Module MAU34201: Algebraic Topology I
Michaelmas Term 2022
Section 6: Discontinuous Group Actions and Orbit Spaces

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6 Discontinuous Group Actions and Orbit Spaces

6.1 Path-Lifting and the Fundamental Group

Let $\tilde{X}$ and X be topological spaces, let $\rho\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X, and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in the base space X which both start at some point b_0 of X and finish at some point b_1 of X, so that

$$\alpha(0) = \beta(0) = b_0 \quad \text{and} \quad \alpha(1) = \beta(1) = b_1.$$

Let $\tilde{b}_0$ be some point of the covering space $\tilde{X}$ that projects down to b_0, so that $\rho(\tilde{b}_0) = b_0$. It follows from the Path-Lifting Theorem (Theorem 4.13) that there exist paths $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ and $\tilde{\beta}\colon [0,1] \to \tilde{X}$ in the covering space $\tilde{X}$ that both start at $\tilde{b}_0$ and are lifts of the paths α and β respectively. Thus

$$\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{b}_0,$$

$$\rho(\tilde{\alpha}(t)) = \alpha(t) \quad \text{and} \quad \rho(\tilde{\beta}(t)) = \beta(t) \quad \text{for all } t \in [0,1].$$

These lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting point $\tilde{b}_0$ (see Proposition 4.11).

Now, though the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β have been chosen such that they start at the same point $\tilde{b}_0$ of the covering space $\tilde{X}$, they need not in general end at the same point of $\tilde{X}$. However we shall prove that if $\alpha \simeq \beta \text{ rel } \{0,1\}$, then the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β respectively that both start at some point $\tilde{b}_0$ of $\tilde{X}$ will both finish at some point $\tilde{b}_1$ of $\tilde{X}$, so that $\tilde{\alpha}(1) = \tilde{\beta}(1) = \tilde{b}_1$. This result is established in Proposition 6.1 below.

Proposition 6.1 *Let $\tilde{X}$ and X be topological spaces, and let $\rho\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Also let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in X, where $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, and let $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ and $\tilde{\beta}\colon [0,1] \to \tilde{X}$ be paths in $\tilde{X}$ such that $\rho \circ \tilde{\alpha} = \alpha$ and $\rho \circ \tilde{\beta} = \beta$. Suppose that $\tilde{\alpha}(0) = \tilde{\beta}(0)$ and that $\alpha \simeq \beta \text{ rel } \{0,1\}$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{\alpha} \simeq \tilde{\beta} \text{ rel } \{0,1\}$.*

Proof Let b_0 and b_1 be the points of X given by

$$b_0 = \alpha(0) = \beta(0), \qquad b_1 = \alpha(1) = \beta(1).$$

Now $\alpha \simeq \beta \text{ rel } \{0,1\}$, and therefore there exists a homotopy $F\colon [0,1] \times [0,1] \to X$ such that

$$F(t,0) = \alpha(t) \quad \text{and} \quad F(t,1) = \beta(t) \quad \text{for all } t \in [0,1],$$

and

$$F(0,\tau) = b_0 \quad \text{and} \quad F(1,\tau) = b_1 \quad \text{for all } \tau \in [0,1].$$

It then follows from the Homotopy-Lifting Theorem (Theorem 4.14) that there exists a continuous map $G\colon [0,1] \times [0,1] \to \tilde{X}$ such that $\rho \circ G = F$ and $G(0,0) = \tilde{\alpha}(0)$. Then $\rho(G(0,\tau)) = b_0$ and $\rho(G(1,\tau)) = b_1$ for all $\tau \in [0,1]$. A straightforward application of Proposition 4.11 shows that any continuous lift of a constant path must itself be a constant path. Therefore $G(0,\tau) = \tilde{b}_0$ and $G(1,\tau) = \tilde{b}_1$ for all $\tau \in [0,1]$, where

$$\tilde{b}_0 = G(0,0) = \tilde{\alpha}(0), \qquad \tilde{b}_1 = G(1,0).$$

However

$$G(0,0) = G(0,1) = \tilde{b}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0).$$

Also

$$\rho(G(t,0)) = F(t,0) = \alpha(t) = \rho(\tilde{\alpha}(t))$$

and

$$\rho(G(t,1)) = F(t,1) = \beta(t) = \rho(\tilde{\beta}(t))$$

for all $t \in [0,1]$. It follows that the map that sends $t \in [0,1]$ to $G(t,0)$ is a lift of the path α that starts at $\tilde{b}_0$, and the map that sends $t \in [0,1]$ to $G(t,1)$ is a lift of the path β that also starts at $\tilde{b}_0$.

However the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting points (see Proposition 4.11). It follows that $G(t,0) = \tilde{\alpha}(t)$ and $G(t,1) = \tilde{\beta}(t)$ for all $t \in [0,1]$. In particular,

$$\tilde{\alpha}(1) = G(1,0) = \tilde{b}_1 = G(1,1) = \tilde{\beta}(1).$$

Moreover the map $G\colon [0,1] \times [0,1] \to \tilde{X}$ is a homotopy between the paths $\tilde{\alpha}$ and $\tilde{\beta}$ which satisfies $G(0,\tau) = \tilde{b}_0$ and $G(1,\tau) = \tilde{b}_1$ for all $\tau \in [0,1]$. It follows that $\tilde{\alpha} \simeq \tilde{\beta}$ rel $\{0,1\}$, as required. ∎

Let $\tilde{X}$ and X be topological spaces, and let $\rho\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Also let $\tilde{b}_0$ be a point of the covering space $\tilde{X}$, and let $b_0 = \rho(\tilde{b}_0)$. Then the covering map ρ induces a group homomorphism

$$\rho_\#\colon \pi_1(\tilde{X}, \tilde{b}_0) \to \pi_1(X, b_0)$$

from the fundamental group $\pi_1(\tilde{X}, \tilde{b}_0)$ of the covering space with basepoint $\tilde{b}_0$ to the fundamental group $\pi_1(X, b_0)$ of the base space with basepoint b_0. This induced homomorphism $\rho_\#$ is defined so that $\rho_\#[\tilde{\gamma}] = [\rho \circ \tilde{\gamma}]$ for all loops $\tilde{\gamma}$ in the covering space $\tilde{X}$ based at the point $\tilde{b}_0$ (see Proposition 5.2).

Proposition 6.2 *Let $\tilde{X}$ and X be topological spaces, and let $\rho\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Also let $\tilde{b}_0$ be a point of the covering space $\tilde{X}$, and let $b_0 = \rho(\tilde{b}_0)$. Then the homomorphism*

$$\rho_\#\colon \pi_1(\tilde{X}, \tilde{b}_0) \to \pi_1(X, b_0)$$

of fundamental groups induced by the covering map ρ is injective.

Proof Let σ_0 and σ_1 be loops in $\tilde{X}$ based at the point $\tilde{b}_0$, representing elements $[\sigma_0]$ and $[\sigma_1]$ of $\pi_1(\tilde{X}, \tilde{b}_0)$. Suppose that $\rho_\#[\sigma_0] = \rho_\#[\sigma_1]$. Then $\rho \circ \sigma_0 \simeq \rho \circ \sigma_1$ rel $\{0, 1\}$. Also $\sigma_0(0) = \tilde{b}_0 = \sigma_1(0)$. It therefore follows (on applying Proposition 6.1) that $\sigma_0 \simeq \sigma_1$ rel $\{0, 1\}$, and thus $[\sigma_0] = [\sigma_1]$. We conclude therefore that the homomorphism $\rho_\#\colon \pi_1(\tilde{X}, \tilde{b}_0) \to \pi_1(X, b_0)$ is injective. ∎

Proposition 6.3 *Let $\tilde{X}$ and X be topological spaces, and let $\rho\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Also let $\tilde{b}_0$ be a point of the covering space $\tilde{X}$, let $b_0 = \rho(\tilde{b}_0)$, and let γ be a loop in X based at b_0. Then $[\gamma] \in \rho_\#\bigl(\pi_1(\tilde{X}, \tilde{b}_0)\bigr)$ if and only if there exists a loop $\tilde{\gamma}$ in $\tilde{X}$, based at the point $\tilde{b}_0$, such that $\rho \circ \tilde{\gamma} = \gamma$.*

Proof If $\gamma = \rho \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{b}_0$ then $[\gamma] = \rho_\#[\tilde{\gamma}]$, and therefore $[\gamma] \in \rho_\#\bigl(\pi_1(\tilde{X}, \tilde{b}_0)\bigr)$.

Conversely suppose that $[\gamma] \in \rho_\#\bigl(\pi_1(\tilde{X}, \tilde{b}_0)\bigr)$. We must show that there exists some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{b}_0$ such that $\gamma = \rho \circ \tilde{\gamma}$. Now there exists a loop σ in $\tilde{X}$ based at the point $\tilde{b}_0$ such that $[\gamma] = \rho_\#([\sigma])$ in $\pi_1(X, b_0)$. Then $\gamma \simeq \rho \circ \sigma$ rel $\{0, 1\}$. It follows from the Path-Lifting Theorem for covering maps (Theorem 4.13) that there exists a unique path $\tilde{\gamma}\colon [0, 1] \to \tilde{X}$ in $\tilde{X}$ for which $\tilde{\gamma}(0) = \tilde{b}_0$ and $\rho \circ \tilde{\gamma} = \gamma$. It then follows from Proposition 6.1 that $\tilde{\gamma}(1) = \sigma(1)$ and $\tilde{\gamma} \simeq \sigma$ rel $\{0, 1\}$. But $\sigma(1) = \tilde{b}_0$. Therefore the path $\tilde{\gamma}$ is the required loop in $\tilde{X}$ based the point $\tilde{b}_0$ which satisfies $\rho \circ \tilde{\gamma} = \gamma$. ∎

Corollary 6.4 *Let $\tilde{X}$ and X be topological spaces, and let $\rho\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Also let q_0 and q_1 be points of $\tilde{X}$ satisfying $\rho(q_0) = \rho(q_1)$, and let $\eta\colon [0, 1] \to \tilde{X}$ be a path in $\tilde{X}$ from q_0 to q_1. Suppose that $[\rho \circ \eta] \in \rho_\#\bigl(\pi_1(\tilde{X}, q_0)\bigr)$. Then the path η is a loop in $\tilde{X}$, and thus $q_0 = q_1$.*

Proof It follows from Proposition 6.3 that there exists a loop σ based at q_0 satisfying $\rho \circ \sigma = \rho \circ \eta$. Then $\eta(0) = \sigma(0)$. Now Proposition 4.11 ensures that the lift to $\tilde{X}$ of any path in X is uniquely determined by its starting point. It follows that $\eta = \sigma$. But then the path η must be a loop in $\tilde{X}$, and therefore $q_0 = q_1$, as required. ∎

Theorem 6.5 *Let $\tilde{X}$ and X be topological spaces and let $\rho\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Suppose that $\tilde{X}$ is path-connected and that X is simply connected. Then the covering map $\rho\colon \tilde{X} \to X$ is a homeomorphism.*

Proof We show that the map $\rho\colon \tilde{X} \to X$ is a bijection. This map is surjective (because covering maps are by definition surjective). We must show that it is injective. Let q_0 and q_1 be points of $\tilde{X}$ with the property that $\rho(q_0) = \rho(q_1)$. Then there exists a path $\eta\colon [0,1] \to \tilde{X}$ with $\eta(0) = q_0$ and $\eta(1) = q_1$, because the covering space $\tilde{X}$ is path-connected. Then $\rho \circ \eta$ is a loop in X based at the point b_0, where $b_0 = \rho(q_0)$. However $\pi_1(X, b_0)$ is the trivial group, because X is simply connected. It follows from Corollary 6.4 that the path η is a loop in $\tilde{X}$ based at q_0, and therefore $q_0 = q_1$. This shows that the covering map $\rho\colon \tilde{X} \to X$ is injective.

Accordingly the map $\rho\colon \tilde{X} \to X$ is a bijection. But any bijective covering map is a homeomorphism (Corollary 4.8). The result follows. ∎

6.2 Discontinuous Group Actions

Definition Let G be a group, and let X be a set. The group G is said to *act* on the set X (on the left) if each element g of G determines a corresponding function $\theta_g\colon X \to X$ from the set X to itself, where

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the function θ_e determined by the identity element e of G is the identity function of X.

Let G be a group acting on a set X. Given any element p of X, the *orbit* $[p]_G$ of p (under the group action) is defined to be the subset $\{\theta_g(p) : g \in G\}$ of X, and the *stabilizer* of p is defined to the subgroup $\{g \in G : \theta_g(p) = p\}$ of the group G. Thus the orbit of an element p of X is the set consisting of all points of X to which p gets mapped under the action of elements of the group G. The stabilizer of p is the subgroup of G consisting of all elements of this group that fix the point p. The group G is said to act *freely* on X if $\theta_g(p) \neq p$ for all $p \in X$ and $g \in G$ satisfying $g \neq e$. Thus the group G acts freely on X if and only if the stabilizer of every element of X is the trivial subgroup of G.

Let e be the identity element of G. Then $p = \theta_e(p)$ for all $p \in X$, and therefore $p \in [p]_G$ for all $p \in X$, where $[p]_G = \{\theta_g(p) : g \in G\}$.

Let p and q be elements of X for which $[p]_G \cap [q]_G$ is non-empty, and let $r \in [p]_G \cap [q]_G$. Then there exist elements h and k of G such that $r = \theta_h(p) = \theta_k(q)$. Then $\theta_g(r) = \theta_{gh}(p) = \theta_{gk}(q)$, $\theta_g(p) = \theta_{gh^{-1}}(r)$ and $\theta_g(q) = \theta_{gk^{-1}}(r)$

for all $g \in G$. Therefore $[p]_G = [r]_G = [q]_G$. It follows from this that the group action partitions the set X into orbits, so that each element of X determines an orbit which is the unique orbit for the action of G on X to which it belongs. We denote by X/G the set of orbits for the action of G on X.

Now suppose that the group G acts on a topological space X. Then there is a surjective function $\rho\colon X \to X/G$, where $\rho(p) = [p]_G$ for all $p \in X$. This surjective function induces a quotient topology on the set of orbits: a subset W of X/G is open in this quotient topology if and only if $\rho^{-1}(W)$ is an open set in X (see Lemma 2.13). We define the *orbit space* X/G for the action of G on X to be the topological space whose underlying set is the set of orbits for the action of G on X, the topology on X/G being the quotient topology induced by the function $\rho\colon X \to X/G$. This function $\rho\colon X \to X/G$ is then an identification map: we shall refer to it as the *quotient map* from X to X/G.

We shall be concerned here with situations in which a group action on a topological space gives rise to a covering map. The relevant group actions are those where the group acts *freely and properly discontinuously* on the topological space.

Definition Let G be a group with identity element e, and let X be a topological space. The group G is said to act *freely and properly discontinuously* on X if each element g of G determines a corresponding continuous map $\theta_g\colon X \to X$, where the following conditions are satisfied:

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the continuous map θ_e determined by the identity element e of G is the identity map of X;

(iii) given any point p of X, there exists an open set V in X such that $p \in V$ and $\theta_g(V) \cap V = \emptyset$ for all $g \in G$ satisfying $g \neq e$.

Let G be a group which acts freely and properly discontinuously on a topological space X. Given any element g of G, the corresponding continuous function $\theta_g\colon X \to X$ determined by g is a homeomorphism. Indeed it follows from conditions (i) and (ii) in the above definition that $\theta_{g^{-1}} \circ \theta_g$ and $\theta_g \circ \theta_{g^{-1}}$ are both equal to the identity map of X, and therefore $\theta_g\colon X \to X$ is a homeomorphism with inverse $\theta_{g^{-1}}\colon X \to X$.

Remark The terminology 'freely and properly discontinuously' is traditional, but is hardly ideal. The adverb 'freely' refers to the requirement that $\theta_g(p) \neq p$ for all $p \in X$ and for all $g \in G$ satisfying $g \neq e$. The adverb 'discontinuously' refers to the fact that, given any point x of X, the elements of the orbit $\{\theta_g(p) : g \in G\}$ of p are separated; it does not signify that the functions defining the action are in any way discontinuous or badly-behaved. The adverb 'properly' refers to the fact that, given any compact subset K of X, the number of elements g of the group G for which $K \cap \theta_g(K) \neq \emptyset$ is finite. Moreover the definitions of *properly discontinuous actions* in textbooks and in sources of reference are not always in agreement: some say that an action of a group G on a topological space X (where each group element determines a corresponding homeomorphism of the topological space) is *properly discontinuous* if, given any $p \in X$, there exists an open set V in X such that the number of elements g of the group for which $g(V) \cap V \neq \emptyset$ is finite; others say that the action is *properly discontinuous* if it satisfies the conditions given in the definition above for a group acting freely and properly discontinuously on the set. William Fulton, in his textbook *Algebraic topology: a first course* (Springer, 1995), introduced the term 'evenly' in place of 'freely and properly discontinuously', but this change in terminology does not appear to have been generally adopted.

6.3 Orbit Spaces

Example The cyclic group C_2 of order 2 consists of a set $\{e, a\}$ with two elements e and a, together with a group multiplication operation defined so that $e^2 = a^2 = e$ and $ea = ae = a$. The identity element of C_2 is thus e.

Let us represent the n-dimensional sphere S^n as the unit sphere in $\mathbb{R}^{n+1}$ centred on the origin. Let $\theta_e\colon S^n \to S^n$ be the identity map of S^n and let $\theta_a\colon S^n \to S^n$ be the antipodal map of S^n, defined such that $\theta_a(\mathbf{p}) = -\mathbf{p}$ for all $\mathbf{p} \in S^n$. Then the group C_2 acts on S^n (on the left) so that elements e and a of S^n correspond under this action to the homeomorphisms θ_e and θ_a respectively. Points $\mathbf{p}$ and $\mathbf{q}$ are said to be *antipodal* to one another if and only if $\mathbf{q} = -\mathbf{p}$. Each orbit for the action of C_2 on S^n thus consists of a pair of antipodal points on S^n.

Let $\mathbf{n}$ be a point on the n-dimensional sphere S^n, and let

$$V = \{\mathbf{p} \in S^n : \mathbf{p} \,.\, \mathbf{n} > 0\}.$$

Then V is open in S^n and $\mathbf{n} \in V$. Also

$$\theta_a(V) = \{\mathbf{p} \in S^n : \mathbf{p} \,.\, \mathbf{n} < 0\},$$

and therefore $V \cap \theta_a(V) = \emptyset$. Consequently the group C_2 acts freely and properly discontinuously on S^n.

Distinct points of S^n belong to the same orbit under the action of C_2 on S^n if and only if the line in $\mathbb{R}^{n+1}$ passing through those points also passes through the origin. It follows that lines in $\mathbb{R}^{n+1}$ that pass through the origin are in one-to-one correspondence with orbits for the action of C_2 on S^n. The orbit space S^n/C_2 thus represents the set of lines through the origin in $\mathbb{R}^{n+1}$. We define n-dimensional *real projective space* $\mathbb{R}P^n$ to be the topological space whose elements are the lines in $\mathbb{R}^{n+1}$ passing through the origin, with the topology obtained on identifying $\mathbb{R}P^n$ with the orbit space S^n/C_2. The quotient map $\rho\colon S^n \to \mathbb{R}P^n$ then sends each point $\mathbf{p}$ of S^n to the orbit consisting of the two points $\mathbf{p}$ and $-\mathbf{p}$. Thus each pair of antipodal points on the n-dimenionsional sphere S^n determines a single point of n-dimensional real projective space $\mathbb{R}P^n$.

Proposition 6.6 *Let G be a group acting freely and properly discontinuously on a topological space X, let X/G denote the resulting orbit space, and let $\rho\colon X \to X/G$ be the quotient map that sends each element of X to its orbit under the action of the group G. Let $\varphi\colon X \to Y$ be a continuous surjective map from X to a topological space Y. Suppose that elements p and q of X satisfy $\varphi(p) = \varphi(q)$ if and only if $\rho(p) = \rho(q)$. Suppose also $\varphi(V)$ is open in Y for every open set V in X. Then the surjective continuous map $\varphi\colon X \to Y$ induces a homeomorphism $\psi\colon X/G \to Y$ between the topological spaces X/G and Y, where $\psi(\rho(p)) = \varphi(p)$ for all $p \in X$.*

Proof The function $\psi\colon X/G \to Y$ is continuous because $\varphi\colon X \to Y$ is continuous and $\rho\colon X \to Y$ is a quotient map (see Lemma 2.14). Moreover it is surjective because $\varphi\colon X \to Y$ is surjective, and it is injective because elements p and q satisfy $\varphi(p) = \varphi(q)$ if and only if $\rho(p) = \rho(q)$. It follows that $\psi\colon X/G \to Y$ is a bijection.

Let W be an open set in X/G. It follows from the definition of the quotient topology that $\rho^{-1}(W)$ is open in X. The map φ maps open sets to open sets. Therefore $\varphi(\rho^{-1}(W))$ is open in Y. But $\varphi(\rho^{-1}(W)) = \psi(W)$. Thus $\psi(W)$ is open in Y for every open set W in X/G, and therefore the inverse of the map ψ is continuous. Thus the continuous bijection $\psi\colon X/G \to Y$ is a homeomorphism, as required. ∎

Corollary 6.7 *Let the group $\mathbb{Z}$ act on the real line $\mathbb{R}$ by translation, where the action sends each integer n to the translation function $\theta_n\colon \mathbb{R} \to \mathbb{R}$ that is defined so that $\theta_n(t) = t + n$ for all real numbers t. Let $\mathbb{R}/\mathbb{Z}$ denote the orbit space for this action, and let $\rho\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map that sends each real number to its orbit under the action of the group $\mathbb{Z}$. Let S^1 denote the unit circle centred on the origin in $\mathbb{R}^2$, let $\kappa\colon \mathbb{R} \to S^1$ be defined such that*

$$\kappa(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all real numbers t, and let $\psi\colon \mathbb{R}/\mathbb{Z} \to S^1$ be the map defined such that $\psi(\rho(t)) = \kappa(t)$ for all real numbers t. Then $\psi\colon \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism.

Proof The map $\kappa\colon \mathbb{R} \to S^1$ maps open sets to open sets. The result therefore follows directly on applying Proposition 6.6. ∎

Proposition 6.8 *Let G be a group acting freely and properly discontinuously on a topological space X, let X/G denote the resulting orbit space, and let $\rho\colon X \to X/G$ be the quotient map that sends each element of X to its orbit under the action of the group G. Let $\varphi\colon X \to Y$ be a continuous surjective map from X to a Hausdorff topological space Y. Suppose that elements p and q of X satisfy $\varphi(p) = \varphi(q)$ if and only if $\rho(p) = \rho(q)$. Suppose also that there exists a compact subset K of X that intersects every orbit for the action of G on X. Then the surjective continuous map $\varphi\colon X \to Y$ induces a homeomorphism $\psi\colon X/G \to Y$ between the topological spaces X/G and Y, where $\psi(\rho(p)) = \varphi(p)$ for all $p \in X$.*

Proof The function $\psi\colon X/G \to Y$ is continuous because $\varphi\colon X \to Y$ is continuous and $\rho\colon X \to X/G$ is a quotient map (see Lemma 2.14). Moreover it is surjective because $\varphi\colon X \to Y$ is surjective, and it is injective because elements p and q satisfy $\varphi(p) = \varphi(q)$ if and only if $\rho(p) = \rho(q)$. It follows that $\psi\colon X/G \to Y$ is a bijection.

The orbit space X/G is compact, because it is the image $\rho(K)$ of the compact set K under the continuous map $\rho\colon X \to X/G$. (see Lemma 1.29). Thus $\psi\colon X/G \to Y$ is a continuous bijection from a compact topological space to a Hausdorff space. This map is therefore a homeomorphism (see Theorem 1.35). ∎

Example Let the group $\mathbb{Z}$ of integers under addition act on the real line $\mathbb{R}$ by translation so that, under this action, an integer n corresponds to the homeomorphism $\theta_n\colon \mathbb{R} \to \mathbb{R}$ defined such that $\theta_n(t) = t + n$ for all real numbers t. Let $\rho\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map onto the orbit space, and let $\kappa\colon \mathbb{R} \to S^1$ be defined such that

$$\kappa(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all real numbers t, and let $\psi\colon \mathbb{R}/\mathbb{Z} \to S^1$ be the map defined such that $\psi(\rho(t)) = \kappa(t)$ for all real numbers t.

Now S^1 is a Hausdorff space, as it is a subset of the metric space $\mathbb{R}^2$. Also the map $\kappa\colon \mathbb{R} \to S^1$ is surjective. Real numbers t_1 and t_2 satisfy $\kappa(t_1) = \kappa(t_2)$ if and only if $t_1 = t_2 + n$ for some integer n. It follows that $\kappa(t_1) = \kappa(t_2)$

if and only if $\rho(t_1) = \rho(t_2)$. The compact subset $[0, 1]$ of $\mathbb{R}$ intersects every orbit for the action of $\mathbb{Z}$ on $\mathbb{R}$. It therefore follows from Proposition 6.8 that $\psi\colon \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism. (This result was also shown to follow from the fact that $\kappa\colon \mathbb{R} \to S^1$ maps open sets to open sets: see Corollary 6.7.)

Proposition 6.9 *Let G be a group acting freely and properly discontinuously on a topological space X. Then the quotient map $\rho\colon X \to X/G$ from X to the corresponding orbit space X/G is a covering map.*

Proof The quotient map $\rho\colon X \to X/G$ is surjective. Let V be an open set in X. Then $\rho^{-1}(\rho(V))$ is the union $\bigcup_{g\in G} \theta_g(V)$ of the open sets $\theta_g(V)$ as g ranges over the group G, because $\rho^{-1}(\rho(V))$ is the subset of X consisting of all elements of X that belong to the orbit of some element of V. Moreover each set $\theta_g(V)$ is an open set in X, because each map θ_g is a homeomorphism mapping the set X onto itself. Also any union of open sets in a topological space is an open set. We conclude therefore that if V is an open set in X then $\rho(V)$ is an open set in X/G.

Let p be a point of X. Then there exists an open set V in X such that $p \in V$ and $\theta_g(V) \cap V = \emptyset$ for all $g \in G$ satisfying $g \neq e$. Now $\rho^{-1}(\rho(V)) = \bigcup_{g\in G} \theta_g(V)$. We claim that the sets $\theta_g(V)$ are pairwise disjoint. Let g and h be elements of G. Suppose that $\theta_g(V) \cap \theta_h(V) \neq \emptyset$. Then $\theta_{h^{-1}}(\theta_g(V) \cap \theta_h(V)) \neq \emptyset$. But $\theta_{h^{-1}}\colon X \to X$ is a bijection. Consequently

$$\theta_{h^{-1}}(\theta_g(V) \cap \theta_h(V)) = \theta_{h^{-1}}(\theta_g(V)) \cap \theta_{h^{-1}}(\theta_h(V)) = \theta_{h^{-1}g}(V) \cap V,$$

and therefore $\theta_{h^{-1}g}(V) \cap V \neq \emptyset$. It follows that $h^{-1}g = e$, where e denotes the identity element of G, and therefore $g = h$. It follows from this that if g and h are elements of the group G, and if $g \neq h$, then $\theta_g(V) \cap \theta_h(V) = \emptyset$. We conclude therefore that the preimage $\rho^{-1}(\rho(V))$ of $\rho(V)$ is indeed the disjoint union of the sets $\theta_g(V)$ as g ranges over the group G. Moreover each of these sets $\theta_g(V)$ is an open set in X.

Now $V \cap [p]_G = \{p\}$ for all $p \in V$, because $[p]_G = \{\theta_g(p) : g \in G\}$ and $V \cap \theta_g(V) = \emptyset$ whenever g is an element of the group G distinct from the identity element of that group. It follows that if p and q are elements of V, and if $\rho(p) = \rho(q)$ then $[p]_G = [q]_G$ and therefore $p = q$. Consequently the restriction $\rho|V\colon V \to X/G$ of the quotient map ρ to V is injective, and therefore ρ maps V bijectively onto $\rho(V)$. But ρ maps open sets onto open sets, and any continuous bijection that maps open sets onto open sets is a homeomorphism. We conclude therefore that the restriction of $\rho\colon X \to X/G$ to the open set V maps V homeomorphically onto $\rho(V)$. Moreover, given any element g of G, the quotient map ρ satisfies $\rho = \rho \circ \theta_{g^{-1}}$, and the homeomorphism $\theta_{g^{-1}}$ maps $\theta_g(V)$ homeomorphically onto V. It follows that the quotient map ρ maps $\theta_g(V)$ homeomorphically onto $\rho(V)$ for all $g \in V$.

We conclude therefore that $\rho(V)$ is an evenly covered open set in X/G whose preimage $\rho^{-1}(\rho(V))$ is the disjoint union of the open sets $\theta_g(V)$ as g ranges over the group G. Consequently the quotient map $\rho\colon X \to X/G$ is a covering map, as required. ∎

6.4 Fundamental Groups of Orbit Spaces

Theorem 6.10 *Let G be a group acting freely and properly discontinuously on a path-connected topological space X, let $\rho\colon X \to X/G$ be the quotient map from X to the orbit space X/G, let b_0 be a point of X, and let $c_0 = \rho(b_0) = [b_0]_G$. Then there exists a surjective homomorphism $\lambda\colon \pi_1(X/G, c_0) \to G$ characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(b_0)$ for any loop γ in X/G based at c_0, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = b_0$ and $\rho \circ \tilde{\gamma} = \gamma$. The kernel of this homomorphism is the subgroup $\rho_\#\bigl(\pi_1(X, b_0)\bigr)$ of $\pi_1(X/G, c_0)$.*

Proof Let $\gamma\colon [0,1] \to X/G$ be a loop in the orbit space with $\gamma(0) = \gamma(1) = c_0$. It follows from the Path-Lifting Theorem for covering maps (Theorem 4.13) that there exists a unique path $\tilde{\gamma}\colon [0,1] \to X$ for which $\tilde{\gamma}(0) = b_0$ and $\rho \circ \tilde{\gamma} = \gamma$. Now $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ must belong to the same orbit under the action of the group G on the topological space X, because

$$\rho(\tilde{\gamma}(0)) = \gamma(0) = \gamma(1) = \rho(\tilde{\gamma}(1)).$$

Therefore there exists some element g of G such that $\tilde{\gamma}(1) = \theta_g(b_0)$. This element g is uniquely determined, because the group G acts freely on X. Moreover the value of g is determined by the based homotopy class $[\gamma]$ of γ in $\pi_1(X/G, c_0)$. Indeed it follows from Proposition 6.1 that if σ is a loop in X/G based at c_0, if $\tilde{\sigma}$ is the lift of σ starting at b_0 (so that $\rho \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = b_0$), and if $[\gamma] = [\sigma]$ in $\pi_1(X/G, c_0)$ (so that $\gamma \simeq \sigma$ rel $\{0,1\}$), then $\tilde{\gamma}(1) = \tilde{\sigma}(1)$. We conclude therefore that there exists a well-defined function

$$\lambda\colon \pi_1(X/G, c_0) \to G,$$

which is characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(b_0)$ for any loop γ in X/G based at c_0, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = b_0$ and $\rho \circ \tilde{\gamma} = \gamma$.

Now let $\alpha\colon [0,1] \to X/G$ and $\beta\colon [0,1] \to X/G$ be loops in X/G based at c_0, and let $\tilde{\alpha}\colon [0,1] \to X$ and $\tilde{\beta}\colon [0,1] \to X$ be the lifts of α and β respectively starting at b_0, so that $\rho \circ \tilde{\alpha} = \alpha$, $\rho \circ \tilde{\beta} = \beta$ and $\tilde{\alpha}(0) = \tilde{\beta}(0) = b_0$. Then

$\tilde{\alpha}(1) = \theta_{\lambda([\alpha])}(b_0)$ and $\tilde{\beta}(1) = \theta_{\lambda([\beta])}(b_0)$. Then the path $\theta_{\lambda([\alpha])} \circ \tilde{\beta}$ is also a lift of the loop β, and is the unique lift of β starting at $\tilde{\alpha}(1)$. Let $\alpha\,.\,\beta$ be the concatenation of the loops α and β, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then the unique lift of $\alpha\,.\,\beta$ to X starting at b_0 is the path $\sigma\colon [0,1] \to X$, where

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \theta_{\lambda([\alpha])}(\tilde{\beta}(2t-1)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

It follows that

$$\begin{aligned} \theta_{\lambda([\alpha][\beta])}(b_0) &= \theta_{\lambda([\alpha.\beta])}(b_0) = \sigma(1) = \theta_{\lambda([\alpha])}(\tilde{\beta}(1)) \\ &= \theta_{\lambda([\alpha])}(\theta_{\lambda([\beta])}(b_0)) = \theta_{\lambda([\alpha])\lambda([\beta])}(b_0). \end{aligned}$$

Consequently $\lambda([\alpha][\beta]) = \lambda([\alpha])\lambda([\beta])$. Thus the function

$$\lambda\colon \pi_1(X/G, c_0) \to G$$

is a homomorphism.

Let $g \in G$. Then there exists a path α in X from b_0 to $\theta_g(b_0)$, because the space X is path-connected. Then $\rho \circ \alpha$ is a loop in X/G based at c_0, and $g = \lambda([\rho \circ \alpha])$. This shows that the homomorphism λ is surjective.

Let $\gamma\colon [0,1] \to X/G$ be a loop in X/G based at c_0. Suppose that $[\gamma] \in \ker \lambda$. Then $\tilde{\gamma}(1) = \theta_e(b_0) = b_0$, and therefore $\tilde{\gamma}$ is a loop in X based at b_0. Moreover $[\gamma] = \rho_\#[\tilde{\gamma}]$. Consequently $[\gamma] \in \rho_\#\bigl(\pi_1(X, b_0)\bigr)$. On the other hand, if $[\gamma] \in \rho_\#\bigl(\pi_1(X, b_0)\bigr)$ then $\gamma = \rho \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in X based at b_0 (see Proposition 6.3). But then $b_0 = \tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(b_0)$, and therefore $\lambda([\gamma]) = e$, where e is the identity element of G. Thus $\ker \lambda = \rho_\#\bigl(\pi_1(X, b_0)\bigr)$, as required. ∎

Corollary 6.11 *Let G be a group acting freely and properly discontinuously on a path-connected topological space X, let $\rho\colon X \to X/G$ be the quotient map from X to the orbit space X/G, and let b_0 be a point of X. Then $\rho_\#\bigl(\pi_1(X, b_0)\bigr)$ is a normal subgroup of the fundamental group $\pi_1(X/G, c_0)$ of the orbit space, and*

$$\frac{\pi_1(X/G, c_0)}{\rho_\#\bigl(\pi_1(X, b_0)\bigr)} \cong G.$$

Proof The subgroup $\rho_\#\bigl(\pi_1(X, b_0)\bigr)$ is the kernel of the homomorphism

$$\lambda\colon \pi_1(X/G, c_0) \to G$$

characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(b_0)$ for any loop γ in X/G based at c_0, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = b_0$ and $\rho \circ \tilde{\gamma} = \gamma$. The image of $\pi_1(X, b_0)$ under the homomorphism $\rho_\#$ of fundamental groups induced by the quotient map ρ is therefore a normal subgroup of $\pi_1(X/G, c_0)$, because the kernel of any homomorphism is a normal subgroup. The homomorphism λ is surjective, and the image of any group homomorphism is isomorphic to the quotient of its domain by its kernel. The result follows. ∎

Corollary 6.12 *Let G be a group acting freely and properly discontinuously on a simply connected topological space X, let $\rho\colon X \to X/G$ be the quotient map from X to the orbit space X/G, and let b_0 be a point of X, and let $c_0 = \rho(b_0) = [b_0]_G$. Then $\pi_1(X/G, c_0) \cong G$.*

Proof This is a special case of Corollary 6.11. ∎