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Section 2: Product and Quotient Topologies

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2 Product and Quotient Topologies

2.1 Topologies on Products of Topological Spaces

A *product topology* is a topology on a Cartesian product of topological spaces that is determined in a suitably natural fashion by the topologies on the spaces that constitute the Cartesian product.

We begin with some preliminary discussion of Cartesian products of sets. Let $X_1, X_2, \ldots, X_n$ be sets. The *Cartesian product* of the sets $X_1, X_2, \ldots, X_n$ consists of all ordered n-tuples $(p_1, p_2, \ldots, p_n)$ in which the ith component p_i is an element, or point, of the set X_i for $i = 1, 2, \ldots, n$.

Let $X_1, X_2, \ldots, X_n$ be sets, and let B_i be a subset of X_i for $i = 1, 2, \ldots, n$. The very definition of a Cartesian product of n sets, representing the elements of the Cartesian product as ordered n-tuples, with components taken from the respective sets, ensures that the Cartesian product $B_1 \times B_2 \times \cdots \times B_n$ of the sets $B_1, B_2, \ldots, B_n$ is a subset of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$.

Lemma 2.1 *Let $X_1, X_2, \ldots, X_n$ be topological spaces and let*

$$X = X_1 \times X_2 \times \cdots \times X_n.$$

Also let τ be the collection of subsets W of X which have the property that, given any point $(p_1, p_2, \ldots, p_n)$ of W, there exist open sets $V_1, V_2, \ldots, V_n$, where $p_i \in V_i$ for $i = 1, 2, \ldots, n$, such that

$$V_1 \times V_2 \times \cdots \times V_n \subset W.$$

Then the collection τ of subsets of the Cartesian product set X is a topology on X.

Proof Let $X = X_1 \times X_2 \times \cdots \times X_n$. For the purposes of this proof we refer to those subsets of X that belong to the collection τ as *open sets*. We must verify that, if open sets are defined in this fashion, then the topological space axioms are all satisfied.

The definition of open sets (i.e., the definition of the collection τ of subsets of X) ensures that the empty set and the whole set X are open in X. We must prove that any union or finite intersection of open sets in X is an open set.

We next show that any union of open sets in X is itself an open set. Let E be a union of some given collection of open sets in X and let $(p_1, p_2, \ldots, p_n)$ be some given point of E. Then $(p_1, p_2, \ldots, p_n) \in D$ for some open set D in the given collection. There then exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $p_i \in V_i$ for $i = 1, 2, \ldots, n$ and

$$V_1 \times V_2 \times \cdots \times V_n \subset D \subset E.$$

Consequently the set E is open in X.

Finally we show that any finite intersection of open sets in X is itself an open set. Let $W_1, W_2, \ldots, W_s$ be open sets in X, and let $W = W_1 \cap W_2 \cap \cdots \cap W_s$. Also let some point p of W be given, and let p_i in X_i be determined for $i = 1, 2, \ldots, n$ so that $p = (p_1, p_2, \ldots, p_n)$. Then there exist open sets $V_{r,i}$ in X_i for $r = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, n$ such that $p_i \in V_{r,i}$ for $r = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, n$ and

$$V_{r,1} \times V_{r,2} \times \cdots \times V_{r,n} \subset W_r$$

for $r = 1, 2, \ldots, s$. Let $V_i = V_{1,i} \cap V_{2,i} \cap \cdots \cap V_{s,i}$ for $i = 1, 2, \ldots, n$. Then $p_i \in V_i$ for $i = 1, 2, \ldots, n$. Also

$$V_1 \times V_2 \times \cdots \times V_n \subset V_{r,1} \times V_{r,2} \times \cdots \times V_{r,n} \subset W_r$$

for $r = 1, 2, \ldots, s$. But then $V_1 \times V_2 \times \cdots \times V_n \subset W$, because the set W is the intersection of the sets W_r for $r = 1, 2, \ldots, s$. It follows that W is open in X, as required. ∎

Definition Let $X_1, X_2, \ldots, X_n$ be topological spaces and let

$$X = X_1 \times X_2 \times \cdots \times X_n.$$

The product topology on the Cartesian product X of the topological spaces $X_1, X_2, \ldots, X_n$ is that topology on X whose open sets are the subsets W characterized by the property that, given any point $(p_1, p_2, \ldots, p_n)$ of W, there exist open sets $V_1, V_2, \ldots, V_n$, where $p_i \in V_i$ for $i = 1, 2, \ldots, n$, such that

$$V_1 \times V_2 \times \cdots \times V_n \subset W.$$

Lemma 2.1 ensures that the collection of open sets in a Cartesian product of topological spaces characterized as set out above is indeed a topology on the Cartesian product of the underlying sets.

Lemma 2.2 *Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let V_i be an open set in X_i for $i = 1, 2, \ldots, n$. Then $V_1 \times V_2 \times \cdots \times V_n$ is open in $X_1 \times X_2 \times \cdots \times X_n$.*

Proof It follows directly from the definition of the product topology on $X_1 \times X_2 \times \cdots \times X_n$. ∎

Lemma 2.3 *Let $X_1, X_2, \ldots, X_n$ be topological spaces, let*

$$X = X_1 \times X_2 \times \cdots \times X_n,$$

let $p_i \in X_i$ for $i = 1, 2, \ldots, n$, and let $p = (p_1, p_2, \ldots, p_n)$. A subset N of X is a neighbourhood of p (with respect to the product topology on X) if and only if there exist open neighbourhoods V_i of p_i in X_i for $i = 1, 2, \ldots, n$ for which

$$V_1 \times V_2 \times \cdots \times V_n \subset N.$$

Proof First suppose that N is a subset of X to which the point p belongs. Suppose also that there exist open neighbourhoods V_i of p_i in X_i for $i = 1, 2, \ldots, n$ for which

$$V_1 \times V_2 \times \cdots \times V_n \subset N.$$

Then the product of the open sets V_i for $i = 1, 2, \ldots, n$ is an open subset of X contained in the set N, and the point p belongs to this product of open sets. The definition of neighbourhoods in a topological space therefore ensures that the set N is a neighbourhood of the point p.

Conversely suppose that N is a subset of the Cartesian product X that is a neighbourhood of the point p (with respect to the product topology on X. Then there exists an open neighbourhood W of p in X that is contained in the neighbourhood N of p. The definition of the product topology then ensures the existence of open neighbourhoods V_i of p_i in X_i for $i = 1, 2, \ldots, n$ for which

$$V_1 \times V_1 \times \cdots \times V_n \subset W \subset N.$$

The result follows. ∎

2.2 Continuity of Maps defined on Product Spaces

Proposition 2.4 *Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let $X = X_1 \times X_2 \times \cdots \times X_n$. Also let $\varphi \colon X \to Y$ be a function mapping the product space X into some topological space Y, let p be a point of X, and let $p = (p_1, p_2, \ldots, p_n)$, where $p_i \in X_i$ for $i = 1, 2, \ldots, n$. Then the function φ is continuous at the point p, if and only if, given any open neighbourhood W of $\varphi(p)$ in Y, there exist neighbourhoods M_i of p_i in X_i for $i = 1, 2, \ldots, n$, where those neighbourhoods M_i are small enough to ensure that $\varphi(M_1 \times M_2 \times \cdots \times M_n) \subset W$.*

Proof First suppose that φ is continuous at the point p. Then given any open neighbourhood W of $\varphi(p)$ in Y, the preimage $\varphi^{-1}(W)$ is a neighbourhood of the point p, and therefore there exist open neighbourhoods V_i of p_i in X_i for $i = 1, 2, \ldots, n$ for which

$$V_1 \times V_2 \times \cdots \times V_n \subset \varphi^{-1}(W)$$

(see Lemma 2.3). The open set V_i is then the required neighbourhood of the point p_i for $i = 1, 2, \ldots, n$. Conversely suppose that $\varphi\colon X \to Y$ is any function from X to Y with the property that, given any open neighbourhood W of $\varphi(p)$, there exist neighbourhoods M_i of p_i in X_i for $i = 1, 2, \ldots, n$ whose Cartesian product is mapped by φ into the given open neighbourhood. Let some open neighbourhood W of $\varphi(p)$ in Y be given, and let $M_1, M_2, \ldots, M_n$ be neighbourhoods of $p_1, p_2, \ldots, p_n$ respectively whose Cartesian product is mapped by φ into the open neighbourhood W of $\varphi(p)$. Then there exist open sets V_i in the topological spaces X_i such that $p_i \in V_i$ and $V_i \subset M_i$ for $i = 1, 2, \ldots, n$. Then

$$V_1 \times V_2 \times \cdots \times V_n \subset \varphi^{-1}(W).$$

It follows from this that the preimage $\varphi^{-1}(W)$ of the open neighbourhood W of $\varphi(p)$ is a neighbourhood of the point p in the Cartesian product space X (see Lemma 2.3). We have now shown that the preimage of any open neighbourhood of the point $\varphi(p)$ in Y is a neighbourhood of the point p. It follows that the function φ is continuous at the point p. This completes the proof. ∎

2.3 Continuity of Maps into Product Spaces

Theorem 2.5 *Let $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are topological spaces and X is given the product topology, and for each i, let $\pi_i\colon X \to X_i$ denote the projection function which sends each point $(p_1, p_2, \ldots, p_n)$ of the product space X to its ith component p_i. Then the functions $\pi_1, \pi_2, \ldots, \pi_n$ are continuous. Moreover a function $\varphi\colon Z \to X$ mapping a topological space Z into X is continuous if and only if $\pi_i \circ \varphi\colon Z \to X_i$ is continuous for $i = 1, 2, \ldots, n$.*

Proof Let V_i be an open set in X_i for some integer i between 1 and n. Then

$$\pi_i^{-1}(V_i) = X_1 \times \cdots \times X_{i-1} \times V_i \times X_{i+1} \times \cdots \times X_n.$$

It follows that $\pi_i^{-1}(V_i)$, being a product of open sets, is itself an open set in X (Lemma 2.2). Thus, for each integer i between 1 and n, the preimage under π_i of any open set in the topological space X_i is open in the product space X, and thus the projection function $\pi_i\colon X \to X_i$ is continuous.

Now let $\varphi\colon Z \to X$ be a continuous function mapping some topological space Z into the product space X. Then, for each integer i between 1 and n, the function $\pi_i \circ \varphi\colon Z \to X_i$ is a composition of continuous functions, and is thus itself continuous.

Conversely suppose that $\varphi\colon Z \to X$ is a function with the property that $\pi_i \circ \varphi$ is continuous for all i. Let W be an open set in X. We must show that $\varphi^{-1}(W)$ is open in Z.

Let q be a point of $\varphi^{-1}(W)$, and let $\varphi(q) = (p_1, p_2, \ldots, p_n)$. Now W is open in X, and therefore there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ respectively such that $p_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset W$. Let

$$N = \varphi_1^{-1}(V_1) \cap \varphi_2^{-1}(V_2) \cap \cdots \cap \varphi_n^{-1}(V_n),$$

where $\varphi_i = \pi_i \circ \varphi$ for $i = 1, 2, \ldots, n$. Now $\varphi_i^{-1}(V_i)$ is an open subset of Z for $i = 1, 2, \ldots, n$, since V_i is open in X_i and $\varphi_i\colon Z \to X_i$ is continuous. Thus N, being a finite intersection of open sets, is itself open in Z. Moreover

$$\varphi(N) \subset V_1 \times V_2 \times \cdots \times V_n \subset W,$$

so that $N \subset \varphi^{-1}(W)$. It follows that the preimage $\varphi^{-1}(W)$ of the open subset W of the product space is a neighbourhood of the point q in Z. But q was an arbitrary point of $\varphi^{-1}(W)$. We conclude therefore that the preimage $\varphi^{-1}(W)$ of W under the function φ is a neighbourhood of each of its points, and is therefore an open set in Z (see Lemma 1.4). We have accordingly shown that the function $\varphi\colon Z \to X$ is continuous, as required. ∎

Proposition 2.6 *Let $X_1, X_2, \ldots, X_n$ be topological spaces, where $n > 2$. Then the product $X_1 \times X_2 \times \cdots \times X_n$ of these topological spaces X_i (with its product topology) is naturally homeomorphic to the product (with the product topology) of the product space $X_1 \times X_2 \times \cdots \times X_{n-1}$ (with its product topology) and the topological space X_n.*

Remark The term *natural* has a technical meaning, in the context of category theory, which we ignore for the purposes of the present discussion, but which is nevertheless valid in the present context, where we take the word to suggest, informally, that the homeomorphism in question is canonical and not arbitrary.

Proof Let functions

$$\lambda\colon (X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n \to X_1 \times X_2 \times \cdots \times X_n$$

and

$$\mu\colon X_1 \times X_2 \times \cdots \times X_n \to (X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$$

be defined so that

$$\lambda((p_1, p_2, \ldots, p_{n-1}), p_n) = (p_1, p_2, \ldots, p_n)$$

and

$$\mu(p_1, p_2, \ldots, p_n) = ((p_1, p_2, \ldots, p_{n-1}), p_n)$$

for all $(p_1, p_2, \ldots, p_n) \in X_1 \times X_2 \times \cdots \times X_n$.

We verify the continuity of the functions λ and μ through repeated applications of Theorem 2.5. For convenience, let

$$\begin{aligned} X &= X_1 \times X_2 \times \cdots \times X_n, \\ Y &= X_1 \times X_2 \times \cdots X_{n-1} \end{aligned}$$

and $Z = Y \times X_n$. The functions $\lambda\colon Z \to X$ and $\mu\colon X \to Z$ are obviously bijections which are inverses of one another. Thus our task is to establish that both of these bijections are continuous.

Now the projection function from Z to Y that maps each element of Z of the form $((p_1, p_2, \ldots, p_{n-1}), p_n)$ to its first component $(p_1, p_2, \ldots, p_{n-1})$ is a continuous function. Therefore, for each integer i between 1 and $n - 1$, the function from Z to X_i that maps each element $((p_1, p_2, \ldots, p_{n-1}), p_n)$ of Z to the ith component p_i of its first component is the composition of two continuous functions, and is therefore continuous. The function from Z to X_n mapping each element $((p_1, p_2, \ldots, p_{n-1}), p_n)$ of Z to its second component p_n is also continuous. Thus the components of the function $\lambda\colon Z \to X$ are continuous functions, and therefore the function $\lambda\colon Z \to X$ is a continuous function from Z to X.

Next we note that if, for any integer i between 1 and $n - 1$, the projection function from X to Y mapping $(p_1, p_2, \ldots, p_n)$ to $(p_1, p_2, \ldots, p_{n-1})$ for all $(p_1, p_2, \ldots, p_n) \in X$ is composed with the projection function mapping $(p_1, p_2, \ldots, p_{n-1})$ to p_i, then the resultant function is the projection function from X to X_i, which, as we have already noted (Theorem 2.5), is continuous. It follows from this that the projection function from X to Y mapping $(p_1, p_2, \ldots, p_n)$ to $(p_1, p_2, \ldots, p_{n-1})$ for all $(p_1, p_2, \ldots, p_n) \in X$ is itself continuous. And also the projection function mapping $(p_1, p_2, \ldots, p_n)$ to p_n is continuous. Thus the two components of the function $\mu\colon X \to Z$ are continuous. It follows that the function $\mu\colon X \to Z$ is itself continuous. Moreover the function μ is the inverse of the continuous function λ. Therefore the functions λ and μ are both homeomorphisms. The result follows. ∎

2.4 Products of Compact Topological Spaces

Proposition 2.7 *Let X and Y be compact topological spaces. Then the Cartesian product $X \times Y$ of the topological spaces X and Y, with the product topology, is a compact topological space.*

Proof Let $\mathcal{C}$ be a collection of open sets in $X \times Y$ which covers $X \times Y$. Then, for each point (p, q) of $X \times Y$, there exist an open set $D_{p,q}$ in X and an open set $E_{p,q}$ in Y whose Cartesian product $D_{p,q} \times E_{p,q}$ is contained in at least one of the members of the collection $\mathcal{C}$ of open sets. Indeed, because the members of this collection cover $X \times Y$, given a point (p, q) of $X \times Y$, some member W of this collection may be chosen for which $(p, q) \in W$. There will then exist an open set $D_{p,q}$ in X and an open set $E_{p,q}$ in Y for which $D_{p,q} \times E_{p,q} \subset W$.

Now, because the topological space Y is compact, we can associate to each point p of the topological space X a finite set $\Gamma(p)$ of points of Y so as to ensure that

$$Y = \bigcup_{q \in \Gamma(p)} E_{p,q}.$$

For each point p of X, having first determined $\Gamma(p)$, let V_p be the intersection of the open sets $D_{p,q}$ in X for which $q \in \Gamma(p)$.

$$V_p = \bigcap_{q \in \Gamma(p)} D_{p,q}.$$

Then

$$V_p \times Y = \bigcup_{q \in \Gamma(p)} V_p \times E_{p,q} \subset \bigcup_{q \in \Gamma(p)} D_{p,q} \times E_{p,q}.$$

The compactness of X then ensures the existence of a finite set Δ of points of X for which the corresponding open sets V_p with $p \in \Delta$ cover X. Then

$$\begin{aligned} X \times Y &= \bigcup_{p \in \Delta} V_p \times Y \\ &= \bigcup_{p \in \Delta} \bigcup_{q \in \Gamma(p)} V_p \times E_{p,q} \\ &\subset \bigcup_{p \in \Delta} \bigcup_{q \in \Gamma(p)} D_{p,q} \times E_{p,q}. \end{aligned}$$

It follows that

$$X \times Y = \bigcup_{(p,q) \in \Lambda} D_{p,q} \times E_{p,q},$$

where

$$\Lambda = \{(p, q) : p \in \Delta \text{ and } q \in \Gamma(p)\}.$$

The set Λ is a finite set of points of the Cartesian product $X \times Y$. For each $(p, q) \in \Lambda$ there exists a member $W_{p,q}$ of the given collection $\mathcal{C}$ of open sets covering $X \times Y$ for which $D_{p,q} \times E_{p,q} \subset W_{p,q}$. Then the sets $W_{p,q}$ with $(p,q) \in \Lambda$ constitute a finite collection of open sets taken from the collection $\mathcal{C}$ which covers the product space $X \times Y$. We have thus shown that every open cover $\mathcal{C}$ of this product space has a finite subcover. Consequently the product of the compact topological spaces X and Y is indeed compact. This completes the proof. ∎

Corollary 2.8 *A Cartesian product of a finite number of compact topological spaces is itself compact.*

Proof The result for Cartesian products of two compact spaces has already been established (see Proposition 2.7). If the number n of compact spaces constituting the product is greater than two, then a product of n compact spaces (with the product topology) is homeomorphic to a product whose first factor is a product of $n-1$ compact spaces and whose second factor is a compact topological space. (This follows on applying Proposition 2.6.) It therefore follows by induction on n that, for any positive integer n, a any product of n compact topological spaces, with the product topology, is itself a compact topological space, which is what we were required to prove. ∎

Theorem 2.9 *Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.*

Proof Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is Hausdorff, and every compact subset of a Hausdorff space is closed (see Corollary 1.33).

For each positive integer m, let V_m be the open cube consisting of all ordered n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$ with the property that $-m < x_i < m$ for each integer i between 1 and n. Then V_m is open in $\mathbb{R}^n$ for each positive integer m, and the collection consisting of all these open sets V_m as m ranges over the set of positive integers is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset V_{m_1} \cup V_{m_2} \cup \cdots \cup V_{m_k}$. But then $K \subset V_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$. Thus the compact set K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact, by the one-dimensional Heine-Borel Theorem (Theorem 1.27). Moreover the closed cube C is the Cartesian product of n copies of this compact set, and any finite product of compact topological spaces is itself compact (Corollary 2.8). Therefore the closed cube C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 1.28. Thus K is compact, as required. ∎

2.5 Homotopies between Continuous Maps

Definition Let $f: X \to Y$ and $g: X \to Y$ be continuous maps between topological spaces X and Y. The maps f and g are said to be *homotopic* if there exists a continuous map $H: X \times [0,1] \to Y$ such that $H(p,0) = f(p)$ and $H(p,1) = g(p)$ for all $p \in X$. If the maps f and g are homotopic then we denote this fact by writing $f \simeq g$. The map H with the properties stated above is referred to as a *homotopy* between f and g.

Continuous maps f and g from X to Y are homotopic if and only if it is possible to 'continuously deform' the map f into the map g.

Let X and Y be topological spaces. The relation of being homotopic to one another is an equivalence relation on the set of continuous functions from the space X to the space Y. This result will eventually be noted (as Corollary 2.11), but as a corollary of a more general result subsequently to be stated and proved.

It is useful to introduce the concept of homotopy *relative* to a subset of the domain of the functions in question. Homotopies between continuous functions relative to a subset of a common domain are employed in defining many of the basic concepts and invariants that are the subject matter of algebraic topology.

Definition Let X and Y be topological spaces, and let A be a subset of X. Let $f: X \to Y$ and $g: X \to Y$ be continuous maps from X to some topological space Y, where $f|A = g|A$ (i.e., $f(s) = g(s)$ for all $s \in A$). We say that f and g are *homotopic relative to* A (denoted by $f \simeq g \text{ rel } A$) if and only if there exists a (continuous) homotopy $H: X \times [0,1] \to Y$ such that $H(p,0) = f(p)$ and $H(p,1) = g(p)$ for all $p \in X$ and $H(s,t) = f(s) = g(s)$ for all $s \in A$ and $t \in [0,1]$.

Proposition 2.10 *Let X and Y be topological spaces, and let A be a subset of X. The relation of being homotopic relative to the subset A is then an equivalence relation on the set of all continuous maps from X to Y.*

Proof Given $f: X \to Y$, let $H_0: X \times [0,1] \to Y$ be defined so that $H_0(p,t) = f(p)$ for all $p \in X$ and $t \in [0,1]$. Then $H_0(p,0) = H_0(p,1) = f(p)$ for all

$p \in X$ and $H_0(s,t) = f(s)$ for all $s \in A$ and $t \in [0,1]$, and therefore $f \simeq f$ rel A. Thus the relation of homotopy relative to A is reflexive.

Let f and g be continuous maps from X to Y that satisfy $f(s) = g(s)$ for all $s \in A$. Suppose that $f \simeq g$ rel A. Then there exists a homotopy $H\colon X \times [0,1] \to Y$ with the properties that $H(p,0) = f(p)$ and $H(p,1) = g(p)$ for all $p \in X$ and $H(s,t) = f(s) = g(s)$ for all $s \in A$ and $t \in [0,1]$. Let $K\colon X \times [0,1] \to Y$ be defined so that $K(p,t) = H(p,1-t)$ for all $t \in [0,1]$. Then K is a homotopy between g and f, and $K(s,t) = g(s) = f(s)$ for all $s \in A$ and $t \in [0,1]$. It follows that $g \simeq f$ rel A. Thus the relation of homotopy relative to A is symmetric. Finally let f, g and h be continuous maps from X to Y with the property that $f(s) = g(s) = h(s)$ for all $s \in A$. Suppose that $f \simeq g$ rel A and $g \simeq h$ rel A. Then there exist homotopies $H_1\colon X \times [0,1] \to Y$ and $H_2\colon X \times [0,1] \to Y$ satisfying the following properties:

$$\begin{aligned} H_1(p,0) &= f(p), \\ H_1(p,1) &= g(p) = H_2(p,0), \\ H_2(p,1) &= h(p) \end{aligned}$$

for all $p \in X$;

$$H_1(s,t) = H_2(s,t) = f(s) = g(s) = h(s)$$

for all $s \in A$ and $t \in [0,1]$. Define $H\colon X \times [0,1] \to Y$ by

$$H(p,t) = \begin{cases} H_1(p,2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ H_2(p,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now $H|X \times [0,\frac{1}{2}]$ and $H|X \times [\frac{1}{2},1]$ are continuous. It follows from the Pasting Lemma (Lemma 1.25) that H is continuous on $X \times [0,1]$. Moreover $H(p,0) = f(p)$ and $H(p,1) = h(p)$ for all $p \in X$. Thus $f \simeq h$ rel A. Thus the relation of homotopy relative to the subset A of X is transitive. This relation has now been shown to be reflexive, symmetric and transitive. It is therefore an equivalence relation. ∎

Remark Let X and Y be topological spaces, and let $H\colon X \times [0,1] \to Y$ be a function whose restriction to the sets $X \times [0,\frac{1}{2}]$ and $X \times [\frac{1}{2},1]$ is continuous. Then the function H is continuous on $X \times [0,1]$. The Pasting Lemma (Lemma 1.25) was applied in the proof of Proposition 2.10 to justify this assertion. We consider in more detail how the Pasting Lemma guarantees the continuity of this function. Let $p \in X$. If $t \in [0,1]$ and $t \neq \frac{1}{2}$ then the point (p,t) is contained in an open subset of $X \times [0,1]$ over which the function H is continuous, and therefore the function H is continuous at (p,t). In order to complete the proof that the function H is continuous everywhere on $X \times [0,1]$ it suffices to verify continuity of H at $(p,\frac{1}{2})$, where $p \in X$.

Let V be an open set in Y for which $H(p, \frac{1}{2}) \in V$. Then the continuity of the restrictions of H to $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ ensures the existence of open sets W_1 and W_2 in $X \times [0, 1]$ such that $(p, \frac{1}{2}) \in W_1 \cap W_2$, $H(W_1 \cap (X \times [0, \frac{1}{2}])) \subset V$ and $H(W_2 \cap (X \times [\frac{1}{2}, 1])) \subset V$. Let $W = W_1 \cap W_2$. Then $H(W) \subset V$. This completes the verification that the function H is continuous at $(p, \frac{1}{2})$.

The Pasting Lemma is a basic tool for establishing the continuity of functions occurring in algebraic topology that are similar in nature to the function H whose continuity was justified in some detail in the foregoing discussion. The continuity of such functions can typically be established directly using arguments analogous to that employed here.

Corollary 2.11 *Let X and Y be topological spaces. The homotopy relation $\simeq$ is an equivalence relation on the set of all continuous maps from X to Y.*

Proof This result follows on applying Proposition 2.10 in the case where homotopies are relative to the empty set. ∎

Proposition 2.12 *Let X and Y be topological spaces, let $H\colon X \times [0, 1] \to Y$ be a continuous map defined on the product space $X \times [0, 1]$, let p be an element of the topological space X and let τ be a real number satisfying $0 \leq \tau \leq 1$. Then, given any open subset W of Y to which the point $H(p, \tau)$ belongs, there exists a neighbourhood N of p in X and a positive real number δ such that $H(p', \tau') \in W$ for all $p' \in N$ and for all $\tau' \in [0, 1]$ satisfying $\tau - \delta < \tau' < \tau + \delta$.*

The result just stated is nothing more than a special case of Proposition 2.4. ∎

2.6 Identification Maps and Quotient Topologies

Definition Let X and Q be topological spaces and let $\chi\colon X \to Q$ be a function from X to Q. The function χ is said to be an *identification map* if and only if the following conditions are satisfied:

- the function $\chi\colon X \to Q$ is surjective,
- a subset W of Q is open in Q if and only if $\chi^{-1}(W)$ is open in X.

It follows directly from the definition that any identification map is continuous. Moreover, in order to show that a continuous surjection $\chi\colon X \to Q$ is an identification map, it suffices to prove that if W is a subset of Q with the property that $\chi^{-1}(W)$ is open in X then W is open in Q.

Example Let S^1 be the unit circle in $\mathbb{R}^2$, and let $\kappa\colon \mathbb{R} \to S^1$ be the continuous map that sends each real number t to $(\cos 2\pi t, \sin 2\pi t)$. Then $\kappa\colon \mathbb{R} \to S^1$ is an identification map.

Indeed let W be a subset of the circle S^1 whose preimage $\kappa^{-1}(W)$ under the map κ is open in the real line, and let $\mathbf{p}$ be a point on the circle S^1 that belongs to W. Then there exists some real number s for which $\mathbf{p} = \kappa(s)$. Then $s \in \kappa^{-1}(W)$, and $\kappa^{-1}(W)$ is open in $\mathbb{R}$, by assumption. Therefore there exists some positive real number δ for which the open interval $(s - \delta, s + \delta)$ is contained in W. Then κ maps that open interval either to an open arc in the circle S^1 that contains the point $\mathbf{p}$ or else (in the case when $\delta > \frac{1}{2}$) to the entire circle. It follows in either case that the set W contains some open neighbourhood of the point $\mathbf{p}$, and is thus itself a neighbourhood of $\mathbf{p}$.

This argument shows that the subset W of the circle S^1 is a neighbourhood of each of its points. It is therefore open in the circle (see Lemma 1.4). Thus if W is a subset of the circle S^1 and if the preimage $\kappa^{-1}(W)$ of W under the map κ is open in the real line $\mathbb{R}$ then W itself is open in the circle S^1. Conversely if W is a subset of the circle S^1 which is open in the circle, then the continuity of the map κ ensures that the preimage $\kappa^{-1}(W)$ of W under the map κ is open in the real line. It follows that the surjective map $\kappa\colon \mathbb{R} \to S^1$ is indeed an identification map.

Example Let S^1 be the unit circle in $\mathbb{R}^2$, and let $\eta\colon [0,1] \to S^1$ be the continuous map that sends each real number t in the closed bounded interval $[0,1]$ to $(\cos 2\pi t, \sin 2\pi t)$. Then $\eta\colon [0,1] \to S^1$ is an identification map.

Let W be a subset of the circle S^1 whose preimage $\eta^{-1}(W)$ under the function η is open in the closed unit interval $[0,1]$. Let $\mathbf{p}$ be a point of the circle belonging to W which is distinct from the point $(1,0)$. Then there exists some real number s satisfying $0 < s < 1$ for which $\eta(s) = \mathbf{p}$. Now $\eta^{-1}(W)$ is open in $[0,1]$, by assumption. It follows that there exists some positive real number δ satisfying the inequalities $0 < s - \delta < s + \delta < 1$ for which the open interval $(s-\delta, s+\delta)$ is contained in $\eta^{-1}(W)$. Then the image of this open interval under the map η is an open set contained in W to which the point $\mathbf{p}$ belongs. It follows that W is a neighbourhood of any point of W that is distinct from the point $(1,0)$ of the circle to which the endpoints of the closed unit interval $[0,1]$ are sent by the map η.

Now suppose that the point $\mathbf{p}_0$ belongs to W, where $\mathbf{p}_0 = (1,0)$. We show that W is a neighbourhood, in the circle S^1, of the point $\mathbf{p}_0$.

Now the points of the closed unit interval $[0,1]$ that are mapped by η to the point $\mathbf{p}_0$ are the endpoints 0 and 1 of the closed unit interval. Now the preimage $\eta^{-1}(W)$ of W under the map η is assumed to be open in the closed unit interval $[0,1]$. The definition of the subspace topology on $[0,1]$ then ensures the existence of real numbers δ_0 and δ_1 with values strictly between 0 and $\frac{1}{2}$ for which $[0, \delta_0) \subset \eta^{-1}(W)$ and $(1-\delta_1, 1] \subset \eta^{-1}(W)$. Then the set W contains the open arc in the circle with endpoints $\eta(1-\delta_1)$ and $\eta(\delta_0)$ that contains the point $\mathbf{p}_0$. It follows that the set W is a neighbourhood of the point $\mathbf{p}_0$ in the circle, as previously claimed.

We have now shown that if W is a subset of the circle S^1 whose preimage $\eta^{-1}(W)$ under the continuous map η is open in the closed unit interval $[0, 1]$ then W itself is a neighbourhood of each of its points. Consequently if $\eta^{-1}(W)$ is open in $[0, 1]$ then W itself is open in the circle S^1. Conversely if W is a subset of the circle that is open in the circle, then the continuity of the map $\eta\colon [0, 1] \to S^1$ ensures that the preimage W under the map η is open in the closed unit interval. It follows that the surjective map $\eta\colon [0, 1] \to S^1$ is indeed an identification map.

Lemma 2.13 *Let X be a topological space, let Q be a set, and let $\chi\colon X \to Q$ be a surjection. Then there is a unique topology on Q that ensures that the function $\chi\colon X \to Q$ mapping the topological space X onto Q is an identification map.*

Proof Let τ be the collection consisting of all subsets W of Q for which $\chi^{-1}(W)$ is open in X. Now $\chi^{-1}(\emptyset) = \emptyset$, and $\chi^{-1}(Q) = X$. Thus the empty set $\emptyset$ and the whole set Q both belong to the collection τ.

Now, given any collection of subsets of Q, the preimage, under the function χ, of the union of those sets is the union of the preimages of the sets (Lemma 1.9). Also the preimages under χ of sets belonging to the collection τ are open sets in X. It follows that, the preimage of any union of subsets of Q belonging to the collection τ is a union of open sets in X, and must therefore itself be an open set in X. Consequently any union of subsets of Q belonging to the collection τ must itself belong to that collection τ.

Furthermore, given any collection of subsets of Q, the preimage, under the function χ, of the intersection of those sets is the intersection of the preimages of the sets (Lemma 1.10). We have moreover already noted that the preimages under χ of sets belonging to the collection τ are open sets in X. It follows that, the preimage of any finite intersection of subsets of Q belonging to the collection τ is a finite intersection of open sets in X, and must therefore itself be an open set in X. Consequently any finite intersection of subsets of Q belonging to the collection τ must itself belong to that collection τ.

We have now shown that the empty set and the whole of the set Q belong to the collection τ, the union of any collection of subsets of Q belonging to τ must itself belong to the collection τ, and intersection of any finite collection of subsets of Q belonging to τ must itself belong to the collection τ. Consequently τ is a topology on Q. Moreover the definition of this topology ensures that the map $\chi\colon X \to Q$ mapping the topological space X onto Q is an identification map when the topology on the set Q is the topology τ. Now the very definition of quotient topologies ensures that if the function mapping the topological space X onto Q is to be an identification map, then the open sets in Q must be those whose preimages are open in the topological space X. It follows that τ is the unique topology on Q that ensures that the function χ mapping the topological space X onto Q is an identification map. ∎

Definition Let X be a topological space, let Q be a set, and let $\chi\colon X \to Q$ be a surjection. The unique topology on Q that ensures that the function χ is an identification map is referred to as the *quotient topology* (or *identification topology*) on Q.

Lemma 2.14 *Let X and Q be topological spaces and let $\chi\colon X \to Q$ be an identification map. Let Z be a topological space, and let $\psi\colon Q \to Z$ be a function from Q to Z. Then the function ψ is continuous if and only if the composition function $\psi \circ \chi\colon X \to Z$ is continuous.*

Proof Suppose that ψ is continuous. Then the composition function $\psi \circ \chi$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $\psi \circ \chi$ is continuous. Let V be an open set in Z. Then $\chi^{-1}(\psi^{-1}(V))$ is open in X, because this subset of X is the preimage of the open set V under the composition function $\psi \circ \chi$, and that composition function is assumed to be continuous. It follows that $\psi^{-1}(V)$ is open in Q, because the function χ is an identification map. Therefore the function ψ is continuous, as required. ∎

Example Let S^n be the n-sphere, consisting of all points $\mathbf{p}$ in $\mathbb{R}^{n+1}$ satisfying $|\mathbf{p}| = 1$. Let $\mathbb{R}P^n$ be the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin (i.e., $\mathbb{R}P^n$ is the set of all one-dimensional vector subspaces of $\mathbb{R}^{n+1}$).

Let $\chi\colon S^n \to \mathbb{R}P^n$ be the function which sends a point $\mathbf{p}$ of S^n to the element of $\mathbb{R}P^n$ represented by the line in $\mathbb{R}^{n+1}$ that passes through both $\mathbf{p}$ and the origin. Note that each element of the set $\mathbb{R}P^n$ is the image (under χ) of exactly two antipodal points $\mathbf{p}$ and $-\mathbf{p}$ of S^n. The function χ induces a corresponding quotient topology on $\mathbb{R}P^n$ which ensures that the surjective function $\chi\colon S^n \to \mathbb{R}P^n$ is an identification map. The set of lines in $(n+1)$-dimensional Euclidean space that pass through the centre of the unit sphere, with the quotient topology just described, is the topological space referred to as n-dimensional real projective space.

The space $\mathbb{R}P^2$ is then the image of the two-dimensional sphere S^2 under the identification map just described that identifies pairs of antipodal points on the sphere. This topological space is referred to as the *real projective plane*.

Note that a function $\psi\colon \mathbb{R}P^n \to Z$ mapping $\mathbb{R}P^n$ into a topological space Z is continuous if and only if the composition function $\psi \circ \chi\colon S^n \to Z$ is continuous. (This follows on applying Lemma 2.14.)

Proposition 2.15 *A continuous surjection $\varphi\colon X \to Q$ from a compact topological space X to a Hausdorff space Q is an identification map.*

Proof Let W be a subset of the Hausdorff space Q. The surjectivity of the map φ ensures that $Q \setminus W = \varphi(\varphi^{-1}(Q \setminus W))$. It follows that

$$Q \setminus W = \varphi(\varphi^{-1}(Q \setminus W)) = \varphi(X \setminus \varphi^{-1}(W)),$$

because the preimage of the complement in Q of the subset W of Q is the complement in X of the preimage of W under the map φ (see Lemma 1.11).

Now suppose that the preimage $\varphi^{-1}(W)$ of W under the map φ is an open set in X. Then its complement is closed in X. But the topological space X is compact, and any closed subset of a compact topological space is itself compact. It follows that $X \setminus \varphi^{-1}(W)$ is a compact set. Now continuous functions map compact sets to compact sets. It follows that the complement $Q \setminus W$ in Q of the subset W of Q is a compact set, being the image of the compact set $X \setminus \varphi^{-1}(W)$ under the continuous map φ. Now compact subsets of Hausdorff spaces are closed. It follows therefore that $Q \setminus W$ is closed in Q, and therefore the set W itself is open in Q. Thus the preimage of a subset of Q under the map φ is open in the topological space X then that subset W of Q is open in Q. We conclude therefore that a continuous surjection $\varphi\colon X \to Q$, mapping a compact space onto a Hausdorff space must necessarily be an identification map, which is what we were required to prove. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined so that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and let $\eta\colon [0, 1] \to S^1$ be defined so that $\eta(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0, 1]$. It has been shown that the map η is an identification map. This also follows directly from the fact that $\eta\colon [0, 1] \to S^1$ is a continuous surjection from the compact space $[0, 1]$ to the Hausdorff space S^1.