

Module MAU34201: Algebraic Topology I
Michaelmas Term 2022
Section 5: The Fundamental Group of a
Topological Space

D. R. Wilkins

© Trinity College Dublin 2020–22

Contents

| | | |
|----------|---|-----------|
| 5 | The Fundamental Group of a Topological Space | 65 |
| 5.1 | The Fundamental Group | 65 |

5 The Fundamental Group of a Topological Space

5.1 The Fundamental Group

Let X be a topological space, and let p and q be points of X . A *path* in X from p to q is represented as a continuous function $\gamma: [0, 1] \rightarrow X$ for which $\gamma(0) = p$ and $\gamma(1) = q$. A *loop* in X based at p is represented as a continuous function $\gamma: [0, 1] \rightarrow X$ for which $\gamma(0) = \gamma(1) = p$.

We can concatenate paths. Let $\alpha: [0, 1] \rightarrow X$ and $\beta: [0, 1] \rightarrow X$ be paths in some topological space X . Suppose that $\alpha(1) = \beta(0)$. We define the *product path* $\alpha \cdot \beta: [0, 1] \rightarrow X$ by

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Let $\gamma: [0, 1] \rightarrow X$ be a path in X . The *inverse path* $\gamma^{-1}: [0, 1] \rightarrow X$ is defined so that $\gamma^{-1}(t) = \gamma(1 - t)$ for all $t \in [0, 1]$. (Thus if γ is a path from the point p to the point q then γ^{-1} is the path from q to p obtained by traversing γ in the reverse direction.)

Let X be a topological space, and let $p \in X$ be some chosen point of X . We define an equivalence relation on the set of all loops based at the basepoint p of X , where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1 \text{ rel } \{0, 1\}$. We denote the equivalence class of a loop $\gamma: [0, 1] \rightarrow X$ based at p by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ . The set of equivalence classes of loops based at p is denoted by $\pi_1(X, p)$. Thus two loops γ_0 and γ_1 represent the same element of $\pi_1(X, p)$ if and only if $\gamma_0 \simeq \gamma_1 \text{ rel } \{0, 1\}$ (i.e., if and only if there exists a homotopy $F: [0, 1] \times [0, 1] \rightarrow X$ that maps $(t, 0)$ and $(t, 1)$ to $\gamma_0(t)$ and $\gamma_1(t)$ for all $t \in [0, 1]$ and also maps $(0, \tau)$ and $(1, \tau)$ to the basepoint p for all $\tau \in [0, 1]$).

Theorem 5.1 *Let X be a topological space, let p be some chosen point of X , and let $\pi_1(X, p)$ be the set of all based homotopy classes of loops based at the point p . Then $\pi_1(X, p)$ is a group, the group operation on $\pi_1(X, p)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2]$ for all loops γ_1 and γ_2 based at p .*

Proof First we show that the product operation on $\pi_1(X, p)$ is well-defined. Let $\gamma_1, \gamma'_1, \gamma_2$ and γ'_2 be loops in X based at the point p . Suppose that $[\gamma_1] = [\gamma'_1]$ and $[\gamma_2] = [\gamma'_2]$. Let $F_1: [0, 1] \times [0, 1] \rightarrow X$ be a homotopy between

γ_1 and γ'_1 , and let $F_2: [0, 1] \times [0, 1] \rightarrow X$ be a homotopy between γ_2 and γ'_2 , and where the homotopies F_1 and F_2 map $(0, \tau)$ and $(1, \tau)$ to p for all $\tau \in [0, 1]$. Then let $F: [0, 1] \times [0, 1] \rightarrow X$ be defined so that

$$F(t, \tau) = \begin{cases} F_1(2t, \tau) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ F_2(2t - 1, \tau) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then F is itself a homotopy from $\gamma_1 \cdot \gamma_2$ to $\gamma'_1 \cdot \gamma'_2$. Moreover $F(0, \tau) = F(1, \tau) = p$ for all $\tau \in [0, 1]$. Thus $[\gamma_1 \cdot \gamma_2] = [\gamma'_1 \cdot \gamma'_2]$. We conclude that the product operation on $\pi_1(X, p)$ is well-defined.

Next we show that the product operation on $\pi_1(X, p)$ is associative. Let γ_1, γ_2 and γ_3 be loops based at p , and let $\alpha = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3$. Then $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3) = \alpha \circ \theta$, where

$$\theta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Let $G: [0, 1] \times [0, 1] \rightarrow X$ be the continuous function defined so that $G(t, \tau) = \alpha((1 - \tau)t + \tau\theta(t))$ for all $t, \tau \in [0, 1]$. Then the continuous function G is a homotopy between $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$ and $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$, and moreover this homotopy maps $(0, \tau)$ and $(1, \tau)$ to p for all $\tau \in [0, 1]$. It follows that $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \simeq \gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \text{ rel } \{0, 1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1](\gamma_2[\gamma_3])$. This shows that the product operation on $\pi_1(X, p)$ is associative.

Let $\varepsilon: [0, 1] \rightarrow X$ denote the constant loop at p , defined by $\varepsilon(t) = p$ for all $t \in [0, 1]$. Let functions θ_0 and θ_1 mapping the closed unit interval $[0, 1]$ onto itself be defined so that

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2t - 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$\theta_1(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for all $t \in [0, 1]$. Then $\varepsilon \cdot \gamma = \gamma \circ \theta_0$ and $\gamma \cdot \varepsilon = \gamma \circ \theta_1$ for any loop γ based at p . But the continuous map $(t, \tau) \mapsto \gamma((1 - \tau)t + \tau\theta_j(t))$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0, 1$ which sends $(0, \tau)$ and $(1, \tau)$ to p for all $\tau \in [0, 1]$. Therefore $\varepsilon \cdot \gamma \simeq \gamma \simeq \gamma \cdot \varepsilon \text{ rel } \{0, 1\}$. Consequently $[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ is the identity element of $\pi_1(X, p)$.

It only remains to verify the existence of inverses. Let the continuous function $K: [0, 1] \times [0, 1] \rightarrow X$ be defined so that

$$K(t, \tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma(2\tau(1 - t)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then continuous function K is a homotopy between the loops ε and $\gamma \cdot \gamma^{-1}$, and moreover this homotopy sends $(0, \tau)$ and $(1, \tau)$ to p for all $\tau \in [0, 1]$. Therefore $\varepsilon \simeq \gamma \cdot \gamma^{-1} \text{ rel } \{0, 1\}$. It follows that $[\gamma][\gamma^{-1}] = [\gamma \cdot \gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1} , we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required. ■

Let p be a point of some topological space X . The group $\pi_1(X, p)$ is referred to as the *fundamental group* of X based at the point p .

Proposition 5.2 *Let $\varphi: X \rightarrow Y$ be a continuous map between topological spaces X and Y , and let p be a point of X . Then φ induces a homomorphism $\varphi_\#: \pi_1(X, p) \rightarrow \pi_1(Y, \varphi(p))$ between the fundamental groups of X and Y with basepoints p and $\varphi(p)$ respectively, where $\varphi_\#([\gamma]) = [\varphi \circ \gamma]$ for all loops $\gamma: [0, 1] \rightarrow X$ based at p .*

Proof Let γ_0 and γ_1 be loops in X based at the point p that belong to the same based homotopy class, and thus represent the same element of the fundamental group of the topological space X with basepoint p . Then there exists a homotopy $H: [0, 1] \times [0, 1] \rightarrow X$ with the properties that $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1(t)$ for all $t \in [0, 1]$ and $H(0, \tau) = H(1, \tau) = p$ for all $\tau \in [0, 1]$.

Let $K = \varphi \circ H$. Then $K(t, 0) = \varphi(\gamma_0(t))$ and $K(t, 1) = \varphi(\gamma_1(t))$ for all $t \in [0, 1]$ and $K(0, \tau) = K(1, \tau) = \varphi(p)$ for all $\tau \in [0, 1]$. It follows that K is a based homotopy between the loops $\varphi \circ \gamma_0$ and $\varphi \circ \gamma_1$, and therefore those loops represent the same element of the fundamental group $\pi_1(Y, \varphi(p))$ of Y with basepoint $\varphi(p)$. Thus the map φ induces a well-defined function

$$\varphi_\#: \pi_1(X, p) \rightarrow \pi_1(Y, \varphi(p)).$$

Now if γ is the concatenation of loops α and β in X , where each of the loops α and β is based at the point p , then $\varphi \circ \gamma$ is the concatenation of the loops $\varphi \circ \alpha$ and $\varphi \circ \beta$. It follows that

$$\varphi_\#([\alpha][\beta]) = \varphi_\#([\gamma]) = [\varphi \circ \gamma] = [\varphi \circ \alpha][\varphi \circ \beta] = \varphi_\#([\alpha])\varphi_\#([\beta]).$$

Consequently the function $\varphi_\#$ is a homomorphism from the fundamental group $\pi_1(X, p)$ of X with basepoint p to the fundamental group $\pi_1(Y, \varphi(p))$ of Y with basepoint $\varphi(p)$. This completes the proof. ■

If p, q and r are points belonging to topological spaces X, Y and Z , and if $\varphi: X \rightarrow Y$ and $\psi: Y \rightarrow Z$ are continuous maps satisfying $\varphi(p) = q$ and $\psi(q) = r$, then the induced homomorphisms $\varphi_\#: \pi_1(X, p) \rightarrow \pi_1(Y, q)$ and $\psi_\#: \pi_1(Y, q) \rightarrow \pi_1(Z, r)$ satisfy $\psi_\# \circ \varphi_\# = (\psi \circ \varphi)_\#$.

The property just described can in particular be applied in the case when $\varphi: X \rightarrow Y$ is a homeomorphism whose inverse is $\psi: Y \rightarrow X$. We can then conclude that $\varphi_\#: \pi_1(X, p) \rightarrow \pi_1(Y, \varphi(p))$ is an isomorphism of groups whose inverse is $\psi_\#: \pi_1(Y, \varphi(p)) \rightarrow \pi_1(X, p)$.

Proposition 5.3 *Let X be a topological space, and let α be a path in X . Then there is a well-defined isomorphism $\Theta_\alpha: \pi_1(X, \alpha(1)) \rightarrow \pi_1(X, \alpha(0))$ between the fundamental groups at the endpoints at that path which sends the based homotopy class $[\gamma]$ of any loop γ based at $\alpha(1)$ to the based homotopy class of the loop $\alpha.\gamma.\alpha^{-1}$ based at $\alpha(0)$, where*

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \text{if } 0 \leq t \leq \frac{1}{3}; \\ \gamma(3t - 1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ \alpha(3 - 3t) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

(Note that $\alpha.\gamma.\alpha^{-1}$ represents ‘ α followed by γ followed by α reversed’).

Proof We first show that the function Θ_α between the fundamental groups of the topological space based at the points $\alpha(1)$ and $\alpha(0)$ is a homomorphism.

Let γ_1 and γ_2 be loops in X based at the point $\alpha(1)$. Then the product $\Theta_\alpha([\gamma_1])\Theta_\alpha([\gamma_2])$ in the fundamental group $\pi_1(X, \alpha(0))$ is represented by the loop $\eta_1: [0, 1] \rightarrow X$ based at $\alpha(0)$ where

$$\eta_1(t) = \begin{cases} \alpha(6t) & \text{if } 0 \leq t \leq \frac{1}{6}, \\ \gamma_1(6t - 1) & \text{if } \frac{1}{6} \leq t \leq \frac{1}{3}, \\ \alpha(3 - 6t) & \text{if } \frac{1}{3} \leq t \leq \frac{1}{2}, \\ \alpha(6t - 3) & \text{if } \frac{1}{2} \leq t \leq \frac{2}{3}, \\ \gamma_2(6t - 4) & \text{if } \frac{2}{3} \leq t \leq \frac{5}{6}, \\ \alpha(6 - 6t) & \text{if } \frac{5}{6} \leq t \leq 1. \end{cases}$$

Also the element $\Theta_\alpha([\gamma_1][\gamma_2])$ in the fundamental group is equal to $\Theta_\alpha([\gamma_1.\gamma_2])$, where $\gamma_1.\gamma_2$ is the concatenation of the loops γ_1 and γ_2 , and therefore $\Theta_\alpha([\gamma_1][\gamma_2])$ is represented by the loop $\eta_2: [0, 1] \rightarrow X$ in X based at $\alpha(0)$, where

$$\eta_2(t) = \begin{cases} \alpha(3t) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \gamma_1(6t - 2) & \text{if } \frac{1}{3} \leq t \leq \frac{1}{2}, \\ \gamma_2(6t - 3) & \text{if } \frac{1}{2} \leq t \leq \frac{2}{3}, \\ \alpha(3 - 3t) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Let $H_1: [0, 1] \times [0, 1] \rightarrow X$ be defined so that

$$H_1(t, \tau) = \begin{cases} \alpha(6t) & \text{if } 0 \leq t \leq \frac{1}{6}, \\ \gamma_1(6t - 1) & \text{if } \frac{1}{6} \leq t \leq \frac{1}{3}, \\ \alpha(1 + 2\tau - 6\tau t) & \text{if } \frac{1}{3} \leq t \leq \frac{1}{2}, \\ \alpha(1 - 4\tau + 6\tau t) & \text{if } \frac{1}{2} \leq t \leq \frac{2}{3}, \\ \gamma_2(6t - 4) & \text{if } \frac{2}{3} \leq t \leq \frac{5}{6}, \\ \alpha(6 - 6t) & \text{if } \frac{5}{6} \leq t \leq 1. \end{cases}$$

Note that $1 + 2\tau - 6\tau t$ decreases from 1 to $1 - \tau$ as t increases from $\frac{1}{3}$ to $\frac{1}{2}$, and $1 - 4\tau + 6\tau t$ increases from $1 - \tau$ to 1 as t increases from $\frac{1}{2}$ to $\frac{2}{3}$. Note also that $H_1(0, \tau) = H_1(1, \tau) = \alpha(0)$ for all $\tau \in [0, 1]$.

Let η_0 be the loop in X based at $\alpha(0)$ defined so that

$$\eta_0(t) = \begin{cases} \alpha(6t) & \text{if } 0 \leq t \leq \frac{1}{6}, \\ \gamma_1(6t - 1) & \text{if } \frac{1}{6} \leq t \leq \frac{1}{3}, \\ \alpha(1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \gamma_2(6t - 4) & \text{if } \frac{2}{3} \leq t \leq \frac{5}{6}, \\ \alpha(6 - 6t) & \text{if } \frac{5}{6} \leq t \leq 1. \end{cases}$$

Then the function H_1 is a based homotopy between the loops η_1 and η_0 , and thus the loops η_1 and η_0 represent the same element of the fundamental group $\pi_1(X, \alpha(0))$ of X based at the point $\alpha(0)$.

But $\eta_0 = \eta_2 \circ \kappa$, where κ is the monotonically increasing function mapping the unit interval $[0, 1]$ onto itself defined so that

$$\kappa(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{6}, \\ t + \frac{1}{6} & \text{if } \frac{1}{6} \leq t \leq \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ t - \frac{1}{6} & \text{if } \frac{2}{3} \leq t \leq \frac{5}{6}, \\ 2t - 1 & \text{if } \frac{5}{6} \leq t \leq 1. \end{cases}$$

Now $0 \leq t - \tau t + \tau \kappa(t) \leq 1$ whenever $0 \leq t \leq 1$ and $0 \leq \tau \leq 1$. Let $H_2: [0, 1] \times [0, 1] \rightarrow X$ be the continuous map defined so that

$$H_2(t, \tau) = \eta_2(t - \tau t + \tau \kappa(t))$$

for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Then $H_2(t, 0) = \eta_2(t)$ and $H_2(t, 1) = \eta_2(\kappa(t)) = \eta_0(t)$ for all $t \in [0, 1]$. Also $H_2(0, \tau) = \eta_2(0) = \alpha(0)$ and

$H_2(1, \tau) = \eta_2(1) = \alpha(0)$ for all $\tau \in [0, 1]$. It follows that H_2 is a based homotopy between the loops η_2 and η_0 . It follows that the loops η_0 and η_2 represent the same element of the fundamental group $\pi_1(X, \alpha(0))$ of X with basepoint $\alpha(0)$.

Combining the results obtained above, we see that the identity

$$\Theta_\alpha([\gamma_1])\Theta_\alpha([\gamma_2]) = [\eta_1] = [\eta_0] = [\eta_2] = \Theta_\alpha([\gamma_1][\gamma_2])$$

holds in the fundamental group $\pi_1(X, \alpha(0))$ of X based at the point $\alpha(0)$. Thus the function $\Theta_\alpha: \pi_1(X, \alpha(1)) \rightarrow \pi_1(X, \alpha(0))$ is a group homomorphism.

Now let $\Theta_{\alpha^{-1}}$ be the corresponding homomorphism from $\pi_1(X, \alpha(0))$ to $\pi_1(X, \alpha(1))$ determined by the inverse α^{-1} of the path α , where $\alpha^{-1}(t) = \alpha(1 - t)$ for all $t \in [0, 1]$, and let γ be a loop in X based at $\alpha(1)$. Then $\Theta_{\alpha^{-1}}(\Theta_\alpha([\gamma]))$ is represented by the loop $\zeta_0: [0, 1] \rightarrow X$ based at $\alpha(1)$, where

$$\zeta_0(t) = \begin{cases} \alpha(1 - 3t) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \alpha(9t - 3) & \text{if } \frac{1}{3} \leq t \leq \frac{4}{9}, \\ \gamma(9t - 4) & \text{if } \frac{4}{9} \leq t \leq \frac{5}{9}, \\ \alpha(6 - 9t) & \text{if } \frac{5}{9} \leq t \leq \frac{2}{3}, \\ \alpha(3t - 2) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Now let $K_1: [0, 1] \times [0, 1] \rightarrow X$ be defined so that

$$K_1(t, \tau) = \begin{cases} \alpha(1 - 3t + 3t\tau) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \alpha(9t - 3 + 4\tau - 9t\tau) & \text{if } \frac{1}{3} \leq t \leq \frac{4}{9}, \\ \gamma(9t - 4) & \text{if } \frac{4}{9} \leq t \leq \frac{5}{9}, \\ \alpha(6 - 9t - 5\tau + 9t\tau) & \text{if } \frac{5}{9} \leq t \leq \frac{2}{3}, \\ \alpha(3t - 2 + 3\tau - 3t\tau) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Now $0 \leq 1 - 3t + 3t\tau \leq 1$ whenever $0 \leq t \leq \frac{1}{3}$ and $0 \leq \tau \leq 1$, $0 \leq 9t - 3 + 4\tau - 9t\tau \leq 1$ whenever $\frac{1}{3} \leq t \leq \frac{4}{9}$ and $0 \leq \tau \leq 1$, $0 \leq 6 - 9t - 5\tau + 9t\tau \leq 1$ whenever $\frac{5}{9} \leq t \leq \frac{2}{3}$ and $0 \leq \tau \leq 1$, and $0 \leq 3t - 2 + 3\tau - 3t\tau \leq 1$ whenever $\frac{2}{3} \leq t \leq 1$ and $0 \leq \tau \leq 1$. Consequently the function K_1 is well-defined. Also $K_1(0, \tau) = \alpha(1)$ and $K_1(1, \tau) = \alpha(1)$ for all $\tau \in [0, 1]$.

Let $\zeta_1: [0, 1] \rightarrow X$ be the loop based at $\alpha(1)$ defined such that

$$\zeta_1 = \begin{cases} \alpha(1) & \text{if } 0 \leq t \leq \frac{4}{9}, \\ \gamma(9t - 4) & \text{if } \frac{4}{9} \leq t \leq \frac{5}{9}, \\ \alpha(1) & \text{if } \frac{5}{9} \leq t \leq 1. \end{cases}$$

Then $K_1(t, 0) = \zeta_0(t)$ and $K_1(t, 1) = \zeta_1(t)$ for all $t \in [0, 1]$. We have previously noted that $K_1(0, \tau) = K_1(1, \tau) = \alpha(1)$. It follows that K_1 is a based homotopy between the loops ζ_0 and ζ_1 , and thus those loops represent the same element of the fundamental group $\pi_1(X, \alpha(1))$.

Now let $\varphi: [0, 1] \rightarrow [0, 1]$ be defined such that

$$\varphi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{4}{9}, \\ 9t - 4 & \text{if } \frac{4}{9} \leq t \leq \frac{5}{9}, \\ 1 & \text{if } \frac{5}{9} \leq t \leq 1. \end{cases}$$

Then let

$$K_2(t, \tau) = \gamma(1 - \tau + \tau\varphi(t))$$

for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Then $K_2(t, 0) = \gamma(t)$ and $K_2(t, 1) = \zeta_1(t)$ for all $t \in [0, 1]$, and $K_2(0, \tau) = K_2(1, \tau) = \alpha(1)$ for all $\tau \in [0, 1]$. It follows that K_2 is a based homotopy between the loops γ and ζ_1 . Consequently

$$\Theta_{\alpha^{-1}}(\Theta_{\alpha}([\gamma])) = [\zeta_0] = [\zeta_1] = [\gamma].$$

This identity holds for all loops γ based at $\alpha(1)$. We conclude therefore that $\Theta_{\alpha}: \pi_1(X, \alpha(1)) \rightarrow \pi_1(X, \alpha(0))$ is an isomorphism of groups whose inverse is the homomorphism $\Theta_{\alpha^{-1}}: \pi_1(X, \alpha(0)) \rightarrow \pi_1(X, \alpha(1))$ determined by the inverse α^{-1} of the path α . The result follows. ■

Corollary 5.4 *Let X be a path-connected topological space. Then the isomorphism class of the fundamental group of the space is independent of the choice of basepoint within the topological space.*

Proposition 5.5 *A path-connected topological space X is simply connected if and only if there exists some point p of X for which the fundamental group $\pi_1(X, p)$ is trivial.*

Proof It follows from Proposition 3.17 and the definition of the fundamental group that a path-connected topological space is simply connected if and only if the fundamental group $\pi_1(X, p)$ of X with basepoint p is the trivial group for all points p of X . It then follows from Proposition 5.3 that this is the case if and only if there exists at least one point p of the path-connected topological space for which $\pi_1(X, p)$ is the trivial group. The result follows. ■