X is connected if and only if the intersection of any two non-empty open sets in X that cover X is non-empty.*

Proof Suppose that the topological space X has the property that the intersection of any two non-empty open sets in X that cover X is non-empty. Let V be a subset of X that is both open and closed in X, and let $W = X \setminus V$. Then $V \cap W = \emptyset$ and $V \cup W = X$. It follows that the open sets V and W cannot both be non-empty, for if they were then V and W would be two disjoint non-empty open sets in X covering X whose intersection is the empty set, contradicting the stated property that, by assumption, is possessed by the topological space X. Moreover $W = \emptyset$ if and only if $V = X$. Thus either $V = \emptyset$ or else $V = X$. We conclude therefore that a topological space having the stated property is connected.

Conversely, suppose that the topological space X is connected. Let V and W be non-empty open subsets of X that cover X. If these sets were disjoint then W would be the complement of X in V, and thus V would be a non-empty open set whose complement is a non-empty open set. Consequently V would be a subset of X that was both open and closed, but that was neither the empty set nor the whole space X, and thus the space X would not be connected. We conclude therefore that the intersection of the sets V and W must be non-empty. This completes the proof. ∎

Lemma 3.2 *A topological space X is connected if and only if the union of any two disjoint non-empty open sets in X is a proper subset of X.*

Proof Suppose that the topological space X has the property that, given any two disjoint non-empty open sets in X, the union of those open sets is a proper subset of X. Let V be a subset of X that is both open and closed in X, and let $W = X \setminus V$. Then V and W are open sets in X for which $V \cup W = X$. Now, by assumption, two disjoint non-empty open subsets of X cannot cover X. But the open sets V and W are disjoint and cover X. Consequently these open sets cannot both be non-empty, and thus either

$V = \emptyset$ or else $W = \emptyset$. Moreover $W = \emptyset$ if and only if $V = X$. Thus either $V = \emptyset$ or $V = X$. We conclude therefore that a topological space having the stated property is connected.

Conversely, suppose that the topological space X is connected. Let V and W be disjoint non-empty open subsets of X. If it were the case that $V \cup W = X$ then $W = X \setminus V$, and thus the sets V and $X \setminus V$ would both be non-empty open sets. Consequently V would be a subset of X that was both open and closed, but that was neither the empty set $\emptyset$ nor the whole space X, and thus the space X would not be connected. We conclude therefore that the set $V \cup W$ cannot be the whole of the topological space X and thus must be a proper subset of X. This completes the proof. ∎

Definition A topological space D is *discrete* if every subset of D is open in D.

Example The set $\mathbb{Z}$ of integers with the usual topology is an example of a discrete topological space. Indeed, given any integer n, the set $\{n\}$ is open in $\mathbb{Z}$, because it is the intersection of $\mathbb{Z}$ with the open ball in $\mathbb{R}$ of radius $\frac{1}{2}$ about n. Any non-empty subset S of $\mathbb{Z}$ is the union of the sets $\{n\}$ as n ranges over the elements of S. Therefore every subset of $\mathbb{Z}$ is open in $\mathbb{Z}$, and thus $\mathbb{Z}$, with the usual topology, is a discrete topological space.

Proposition 3.3 *Let X be a non-empty topological space, and let D be a discrete topological space with at least two elements. Then X is connected if and only if every continuous function from X to D is constant.*

Proof Suppose that X is connected. Let $f\colon X \to D$ be a continuous function from X to D, let $d \in f(X)$, and let $Z = f^{-1}(\{d\})$. Now $\{d\}$ is both open and closed in D. It follows from the continuity of $f\colon X \to D$ that Z is both open and closed in X. Moreover Z is non-empty. It follows from the connectedness of X that $Z = X$, and thus $f\colon X \to D$ is constant.

Now suppose that X is not connected. Then there exists a non-empty proper subset Z of X that is both open and closed in X. Let d and e be elements of D, where $d \neq e$, and let $f\colon X \to D$ be defined so that

$$f(x) = \begin{cases} d & \text{if } x \in Z; \\ e & \text{if } x \in X \setminus Z. \end{cases}$$

If V is a subset of D then $f^{-1}(V)$ is one of the following four sets: $\emptyset$; Z; $X \setminus Z$; X. It follows that $f^{-1}(V)$ is open in X for all subsets V of D. Therefore $f\colon X \to D$ is continuous. But the function $f\colon X \to D$ is not constant, because Z is a non-empty proper subset of X. The result follows. ∎

The following results follow immediately from Proposition 3.3.

Corollary 3.4 *A non-empty topological space X is connected if and only if every continuous function $f\colon X \to \{0,1\}$ from X to the discrete topological space $\{0,1\}$ is constant.*

Corollary 3.5 *A non-empty topological space X is connected if and only if every continuous function $f\colon X \to \mathbb{Z}$ from X to the set $\mathbb{Z}$ of integers is constant.*

Example Let $X = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. The topological space X is not connected. Indeed let $f\colon X \to \mathbb{Z}$ be defined such that

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

Then the function f is continuous on X but is not constant.

Lemma 3.6 *Let X be a topological space, and let A be a subset of X. Then A is connected if and only if, whenever open sets V and W in X cover the subset A and have non-empty intersections with the set A, the intersection of the three sets A, V and W is non-empty,*

Proof A subset of A is open in the subspace topology on A if and only if it is the intersection with A of some open set in the topological space X. It therefore follows that the subset A of the topological space X is connected if and only if, given any open sets V and W in X, where $A \cap V \neq \emptyset$ and $A \cap W \neq \emptyset$ and $(A \cap V) \cup (A \cap W) = A$, it is also the case that $(A \cap V) \cap (A \cap W) \neq \emptyset$ (see Lemma 3.1). But standard set identities ensure that

$$(A \cap V) \cup (A \cap W) = A \cap (V \cup W)$$

and

$$(A \cap V) \cap (A \cap W) = A \cap V \cap W.$$

It follows that

$$(A \cap V) \cup (A \cap W) = A$$

if and only if $A \subset V \cup W$. Also

$$(A \cap V) \cap (A \cap W) \neq \emptyset$$

if and only if $A \cap V \cap W \neq \emptyset$. Thus the set A is connected if and only if, given open sets V and W in X covering the set A whose intersections with the set A are non-empty, the intersection of the three sets A, V and W is non-empty, which is what we were required to prove. ∎

Lemma 3.7 *Let X be a topological space and let A be a connected subset of X. Then the closure $\overline{A}$ of A is connected.*

Proof The intersection of an open set in X with the set A is non-empty if and only if the intersection of that open set with the closure $\overline{A}$ of A in X is non-empty (see Lemma 1.6). Let V and W be open sets in X for which $V \cap \overline{A} \neq \emptyset$, $W \cap \overline{A} \neq \emptyset$, and $\overline{A} \subset V \cup W$. Then $V \cap A \neq \emptyset$, $W \cap A \neq \emptyset$, and $A \subset V \cup W$. It follows from the connectness of the set A that $A \cap V \cap W \neq \emptyset$ (see Lemma 3.6). But then $\overline{A} \cap V \cap W \neq \emptyset$. Consequently we have established that the closure $\overline{A}$ of A in X is connected, as required. ∎

Lemma 3.8 *Let $f\colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a connected subset of X. Then $f(A)$ is connected.*

Proof Let V and W be open sets in Y for which $V \cap f(A) \neq \emptyset$, $W \cap f(A) \neq \emptyset$ and $f(A) \subset V \cup W$. Then $A \cap f^{-1}(V) \neq \emptyset$, $A \cap f^{-1}(W) \neq \emptyset$ and $A \subset f^{-1}(V) \cup f^{-1}(W)$. It follows from the connectedness of A that $A \cap f^{-1}(V) \cap f^{-1}(W) \neq \emptyset$. Let $p \in A \cap f^{-1}(V) \cap f^{-1}(W)$. Then $f(p) \in V \cap W$, and therefore $f(A) \cap V \cap W \neq \emptyset$. Applying Lemma 3.6, we conclude that the subset $f(A)$ of Y is connected, as required. ∎

Lemma 3.9 *Let X be a topological space, and let A and B be connected subsets of X. Suppose that the intersection of the sets A and B is non-empty. Then the union $A \cup B$ of the sets A and B is connected.*

Proof Let V and W be open sets in X with the properties that $A \cup B \subset V \cup W$ and $(A \cup B) \cap V \cap W = \emptyset$. Now the intersection of $A \cap B$ with at least one of the sets V and W must be non-empty. We may suppose, without loss of generality, that the set $A \cap B \cap V$ is non-empty. Now $A \cap V$ and $A \cap W$ are disjoint subsets of A whose union is the set A itself. These subsets of A are then complements of one another. Moreover they are open with respect to the subspace topology on A. It follows in particular that the set $A \cap V$ is a subset of A that is both open and closed in the subspace topology on A. Moreover this set is non-empty, because $A \cap B \cap V$ is non-empty. It follows from the connectedness of the set A that $A \subset V$. Similarly $B \subset V$. It follows therefore that $A \cup B \subset V$, and consequently $(A \cup B) \cap W = \emptyset$.

We have now shown that there cannot exist open sets V and W in X for which $(A \cup B) \subset V \cup W$, $(A \cup B) \cap V \cap W = \emptyset$, $(A \cup B) \cap V \neq \emptyset$ and $(A \cup B) \cap W \neq \emptyset$. Consequently the set $A \cup B$ cannot be expressed as the union of two disjoint non-empty subsets that are both open in the subspace topology on $A \cup B$. We conclude therefore that the set $A \cup B$ is connected, which is what we set out to prove. ∎

3.2 Connected Components of Topological Spaces

Proposition 3.10 *Let X be a topological space. For each $p \in X$, let S_p be the union of all connected subsets of X that contain p. Then*

(i) S_p is connected,

(ii) S_p is closed,

(iii) if $p, q \in X$, then either $S_p = S_q$, or else $S_p \cap S_q = \emptyset$.

Proof Let p be a point of the topological space X, and let V and W be open sets in X whose union contains the set S_p. Suppose that $p \in V$ and that the intersection of the three sets S_p, V and W is the empty set. Then S_p is the union of the sets $S_p \cap V$ and $S_p \cap W$. Moreover these sets $S_p \cap V$ and $S_p \cap W$ are disjoint subsets of S_p.

Now let A be a connected subset of X which the point p belongs. The definition of the set S_p ensures that $A \subset S_p$. Consequently the set A is the disjoint union of the sets $A \cap V$ and $A \cap W$. Moreover these sets $A \cap V$ and $A \cap W$ are both open in A relative to the subspace topology on A, and they are complements in the set A of one another. It follows that $A \cap V$ is a subset of A that is both open and closed in A. Moreover the point p belongs to the set $A \cap V$. It follows from the connectedness of the set A that $A = A \cap V$, and therefore $A \subset V$.

Now the set S_p is by definition the union of all connected subsets of X that contain the point p. It follows from what has already been shown that each of those connected subsets of X is contained in the open set V. Therefore $S_p \subset V$. We conclude therefore that if V and W are open subsets of X for which $S_p \subset V \cup W$ and $S_p \cap V \cap W = \emptyset$, and if $p \in V$, then $S_p \subset V$ and $S_p \cap W = \emptyset$. It follows from this that the set S_p is connected. This establishes (i).

Now the closure $\overline{S_p}$ of S_p is connected (see Lemma 3.7). It follows from the definition of the set S_p that $\overline{S_p} \subset S_p$, and therefore $\overline{S_p} = S_p$. Consequently the set S_p is closed. This establishes (ii).

Finally, suppose that p and q are points of X for which $S_p \cap S_q \neq \emptyset$. The sets S_p and S_q are connected, and their intersection is non-empty. It follows that $S_p \cup S_q$ is connected (see Lemma 3.9). It then follows from the definition of the sets S_p and S_q that $S_p \cup S_q \subset S_p$ and $S_p \cup S_q \subset S_q$, and consequently $S_p = S_q$. This establishes (iii), completing the proof. ∎

Given any topological space X, the connected subsets S_p of X defined as in the statement of Proposition 3.10 are referred to as the *connected components* of X. Now a point p of X belongs to at least one connected component because it belongs to the connected component S_p that it determines. Also we see from Proposition 3.10, part (iii) that the point p cannot belong to more than one distinct connected component, because two distinct connected components cannot have non-empty intersection. It follows that the topological space X is the disjoint union of its connected components.

Example Let X be the subset of $\mathbb{R}^2$ defined so that $X = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ Then the connected components of X are the sets

$$\{(x, y) \in \mathbb{R}^2 : x > 0\} \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Example Let Y be the open subset of the real line defined so that

$$Y = \{x \in \mathbb{R} : |x - n| < \tfrac{1}{2} \text{ for some integer } n\}.$$

Then the connected components of the set Y are the sets J_n for all integers n, where, for each integer n, J_n is the open interval with endpoints $n - \frac{1}{2}$ and $n + \frac{1}{2}$.

3.3 Products of Connected Topological Spaces

Lemma 3.11 *A Cartesian product $X \times Y$ of two connected topological spaces X and Y is itself connected.*

Proof Let (p, q) and (r, s) be points of $X \times Y$. Then the sets

$$\{(x, y) \in X \times Y : y = q\} \quad \text{and} \quad \{(x, y) \in X \times Y : x = r\}$$

are connected subsets of $X \times Y$, being homeomorphic to X and Y respectively. Morever the point (r, q) of $X \times Y$ belongs to both sets. It follows that both points (p, q) and (r, s) belong to the same connected component of $X \times Y$ as the point (r, q). We conclude therefore that any two points of the product space $X \times Y$ belong to the same connected component of that space, and therefore the space is connected.

We deduce immediately that a finite Cartesian product of connected topological spaces is connected.

3.4 Path-Connected Topological Spaces

A concept closely related to that of connectedness is *path-connectedness*.

Definition Let X be a topological space, and let p and q be points of X. A *path* in X from p to q is defined to be a continuous function $\gamma\colon [a,b] \to X$, defined over some closed interval with endpoints a and b, where $a < b$, and mapping that closed interval into the topological space X so that $\gamma(a) = p$ and $\gamma(b) = q$.

We shall usually take the domain of a path to be the closed unit interval $[0,1]$ with endpoints 0 and 1.

Definition A topological space X is said to be *path-connected* if and only if, given any two points p and q of X, there exists a path $\gamma\colon [0,1] \to X$ in X from the point p to the point q.

Proposition 3.12 *Every path-connected topological space is connected.*

Proof Let X be a path-connected topological space, and let V and W be disjoint non-empty open sets in X. We show that the union $V \cup W$ of the open sets V and W must then be a proper subset of X.

Now X is path-connected and the open sets V and W are, by assumption, non-empty. Therefore there exists a path $\gamma\colon [0,1] \to X$ in X for which $\gamma(0) \in V$ and $\gamma(1) \in W$. Let D and E denote the preimages of the open sets V and W respectively under the map γ, so that

$$D = \{t \in [0,1] : \gamma(t) \in V\} \quad \text{and} \quad E = \{t \in [0,1] : \gamma(t) \in W\}.$$

Then the subsets D and E of the closed unit interval $[0,1]$ are open in that interval, because the path γ is a continuous map from the closed unit interval $[0,1]$ to the topological space X. Also $0 \in D$, $1 \in E$, and $D \cap E = \emptyset$.

Let s be the least upper bound of the set D. Now D and E are open in $[0,1]$, and the endpoints 0 and 1 of that interval belong to D and E respectively. It follows that there exist positive real numbers δ_0 and δ_1 for which $[0,\delta_0) \subset D$ and $(1-\delta_1, 1] \subset E$. It follows that $0 < s < 1$. Now if s were to belong to the set D then there would exist some positive real number δ small enough to ensure that $s - \delta > 0$, $s + \delta < 1$ and $(s-\delta, s+\delta) \subset D$. But then s would not be an upper bound of the set D, contradicting the choice of s as the least upper bound of D. Next we note that if s were to belong to the set E then there would exist some positive real number δ small enough to ensure that $s - \delta > 0$, $s + \delta < 1$ and $(s-\delta, s+\delta) \subset E$. But then real numbers t strictly between $s - \delta$ and $s + \delta$ would not belong to the set D, because the open sets D and E are disjoint, and real numbers t greater than s could not belong to the set D, being greater than the least upper bound of that set, and therefore $s - \delta$ would be an upper bound of the set D, contradicting the choice of s as the least upper bound of that set. We conclude therefore that the least upper bound s of the set D cannot belong to either of the sets D and E, and therefore the point $\gamma(s)$ of the topological space X cannot belong to either of the open sets V and W. Thus the union of the disjoint non-empty open sets V and W must be a proper subset of the topological space X. The result follows. ∎

The topological spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ are all path-connected. Indeed, given any two points of one of these spaces, the straight line segment joining these two points is a continuous path from one point to the other. Also the n-dimensional sphere S^n is path-connected for all positive integers n. We conclude that these topological spaces are connected.

Definition A subset X of a real vector space is said to be *convex* if, given points $\mathbf{u}$ and $\mathbf{v}$ of X, the point $(1-t)\mathbf{u}+t\mathbf{v}$ belongs to X for all real numbers t satisfying $0 \leq t \leq 1$.

Corollary 3.13 *All convex subsets of real vector spaces are connected, and are path-connected.*

Remark Proposition 3.12 generalizes the Intermediate Value Theorem of real analysis. Indeed let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on an interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. The range $f([a,b])$ is then a path-connected subset of $\mathbb{R}$. It follows from Proposition 3.12 that this set is connected. Let c be a real number that lies strictly between $f(a)$ and $f(b)$ and let

$$V = \{y \in f([a,b]) : y < c\} \quad \text{and} \quad W = \{y \in f([a,b]) : y > c\}.$$

Then V and W are non-empty open subsets of $f([a,b])$, and $V \cap W = \emptyset$. It follows from the connectness of $f([a,b])$ that $V \cup W$ must be a proper subset of $f([a,b])$ (see Lemma 3.2), and therefore $c \in f([a,b])$. Thus the range of the function f contains all real numbers between $f(a)$ and $f(b)$.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined so that

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and let

$$X = \{(x,y) \in \mathbb{R}^2 : y = f(x)\}.$$

We show that X is a connected set. Let

$$X_+ = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = f(x)\}$$

and

$$X_- = \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y = f(x)\}.$$

Now the restriction of the function f to the set of (strictly) positive real numbers is continuous on the set of positive real numbers. It follows from this that the set X_+ is path-connected. It then follows that the set X_+ is connected (see Proposition 3.12). The connectedness of X_+ can also be verified by noting that it is the image of the connected space $\{x \in \mathbb{R} : x > 0\}$ under a continuous map and is therefore itself connected (see Lemma 3.8). Similarly the set X_- is path-connected, and is therefore connected.

For each positive integer n, let $\mathbf{p}_n = ((n\pi)^{-1}, 0)$. Then $\mathbf{p}_n \in X_+$ for all positive integers n, and $\mathbf{p}_n \to (0, 0)$ as $n \to +\infty$. It follows that $(0, 0)$ belongs to the closure $\overline{X}_+$ of X_+ in X. Connected components of a topological space are closed (see Proposition 3.10). Thus the connected component of X that includes the connected subset X_+ also contains the point $(0, 0)$. Similarly the connected component of X that includes X_- also contains the point $(0, 0)$. Therefore the unique connected component of X that contains the point $(0, 0)$ is the whole of X and thus X is a connected topological space.

However X is not a path-connected topological space. If $\gamma\colon [0, 1] \to X$ is a continuous map from the closed unit interval $[0, 1]$ into X, and if $\gamma(0) = (0, 0)$, then $\gamma(t) = (0, 0)$ for all $t \in [0, 1]$. Indeed let

$$s = \sup\{t \in [0, 1] : \gamma(t) = (0, 0)\}.$$

It follows from the continuity of γ that $\gamma(s) = (0, 0)$. There then exists some positive real number δ such that $|\gamma(t) - (0, 0)| < \frac{1}{2}$ for all $t \in [0, 1]$ satisfying $|t - s| < \delta$. But $\gamma([0, 1] \cap [s, s + \delta))$ must also be a connected subset of X. It follows that $\gamma(t) = (0, 0)$ for all $t \in [0, 1]$ satisfying $s \leq t < s + \delta$. Consequently $s = 1$ and $\gamma(t) = (0, 0)$ for all $t \in [0, 1]$. (Essentially, the path γ cannot get from $(0, 0)$ to any other point of X because continuity prevents the path from getting over intervening humps where the function f takes values such as ± 1.) We conclude that the connected topological space X is not path-connected.

3.5 Locally Path-Connected Topological Spaces

Definition A topological space X is said to be *locally connected* if, given any point p of X, and given any open set N in X for which $p \in N$, there exists some connected open set V in X such that $p \in V$ and $V \subset N$.

Definition A topological space X is said to be *locally path-connected* if, given any point p of X, and given any open set N in X for which $p \in N$, there exists some path-connected open set V in X such that $p \in V$ and $V \subset N$.

Every path-connected subset of a topological space is connected. (This follows directly from Proposition 3.12.) Therefore every locally path-connected topological space is locally connected.

Proposition 3.14 *Let X be a connected, locally path-connected topological space. Then X is path-connected.*

Proof Choose a point p of X. Let Z be the subset of X consisting of all points q of X with the property that q can be joined to p by a path. We show that the subset Z is both open and closed in X.

Now, given any point q of X there exists a path-connected open set N_q in X such that $q \in N_q$. We claim that if $q \in Z$ then $N_q \subset Z$, and if $q \notin Z$ then $N_q \cap Z = \emptyset$.

Suppose first that $q \in Z$. Then, given any point r of N_q, there exists a path in N_q from r to q. Moreover it follows from the definition of the set Z that there exists a path in X from q to p. These two paths can be concatenated to yield a path in X from r to p, and therefore $r \in Z$. This shows that $N_q \subset Z$ whenever $q \in Z$.

Next suppose that $q \notin Z$. Let $r \in N_q$. If it were the case that $r \in Z$, then we would be able to concatenate a path in N_q from q to r with a path in X from r to p in order to obtain a path in X from q to p. But this is impossible, as $q \notin Z$. Therefore $N_q \cap Z = \emptyset$ whenever $q \notin Z$.

Now the set Z is the union of the open sets N_q as q ranges over all points of Z. It follows that Z is itself an open set. Similarly $X \setminus Z$ is the union of the open sets N_q as q ranges over all points of $X \setminus Z$, and therefore $X \setminus Z$ is itself an open set. It follows that Z is a subset of X that is both open and closed. Moreover $p \in Z$, and therefore Z is non-empty. But the only subsets of X that are both open and closed are $\emptyset$ and X itself, because X is connected. Therefore $Z = X$, and thus every point of X can be joined to the point p by a path in X. We conclude that X is path-connected, as required. ∎

3.6 Simply Connected Topological Spaces

Definition A topological space X is said to be *simply connected* if it is both path-connected and also has the property that any continuous function mapping the boundary circle of a closed disc into X can be extended continuously over the whole of the disk.

Example Euclidean space $\mathbb{R}^n$ of dimension n is simply connected for all positive integers n. Indeed any continuous map $f\colon C \to \mathbb{R}^n$ mapping the boundary circle C of the closed unit disk D into $\mathbb{R}^n$ can be extended to a continuous map $F\colon D \to \mathbb{R}^n$ mapping the whole disk into $\mathbb{R}^n$ by setting

$$F(r\mathbf{p}) = rf(\mathbf{p})$$

for all $\mathbf{p} \in C$ and $r \in [0,1]$.

Lemma 3.15 *Let A be a topological space that is homeomorphic to the closed unit disk D, and let B be a subset of A that is the image of the boundary circle C of the closed unit disk D under some homeomorphism between the closed unit disk D and the topological space A. Then a path-connected topological space X is simply connected if and only if every continuous function mapping the set B into the topological space X can be extended to a continuous function mapping the whole of the topological space A into X.*

Proof Let $h\colon D \to A$ be a homeomorphism from the closed unit disk D to the topological space A that maps the boundary circle C of the unit disk onto some subset B of A.

Let us suppose first that every continuous map from B to the topological space X extends to a continuous map from A to X. Let $f\colon C \to X$ be a function mapping the boundary circle C of the closed unit disk D into the topological space X. Then f determines a corresponding continuous map $g\colon B \to X$ mapping the set B into X, where $g(h(\mathbf{p})) = f(\mathbf{p})$ for all points $\mathbf{p}$ of the boundary circle C of the closed unit disk D. The map g extends, by assumption, to a continuous map $G\colon A \to X$ mapping the whole of the topological space A into the topological space X. Let $F\colon D \to X$ be the continuous map from D to X defined so that $F(\mathbf{x}) = G(h(\mathbf{x}))$ for all $\mathbf{x} \in D$. Then the function F extends the the map f to a continuous map defined over the entire unit disk. We conclude therefore that any continuous map from the boundary circle of the closed unit disk to the topological space X can be extended continuously over the whole of the disk, and therefore the path-connected topological space X is simply connected.

Conversely suppose that the topological space X is simply connected. Let B be be the image of the boundary circle C of the unit disk under the homeomorphism h, and let $g\colon B \to X$ be a continuous map defined over B and mapping B into the topological space X. There is then a corresponding map $f\colon C \to X$ mapping the boundary circle C of the closed unit disk into X which is defined so that $f(\mathbf{p}) = g(h(\mathbf{p}))$ for all points $\mathbf{p}$ of that boundary circle. This map f extends to a continuous map $F\colon D \to X$ defined over the entire closed unit disk D and mapping that closed disk into the topological space X. This continuous map F then corresponds to a continuous map $G\colon A \to X$ between the topological spaces A and X defined so that $G(h(\mathbf{x})) = F(\mathbf{x})$ for all points $\mathbf{x}$ of the unit disk D. The result follows. ∎

Proposition 3.16 *Let K be a closed bounded subset of some finite-dimensional Euclidean space, and let $\varphi\colon K \to L$ be a continuous function mapping K onto a subset L of some Euclidean space. Then $\varphi\colon K \to L$ is an identification map.*

Proof Let W be a subset of L whose preimage $\varphi^{-1}(W)$ under the map φ is open in K, and let $\mathbf{q} \in W$. We claim that W is then a neighbourhood in L of the point $\mathbf{q}$.

Let $G = K \setminus \varphi^{-1}(W)$. Suppose that the subset W of L were not a neighbourhood in L of the point $\mathbf{q}$. The surjectivity of the map φ would then ensure the existence of an infinite sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ of points of the closed bounded set G for which $\lim\limits_{j\to+\infty} \varphi(\mathbf{p}_j) = \mathbf{q}$. The multidimensional Bolzano-Weierstrass Theorem would then ensure the existence of a convergent subsequence $\mathbf{p}_{k_1}, \mathbf{p}_{k_2}, \mathbf{p}_{k_3}, \ldots$ of the infinite sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$. Let $\mathbf{r} = \lim\limits_{j\to+\infty} \mathbf{p}_{k_j}$.

Now $\mathbf{p}_{k_j}$ would belong to the closed set G for all positive integers j. and therefore $\mathbf{r} \in G$. But the continuity of the map φ would ensure that

$$\varphi(\mathbf{r}) = \varphi\left(\lim_{j\to+\infty} \mathbf{p}_{k_j}\right) = \lim_{j\to+\infty} \varphi(\mathbf{p}_{k_j}) = \mathbf{p}.$$

Moreover $\mathbf{p} \in W$. It would therefore follow that $\mathbf{r} \in \varphi^{-1}(W)$. But this would be an impossibility, because $\mathbf{r} \in G$ and $G \cap \varphi^{-1}(W) = \emptyset$. Thus the assumption that W was not a neighbourhood of the point $\mathbf{p}$ would lead to a contradiction. We conclude therefore that the set W must be a neighbourhood of the point $\mathbf{p}$, this point $\mathbf{p}$ being an arbitrary point chosen from the set W. Thus the set W, being a neighbourhood of each of its points, must be open in the set L.

We have thus shown that if the preimage of a subset of L is open in K then that subset is open in L. The converse follows immediately from the continuity of the map φ. We can conclude therefore that the function $\varphi\colon K \to L$ is an identification map, which is what we were required to prove. ∎

Alternative Proof The closed unit square, being a closed and bounded subset of the plane, is a compact topological space (see Theorem 2.9). The closed unit disk is a Hausdorff space, because any subset of a Euclidean space is a metric space and is thus a Hausdorff space. Now any continuous surjection from a compact topological space to a Hausdorff space is an identification map (see Proposition 2.15). The result follows. ∎

Proposition 3.17 *A path-connected topological space X is simply connected if and only if, given any loop $\gamma\colon [0,1] \to X$ in X, there exists a homotopy between the loop γ and the constant loop at the point $\gamma(0)$ of X where the loop γ starts and ends, where this homotopy is a homotopy relative to the set $\{0,1\}$ of endpoints of the closed unit interval over which the continuous function γ is defined.*

Proof First suppose that the space X is simply connected. Let $\gamma\colon [0,1] \to X$ be a loop in X based at some point p of X. Now the unit square is homeomorphic to the unit disk, and therefore any continuous map defined over the boundary of the square can be continuously extended over the whole of the square. It follows that there exists a continuous map $H\colon [0,1]\times[0,1] \to X$ such that $H(t,0) = \gamma(t)$ and $H(t,1) = p$ for all $t \in [0,1]$, and $H(0,\tau) = H(1,\tau) = p$ for all $\tau \in [0,1]$. The map H is then the required homotopy between the loop γ and the constant loop at the point p.

Conversely suppose that, given any loop $\gamma\colon [0,1] \to X$ in the topological space X, there exists a homotopy between the loop γ and the constant loop at the point $\gamma(0)$ of X where the loop γ starts and ends, where this homotopy is a homotopy relative to the set $\{0,1\}$ of endpoints of the closed unit interval. Let $f\colon C \to X$ be a continuous function defined on the boundary circle C of the closed unit disk D in $\mathbb{R}^2$, let γ be the function from the closed unit interval to the topological space X defined so that $\gamma(t) = f(\cos(2\pi t), \sin(2\pi t))$ for all $t \in [0,1]$, and let $p = f(1,0)$. We must show that f can be extended continuously over the whole of D. Now there exists a homotopy G between the loop γ and the constant loop at p, this homotopy G being a homotopy relative to the set of endpoints of the closed unit interval. Then $G\colon [0,1]\times[0,1] \to X$ is a continuous map, defined over the closed unit square $[0,1]\times[0,1]$, and mapping that square into the topological space X, with the properties that $G(t,0) = f(\cos(2\pi t), \sin(2\pi t))$ and $G(t,1) = p$ for all $t \in [0,1]$ and $G(0,\tau) = G(1,\tau) = p$ for all $\tau \in [0,1]$ (see Proposition 3.17). Moreover $G(t_1,\tau_1) = G(t_2,\tau_2)$ whenever $q(t_1,\tau_1) = q(t_2,\tau_2)$, where

$$q(t,\tau) = \bigl((1-\tau)\cos(2\pi t) + \tau, (1-\tau)\sin(2\pi t)\bigr)$$

for all $t,\tau \in [0,1]$. It follows that there is a well-defined function $F\colon D \to X$ mapping the closed unit disk D into the topological space X, defined so as to ensure that $F \circ q = G$.

Now the function $q\colon [0,1] \times [0,1] \to D$ is a continuous surjection from the closed unit square to the closed unit disk. Moreover the closed unit square is a closed bounded subset of the plane. It follows that the map q is an identification map (see Proposition 3.16). Moreover the composition function $F \circ q$ is the continuous function G. It follows that the function F must itself be continuous (see Lemma 2.14). We conclude therefore that the function $F\colon D \to X$ is a continuous function from the closed unit disk D to the topological space X that extends the continuous function $f\colon C \to X$ defined on the boundary circle C of the closed unit disk. We can now conclude from the result just established that the path-connected topological space X is simply connected, which is what we were required to prove. ∎

Theorem 3.18 *Let X be a topological space, and let V and W be open subsets of X, with $V \cup W = X$. Suppose that V and W are simply connected, and that $V \cap W$ is non-empty and path-connected. Then X is itself simply connected.*

Proof We must show that any continuous function $f\colon C \to X$ defined on the unit circle C can be extended continuously over the closed unit disk D. Now the preimages $f^{-1}(V)$ and $f^{-1}(W)$ of V and W are open in the circle C (because the function f is continuous), and $C = f^{-1}(V) \cup f^{-1}(W)$. It follows from the Lebesgue Lemma (Lemma 1.36) that there exists some positive real number δ which is small enough to ensure that any arc in the circle C whose length is less than δ is entirely contained in one or other of the sets $f^{-1}(V)$ and $f^{-1}(W)$, and is therefore mapped by the function f into one or other of the open sets V and W.

Choose points $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ around the circle C to ensure that the length of the arc joining $\mathbf{q}_{i-1}$ to $\mathbf{q}_i$ is less than δ for each integer i between 2 and n and similarly the length of the arc joining $\mathbf{q}_n$ to $\mathbf{q}_1$ is less than δ. Then, for each integer i between 2 and n, the short arc joining $\mathbf{q}_{i-1}$ to $\mathbf{q}_i$ is mapped by f into one or other of the open sets V and W, and similarly the short arc joining $\mathbf{q}_n$ to $\mathbf{q}_1$ is also mapped by f into one or other of the open sets V and W.

Let p be some point of $V \cap W$. Now the sets V, W and $V \cap W$ are all path-connected. Therefore we can choose paths $\alpha_i\colon [0,1] \to X$ for $i = 1, 2, \ldots, n$ so as to satisfy the following properties: $\alpha_i(0) = p$ for $i = 1, 2, \ldots, n$; $\alpha_i(1) = f(\mathbf{q}_i)$ for $i = 1, 2, \ldots, n$; $\alpha_i([0,1]) \subset V$ for those integers i for which $f(\mathbf{q}_i) \in V$; $\alpha_i([0,1]) \subset W$ for those integers i for which $f(\mathbf{q}_i) \in W$. For convenience in what follows, let $\mathbf{q}_0 = \mathbf{q}_n$ and $\alpha_0 = \alpha_n$.

Now, for each integer i between 1 and n, consider the sector T_i of the closed unit disk bounded by the line segments joining the centre of the disk to the points $\mathbf{q}_{i-1}$ and $\mathbf{q}_i$ and by the short arc joining $\mathbf{q}_{i-1}$ to $\mathbf{q}_i$. Now this sector is homeomorphic to the closed unit disk, and therefore any continuous function mapping the boundary B_i of T_i into a simply connected space can be extended continuously over the whole of T_i. In particular, let h_i be the function on B_i defined so that

$$h_i(\mathbf{q}) = \begin{cases} f(\mathbf{q}) & \text{if } \mathbf{q} \in T_i \cap C, \\ \alpha_{i-1}(t) & \text{if } \mathbf{q} = t\mathbf{q}_{i-1} \text{ for some } t \in [0,1], \\ \alpha_i(t) & \text{if } \mathbf{q} = t\mathbf{q}_i \text{ for some } t \in [0,1]. \end{cases}$$

Note that $h_i(B_i) \subset V$ whenever the short arc joining $\mathbf{q}_{i-1}$ to $\mathbf{q}_i$ is mapped by the function f into V, and $h_i(B_i) \subset W$ whenever that short arc is mapped by f into W.

Now the open sets V and W are both simply connected. It follows that each of the functions h_i can be extended continuously to a function F_i, defined over the whole of the sector T_i, which maps that sector into one or other of the open sets V and W. Moreover the functions defined in this fashion on each of the sectors T_i agree with one another wherever the sectors intersect. It follows from the Pasting Lemma (Lemma 1.25) that there exists a continuous map F from the closed unit disk D to the topological space X that coincides with the function F_i on the sector T_i for each integer i between 1 and n. This map F extends the given map f defined over the boundary circle of the disk. The required result follows. ∎

The *n-dimensional sphere* S^n is the unit sphere in $\mathbb{R}^{n+1}$, defined so that

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

Corollary 3.19 *The n-dimensional sphere S^n is simply connected for all integers n satisfying $n > 1$.*

Proof Let

$$V = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n : x_{n+1} > -\tfrac{1}{2}\}$$

and

$$W = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n : x_{n+1} < \tfrac{1}{2}\}.$$

Then V and W are homeomorphic to an n-dimensional ball, and are therefore simply connected. Moreover $V \cap W$ is path-connected, provided that $n > 1$. It follows that the n-dimensional sphere S^n is simply connected for all integers n for which $n > 1$. ∎

3.7 Local and Semi-Local Simple Connectedness

Definition A topological space X is said to be *locally simply connected* if, given any point p of X, and given any open set N in X for which $p \in N$, there exists some simply connected open set V in X such that $p \in V$ and $V \subset N$.

Definition A topological space X is said to be *semi-locally simply connected* if, given any point p of X there exists an open set V for which $p \in V$, where that open set V satisfies the following property: given any continuous function $f\colon C \to V$ mapping the boundary circle C of the closed unit disk D into the open set V, there exists a continuous function $F\colon D \to X$ from the closed unit disk D into the topological space X whose restriction to the boundary circle C of that disk coincides with the function f.

Remark There is a classification theorem for covering maps over topological spaces that are connected, locally path-connected and semi-locally simply connected, which establishes that isomorphism classes of covering maps over such a topological space are in one-to-one correspondence with conjugacy classes of subgroups of the fundamental group of the topological space at some chosen basepoint of that topological space. In subsequent lectures we shall establish the definitions and basic properties of *covering maps* and the *fundamental group* and develop some portion of the theory which ultimately yields the classification theorem for covering maps over connected, locally path-connected and semi-locally simply connected topological spaces just alluded to.