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Section 4: Local Homeomorphisms and Covering Maps

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4 Local Homeomorphisms and Covering Maps

4.1 Local Homeomorphisms

Lemma 4.1 *Let X and Y be topological spaces, and let $\varphi\colon X \to Y$ be a function from X to Y. Then the function φ maps an open subset M of X* homeomorphically *onto an open subset N of Y if and only if the following conditions are satisfied:—*

- *each point of M is mapped by φ to a point of N;*
- *for each point q of N, there exists exactly one point p of M for which $\varphi(p) = q$.*
- *a subset V of M is an open set in X if and only if it is mapped by φ onto an open set in Y.*

Proof The first two conditions ensure that the function φ restricts to a bijection between the sets M and N. Now, because the subset M of X is itself open in X, a subset V of M is open in M, relative to the subspace topology on M, if and only if V is open in X (see Lemma 1.14). Similarly a subset of N is open in N if and only if it is open in Y. It follows that the final listed condition is equivalent to requiring that the function induced between the sets M and N by the map φ is a homeomorphism with respect to the subspace topologies on those two open sets. The result follows. ∎

Definition A function $\varphi\colon X \to Y$ between topological spaces X and Y is said to be a *local homeomorphism* if, given any point p of X, there exists some open set M in X to which the point p belongs which is mapped by the function φ homeomorphically onto an open set in Y.

Lemma 4.2 *Let X and Y be topological spaces and let $\varphi\colon X \to Y$ be a local homeomorphism from X to Y. Then the function φ is a continuous map.*

Proof Let W be an open set in Y. We must show that the preimage $\varphi^{-1}(W)$ of W under the function φ is an open set in X. Let p be a point of $\varphi^{-1}(W)$. The definition of local homeomorphisms ensures the existence of an open neighbourhood M in X of the point p which is mapped homeomorphically by φ onto an open set N in Y. Now $W \cap N$ is open in Y. The subset of M that corresponds to this set with respect to the map φ is the subset $M \cap \varphi^{-1}(W)$ of M. It follows that $M \cap \varphi^{-1}(W)$ must be an open set in X.

Moreover the point p belongs to this open set. It follows that $\varphi^{-1}(W)$ is a neighbourhood of the point p.

Now the point p of the preimage $\varphi^{-1}(W)$ was arbitrarily chosen. We have therefore established that the preimage $\varphi^{-1}(W)$ of the open set W under the map φ is a neighbourhood of each of its points. It follows that the preimage of the open subset W of Y under the function φ is an open set in X (see Lemma 1.4). Consequently $\varphi\colon X \to Y$ is continuous. ∎

Lemma 4.3 *Let $\varphi\colon X \to Y$ be a local homeomorphism between topological spaces X and Y, and let V be an open set in X. Then $\varphi(V)$ is an open set in Y.*

Proof Let p be a point belonging to the open set V. Then there exists an open neighbourhood M of the point p that is mapped by the local homeomorphism φ homeomorphically onto an open set N in Y. Let $E = \varphi(V \cap M)$. Now the set $V \cap M$ is an open set in X. It is therefore mapped by φ onto an open set in Y. Thus the set E is an open subset of Y. Moreover $\varphi(p) \in E$. We have thus shown that the subset $\varphi(V)$ contains an open neighbourhood E of the point $\varphi(p)$. It follows that $\varphi(V)$ is itself a neighbourhood of the point $\varphi(p)$.

Now the point p was an arbitrary point chosen from the set V. We conclude therefore that the subset $\varphi(V)$ of Y is a neighbourhood of each of its points and is therefore an open set in Y (see Lemma 1.4). The result follows. ∎

Proposition 4.4 *Let φ be a local homeomorphism from a topological space X to a topological space Y, and let V and W be open sets in X and Y respectively. Suppose that φ maps V bijectively onto W. Then φ maps V homeomorphically onto W.*

Proof The function φ maps V bijectively onto W, and V and W are open sets in X and Y respectively. Therefore in order to show that V is mapped homeomorphically onto W it suffices to verify that a subset of V is open in X if and only if $\varphi(V)$ is open in Y (see Lemma 4.1). Let D be a subset of V. Then $D = V \cap \varphi^{-1}(\varphi(D))$. It follows from the continuity of the local homeomorphism that if $\varphi(D)$ is open in Y then D is itself open in X. Conversely if D is open in X then $\varphi(D)$ is open in Y because local homeomorphisms map open sets to open sets (see Lemma 4.3). Thus the subset D of V is open in X if and only if $\varphi(D)$ is open in Y. The result follows. ∎

Lemma 4.5 *Any surjective local homeomorphism between topological spaces is an identification map.*

Proof Let X and Y be topological spaces and let $\varphi\colon X \to Y$ be a surjective local homeomorphism from X to Y. Let W be a subset of Y whose preimage $\varphi^{-1}(W)$ with respect to the function φ is open in X. Then $W = \varphi(\varphi^{-1}(W))$, because the map φ is surjective. Also any local homeomorphism maps open sets to open sets (Lemma 4.3). It follows that the set W is open in Y. The result follows. ∎

4.2 Evenly-Covered Open Sets and Covering Maps

Definition Let $\pi\colon \tilde{X} \to X$ be a continuous map between topological spaces $\tilde{X}$ and X. An open subset V of X is said to be *evenly covered* by the map π if and only if the preimage $\pi^{-1}(V)$ of the open set V under the map π is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto V by π.

Example Let $\kappa\colon \mathbb{R} \to S^1$ be the continuous function from the real line $\mathbb{R}$ to the unit circle S^1 in the plane $\mathbb{R}^2$ that sends each real number t to the point $(\cos(2\pi t), \sin(2\pi t))$ of the unit circle. Also let u and v be real numbers for which $u < v < u + 1$, and let V be the open arc in the unit circle S^1 consisting of all points of the unit circle S^1 that can be expressed in the form $(\cos(2\pi t), \sin(2\pi t))$ for some real number t satisfying $u < t < v$. Then the open arc V is evenly covered by the map κ. Indeed $\kappa^{-1}(V)$ is the disjoint union of the open intervals J_n as n ranges over the set of integers, where each J_n is the open interval in the real line with endpoints $u+n$ and $v+n$. Moreover the function κ maps each of these open intervals J_n homeomorphically onto the open arc V in the unit circle.

Definition A continuous map $\pi\colon \tilde{X} \to X$ from a topological space $\tilde{X}$ to a topological space X is said to be a *covering map* if $\pi\colon \tilde{X} \to X$ is surjective and in addition every point of X belongs to some open set in X that is evenly covered by the map π.

If a continuous map $\pi\colon \tilde{X} \to X$ from a topological space $\tilde{X}$ to a topological space X is a covering map, then we say that the domain $\tilde{X}$ of this covering map is a *covering space* of X. The codomain X of the covering map is often referred to as the *base space* of the covering map.

Example Let $\kappa\colon \mathbb{R} \to S^1$ be the continuous function from the real line $\mathbb{R}$ to the unit circle S^1 in the plane $\mathbb{R}^2$ that sends each real number t to the point $(\cos(2\pi t), \sin(2\pi t))$ of the unit circle. Indeed this function is surjective. Moreover any point in the unit circle is the image $\kappa(s)$ of some real number s. One can then choose real numbers u and v satisfying $u < v < u+1$ for which

$u < s < v$. Then the open interval in the real line with endpoints u and v is mapped by κ onto an open arc in the unit circle. That open arc is evenly covered by the map κ, and moreover the point $\kappa(s)$ of the unit circle lies within this open arc. We conclude therefore that the continuous function κ from the real line to the unit circle is a covering map.

Proposition 4.6 *Any covering map is a local homeomorphism.*

Proof Let $\pi\colon \tilde{X} \to X$ be a covering map, and let q be a point of $\tilde{X}$. Then $\pi(q)$ belongs to some evenly covered open set V in the base space X of the covering map. The preimage $\pi^{-1}(V)$ of V in the covering space $\tilde{X}$ is then a disjoint union of open subsets of $\tilde{X}$, each of which is mapped homeomorphically onto V by the covering map π. The point q then belongs to exactly one of these open sets constituting the disjoint union: let that open set be $\tilde{V}$. Then $\tilde{V}$ is an open set in the covering space $\tilde{X}$ to which the point q belongs, and this open set is mapped by π homeomorphically onto the open set V in the base space. This argument shows that every point of the covering space $\tilde{X}$ belongs to some open set in the covering space that is mapped homeomorphically onto an open set in the base space X. It follows that the covering map $\pi\colon \tilde{X} \to X$ is a local homeomorphism, as required. ∎

Corollary 4.7 *Let $\pi\colon \tilde{X} \to X$ be a covering map. Then $\pi(V)$ is open in X for every open set V in $\tilde{X}$.*

Proof Any covering map is a local homeomorphism (see Proposition 4.6). Moreover a local homeomorphism maps open subsets of its domain onto open sets (see Lemma 4.3). The result follows. ∎

Corollary 4.8 *A bijective covering map is a homeomorphism.*

Proof Any covering map is a local homeomorphism (see Proposition 4.6). Moreover if a local homeomorphism maps an open subset of its domain bijectively onto an open set in its codomain, then it maps the open set in the domain homeomorphically onto the open set in the codomain (see Proposition 4.4). The required result follows on applying this general property of local homeomorphisms in the case where the open set in the domain of the local homeomorphism is the covering space of the covering map. ∎

Proposition 4.9 *Let $\pi\colon \tilde{X} \to X$ be a local homeomorphism between topological spaces $\tilde{X}$ and X. An open set V in the codomain X of the function π is then evenly covered by the local homeomorphism π if and only if its preimage $\pi^{-1}(V)$ under the map π is a disjoint union of open sets in $\tilde{X}$ each of which is mapped bijectively onto V by the local homeomorphism π.*

Proof This result follows from the definition of evenly covered open sets in the codomain of a continuous map, on applying the result that a local homeomorphism maps an open subset of its domain homeomorphically onto an open set in its codomain if and only if it maps the open set in the domain bijectively onto the open set in the codomain (see Proposition 4.4). ∎

Proposition 4.10 *Let $\tilde{X}$ and X be topological spaces, and let $\pi\colon \tilde{X} \to X$ be a covering map. Let Y be a subset of X and let $\tilde{Y}$ be the preimage $\pi^{-1}(Y)$ of the set Y under the covering map π. Then the restriction of the map π to the subset $\tilde{Y}$ of $\tilde{X}$ constitutes a covering map from $\tilde{Y}$ to Y.*

Proof The restriction of the map π to $\tilde{Y}$ is a surjective map from $\tilde{Y}$ to Y. Let p be a point of Y. Then there exists an open set V in X which is evenly covered by the covering map π. Then the intersection of the open set V with the set Y is open in the subspace topology on Y, and this intersection is evenly covered by the restriction to $\tilde{Y}$ of the covering map π. The result follows. ∎

4.3 Uniqueness of Lifts into Covering Spaces

Definition Let $\tilde{X}$ and X be topological spaces, let $\pi\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X, let Z be a topological space, and let $\varphi\colon Z \to X$ be a continuous map from Z to the base space X of the covering map. A continuous map $\tilde{\varphi}\colon Z \to \tilde{X}$ from Z to the covering space $\tilde{X}$ is then said to be a *lift* of $\varphi\colon Z \to X$ to the covering space $\tilde{X}$ if $\pi \circ \tilde{\varphi} = \varphi$.

Much of the general theory of covering maps is concerned with the development of necessary and sufficient conditions to determine whether or not maps into the base space of a covering map can be lifted to the covering space.

We prove that any lift of a given map from a connected topological space into the base space of a covering map is determined by its value at a single point of its domain.

Proposition 4.11 *Let $\tilde{X}$ and X be topological spaces, and let $\pi\colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Let Z be a connected topological space, and let $\theta\colon Z \to \tilde{X}$ and $\psi\colon Z \to \tilde{X}$ be continuous maps. Suppose that $\pi \circ \theta = \pi \circ \psi$ and that $\theta(z) = \psi(z)$ for at least one point z of Z. Then $\theta = \psi$.*

Proof Let $Z_0 = \{z \in Z : \theta(z) = \psi(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of the topological space Z. There exists an open set V in X containing the point $\pi(\theta(z))$ which is evenly covered by the covering map π. Then $\pi^{-1}(V)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto V by the covering map π. One of these open sets contains $\theta(z)$; let this set be denoted by $\tilde{V}$. Also one of these open sets contains $\psi(z)$; let this open set be denoted by $\tilde{W}$. Let $N_z = \theta^{-1}(\tilde{V}) \cap \psi^{-1}(\tilde{W})$. Then N_z is an open set in the topological space Z to which the point z belongs.

Consider the case when $z \in Z_0$. Then $\theta(z) = \psi(z)$, and therefore $\tilde{V} = \tilde{W}$. It follows from this that both θ and ψ map the open set N_z into $\tilde{V}$. But $\pi \circ \theta = \pi \circ \psi$, and $\pi|\tilde{V} \colon \tilde{V} \to V$ is a homeomorphism. Therefore $\theta|N_z = \psi|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{V} \cap \tilde{W} = \emptyset$, because $\theta(z) \neq \psi(z)$. But $\theta(N_z) \subset \tilde{V}$ and $\psi(N_z) \subset \tilde{W}$. Therefore $\theta(w) \neq \psi(w)$ for all $w \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of the topological space Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. It therefore follows from the connectedness of Z that $Z_0 = Z$, and thus $\theta = \psi$, as required. ∎

Corollary 4.12 *Let $\tilde{X}$ and X be topological spaces, and let $\pi \colon \tilde{X} \to X$ be a covering map from $\tilde{X}$ to X. Let Z be a connected topological space, and let $\psi \colon Z \to \tilde{X}$ be a continuous map from the topological space Z to the covering space $\tilde{X}$. Suppose that $\pi(\psi(z)) = p$ for all $z \in Z$, where p is some point of X. Then the whole of the topological space Z is mapped by ψ to a single point of the covering space $\tilde{X}$.*

Proof Let q be the value of the function ψ at some chosen point of the topological space Z, and let $\theta \colon Z \to \tilde{X}$ be the constant map that maps all points of the topological space Z to the point q. Then the maps θ and ψ coincide at the chosen point of Z. Moreover the compositions of these maps with the covering map π each send the whole of the topological space Z to the point p. It follows from Proposition 4.11 that $\psi = \theta$. The result follows. ∎

4.4 The Path-Lifting Theorem

Theorem 4.13 (Path-Lifting Theorem) *Let $\tilde{X}$ and X be topological spaces, and let $\pi \colon \tilde{X} \to X$ be a covering map. Let $\gamma \colon [a, b] \to X$ be a continuous function mapping the closed interval $[a, b]$ into the base space X*

of the covering map, and let q be a point of $\tilde{X}$ for which $\pi(q) = \gamma(a)$. Then there exists a unique continuous function $\tilde{\gamma}\colon [a,b] \to \tilde{X}$ mapping the closed interval $[a,b]$ into the covering space $\tilde{X}$ for which $\tilde{\gamma}(a) = q$ and $\pi \circ \tilde{\gamma} = \gamma$.

Proof Let S be the subset of $[a,b]$ defined as follows: an element c of $[a,b]$ belongs to S if and only if there exists a continuous map $\eta_c\colon [a,c] \to \tilde{X}$ such that $\eta_c(a) = q$ and $\pi(\eta_c(t)) = \gamma(t)$ for all $t \in [a,c]$. Note that S is non-empty, because a belongs to S. Let $s = \sup S$.

There exists an open neighbourhood V of $\gamma(s)$ which is evenly covered by the map π, because $\pi\colon \tilde{X} \to X$ is a covering map. It then follows from the continuity of the path γ that there exists some positive real number δ such that $\gamma(J(s,\delta)) \subset V$, where

$$J(s,\delta) = \{t \in [a,b] : |t - s| < \delta\}.$$

Now $S \cap J(s,\delta)$ is non-empty, because s is the supremum of the set S. Choose some element c of $S \cap J(s,\delta)$. Then there exists a continuous map $\eta_c\colon [a,c] \to \tilde{X}$ such that $\eta_c(a) = q$ and $\pi(\eta_c(t)) = \gamma(t)$ for all $t \in [a,c]$. Now the open set V is evenly covered by the map π. Therefore $\pi^{-1}(V)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto V by the covering map π. One of these open sets contains the point $\eta_c(c)$; let this open set be denoted by $\tilde{V}$. There then exists a unique continuous map $\sigma\colon V \to \tilde{X}$ defined such that, for all $x \in V$, $\sigma(x)$ is the unique element of $\tilde{V}$ for which $\pi(\sigma(x)) = x$. Then $\sigma(\gamma(c)) = \eta_c(c)$.

Then, given any $d \in J(s,\delta)$, let $\eta_d\colon [a,d] \to \tilde{X}$ be the function from $[a,d]$ to $\tilde{X}$ defined so that

$$\eta_d(t) = \begin{cases} \eta_c(t) & \text{if } a \le t \le c; \\ \sigma(\gamma(t)) & \text{if } c \le t \le d. \end{cases}$$

Then $\eta_d(a) = q$ and $\pi(\eta_d(t)) = \gamma(t)$ for all $t \in [a,d]$. The restrictions of the function $\eta_d\colon [a,d] \to \tilde{X}$ to the subintervals $[a,c]$ and $[c,d]$ of $[a,d]$ are continuous. It follows from the Pasting Lemma (Lemma 1.25) that η_d is continuous on $[a,d]$. Thus $d \in S$. We conclude from this that $J(s,\delta) \subset S$. However s is defined to be the supremum of the set S. Therefore $s = b$, and b belongs to S. It follows that that there exists a continuous map $\tilde{\gamma}\colon [a,b] \to \tilde{X}$ for which $\tilde{\gamma}(a) = q$ and $\pi \circ \tilde{\gamma} = \gamma$, as required. ∎

4.5 The Homotopy-Lifting Theorem

Theorem 4.14 (Homotopy-Lifting Theorem) *Let $\tilde{X}$ and X be topological spaces, and let $\pi\colon \tilde{X} \to X$ be a covering map. Let Z be a topological* *space, and let* $F\colon Z \times [0,1] \to X$ *and* $\psi\colon Z \to \tilde{X}$ *be continuous functions with the property that* $\pi(\psi(z)) = F(z,0)$ *for all* $z \in Z$*. Then there exists a unique continuous function* $G\colon Z \times [0,1] \to \tilde{X}$ *such that* $G(z,0) = \psi(z)$ *for all* $z \in Z$ *and* $\pi \circ G = F$*.*

Proof For each $z \in Z$, consider the path $\gamma_z\colon [0,1] \to Z$ defined by $\gamma_z(t) = F(z,t)$ for all $t \in [0,1]$. Note that $\pi(\psi(z)) = \gamma_z(0)$. It follows from the Path-Lifting Theorem (Theorem 4.13) that there exists a unique continuous path $\tilde{\gamma}_z\colon [0,1] \to \tilde{X}$ such that $\tilde{\gamma}_z(0) = \psi(z)$ for all $z \in Z$ and $\pi \circ \tilde{\gamma}_z = \gamma_z$. Let the function $G\colon Z \times [0,1] \to \tilde{X}$ be defined by $G(z,t) = \tilde{\gamma}_z(t)$ for all $z \in Z$ and $t \in [0,1]$. Then $G(z,0) = \psi(z)$ for all $z \in Z$ and

$$\pi(G(z,t)) = \pi(\tilde{\gamma}_z(t)) = \gamma_z(t) = F(z,t)$$

for all $z \in Z$ and $t \in [0,1]$. It remains to show that the function $G\colon Z \times [0,1] \to \tilde{X}$ is continuous and that it is unique.

Given any $z \in Z$, let S_z denote the set of all real numbers c belonging to the closed interval $[0,1]$ which have the following property:

> there exists an open set N in Z such that $z \in N$ and the function G is continuous on $N \times [0,c]$.

Let s_z be the supremum $\sup S_z$ (i.e., the least upper bound) of the set S_z. We prove that s_z belongs to the set S_z and that $s_z = 1$.

Choose some $z \in Z$, and let $q \in \tilde{X}$ be given by $q = G(z, s_z)$. There exists an open neighbourhood V of $\pi(q)$ in X which is evenly covered by the map π. Thus $\pi^{-1}(V)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto V by the covering map π. One of these open sets contains the point q; let this open set be denoted by $\tilde{V}$. Then there exists a unique continuous map $\sigma\colon V \to \tilde{X}$ defined such that, for all $x \in V$, $\sigma(x)$ is the unique element of $\tilde{V}$ for which $\pi(\sigma(x)) = x$. Then $\sigma(F(z,s_z)) = q$. Now $F(z,s_z) = \pi(q)$. It follows from the continuity of the map F that there exists some positive real number δ and some open set M_1 in Z such that $z \in M_1$ and $F(M_1 \times J(s_z,\delta)) \subset V$, where

$$J(s_z,\delta) = \{t \in [0,1] : s_z - \delta < t < s_z + \delta\}.$$

Now we can choose some real number c belonging to S_z which satisfies $s_z - \delta < c \leq s_z$, because s_z is the least upper bound of the set S_z. It then follows from the definition of the set S_z that there exists an open set M_2 in Z such that $z \in M_2$ and the function G is continuous on $M_2 \times [0,c]$. Let

$$N = \{w \in M_1 \cap M_2 : G(w,c) \in \tilde{V}\}.$$

Then $z \in N$, and the continuity of the function G on $M_2 \times [0, c]$ ensures that N is open in Z. Moreover the function G is continuous on $N \times [0, c]$. Also $F(N \times J(s_z, \delta)) \subset V$.

Let $w \in N$. Then $G(w, c) \in \tilde{V}$ and $\pi(G(w, c)) = F(w, c)$. It follows from the definition of the map $\sigma \colon V \to \tilde{X}$ that $G(w, c) = \sigma(F(w, c))$. Also the interval $J(s_z, \delta)$ is connected, and

$$\pi(G(w, t)) = F(w, t) = \pi(\sigma(F(w, t)))$$

for all $t \in J(s_z, \delta)$. It follows from Proposition 4.11 that $G(w, t) = \sigma(F(w, t))$ for all $t \in J(s_z, \delta)$. We have thus shown that the function G is equal to the continuous function $\sigma \circ F$ on $N \times J(s_z, \delta)$. The function G is therefore continuous on both $N \times [0, c]$ and $N \times [c, t]$ for all $t \in J(s_z, \delta)$ satisfying $t \geq c$. It then follows from the Pasting Lemma (Lemma 1.25) that the function G is continuous on $N \times [0, t]$ for all $t \in J(s_z, \delta)$, and thus $J(s_z, \delta) \subset S_z$. This however contradicts the definition of S_z unless $s_z \in S_z$ and $s_z = 1$. We conclude therefore that $1 \in S_z$, and thus there exists an open set N in Z such that $z \in N$ and $G|N \times [0, 1]$ is continuous.

We conclude from this that every point of $Z \times [0, 1]$ is contained in some open subset of $Z \times [0, 1]$ on which that function G is continuous. It follows that $G \colon Z \times [0, 1] \to \tilde{X}$ is continuous (see Proposition 1.24).

The uniqueness of the map $G \colon Z \times [0, 1] \to \tilde{X}$ follows directly from the fact that for any $z \in Z$ there is a unique continuous path $\tilde{\gamma}_z \colon [0, 1] \to \tilde{X}$ such that $\tilde{\gamma}_z(0) = \psi(z)$ and $\pi(\tilde{\gamma}_z(t)) = F(z, t)$ for all $t \in [0, 1]$. ∎