MAU34201 Algebraic Topology I
Connectedness: An Alternative Approach

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Lemma A *Let X be a topological space, let A and B be subsets of X, where $A \subset B$, and let $i\colon A \hookrightarrow B$ be the inclusion function mapping A into B, defined so that $i(x) = x$ for all $x \in A$. Then the function $i\colon A \hookrightarrow B$ is continuous with respect to the subspace topologies on A and B.*

Proof Let W be a subset of B that is open with respect to the subspace topology on B. It follows immediately from the definition of the subspace topology on B that there exists some open set V in X for which $W = V \cap B$. Then $i^{-1}(W) = (V \cap B) \cap A = V \cap A$. It follows that the preimage $i^{-1}(W)$ of the set W under the function i is open with respect to the subspace topology on A. This shows that the function $i\colon A \hookrightarrow B$ is continuous, as required. ∎

Lemma B *Let X and Y be topological spaces, let A and B be subsets of X, where $A \subset B$, and let $\varphi\colon B \to Y$ be a continuous function mapping the set B into Y. Then the restriction $\varphi|A\colon A \to Y$ of the function φ to the set A is continuous on A.*

Proof The restriction function $\varphi|A$ is determined so that $\varphi|A = \varphi \circ i$, where $i\colon A \hookrightarrow B$ is the inclusion function mapping A into B. This inclusion function is continuous (Lemma A). Consequently the restriction function $\varphi|A\colon A \to Y$, being the composition of two continuous functions, is itself a continuous function. ∎

Proposition C *Let X be a topological space, let Y be a Hausdorff space, let A be a subset of X, and let $\overline{A}$ denote the closure of the set A in X. Let $\varphi\colon \overline{A} \to Y$ and $\psi\colon \overline{A} \to Y$ be continuous functions mapping the closure of the set A into the Hausdorff space Y. Suppose that $\varphi(x) = \psi(x)$ for all $x \in A$. Then $\varphi(x) = \psi(x)$ for all $x \in \overline{A}$.*

Proof Suppose that there were to exist some point p of $\overline{A}$ with the property that $\varphi(p) \neq \psi(p)$. Now the codomain Y of the functions f and g is a Hausdorff space. Consequently there would exist open sets V and W in Y such that $\varphi(p) \in V$, $\psi(p) \in W$ and $V \cap W = \emptyset$. Let $M = \varphi^{-1}(V) \cap \psi^{-1}(W)$. Now the continuity of the functions φ and ψ would ensure that the sets $\varphi^{-1}(V)$ and $\psi^{-1}(W)$ would be open with respect to the subpace topology on $\overline{A}$. Consequently the set M, being the intersection of two open sets, would itself be open with respect to the subspace topology on $\overline{A}$, and therefore there would exist some open set N in X with the property that $M = N \cap \overline{A}$.

Now it would then follow that $p \in N \cap \overline{A}$, and therefore $N \cap \overline{A}$ would be non-empty. Then $N \cap A$ would also be non-empty (see Lemma 1.6). Suppose that q were a point of $N \cap A$. Then $\varphi(q) \in V$ and $\psi(q) \in W$, and therefore $\varphi(q) \neq \psi(q)$, contradicting the condition that $\varphi(x) = \psi(x)$ for all $x \in A$. We conclude therefore that $\varphi(x) = \psi(x)$ for all $x \in \overline{A}$, as required. ∎

Corollary D *Let X be a topological space, let Y be a Hausdorff space, let A be a subset of X, and let $\overline{A}$ denote the closure of the set A in X. Let $\varphi\colon \overline{A} \to Y$ be a continuous function mapping the closure of the set A into the Hausdorff space Y. Suppose that the function φ is constant throughout A. Then it is constant throughout the closure $\overline{A}$ of A.*

Proof This result follows from an immediate application of Proposition C. Indeed if the given function φ takes some value y throughout the set A then the function φ is equal to the constant function with value y throughout the closure of the set A. ∎

Definition A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

Definition A topological space D is *discrete* if every subset of D is open in D.

Example The set $\mathbb{Z}$ of integers with the usual topology is an example of a discrete topological space. Indeed, given any integer n, the set $\{n\}$ is open in $\mathbb{Z}$, because it is the intersection of $\mathbb{Z}$ with the open ball in $\mathbb{R}$ of radius $\frac{1}{2}$ about n. Any non-empty subset S of $\mathbb{Z}$ is the union of the sets $\{n\}$ as n ranges over the elements of S. Therefore every subset of $\mathbb{Z}$ is open in $\mathbb{Z}$, and thus $\mathbb{Z}$, with the usual topology, is a discrete topological space.

Lemma E *Every discrete topological space is Hausdorff*

Proof Let D be a discrete topological space, and let v and w be distinct elements of D. Also let $V = \{v\}$ and $W = \{w\}$. Then the sets V and W are open in D, $v \in V$, $w \in W$ and $V \cap W = \emptyset$. We conclude therefore that the discrete topological space D is indeed Hausdorff. $\blacksquare$

Proposition F *Let X be a non-empty topological space. Then the following conditions are equivalent:—*

(i) *X is connected;*

(ii) *every continuous integer-valued function on X is constant;*

(iii) *every continuous function from X to the discrete topological space $\{0, 1\}$ is constant.*

Proof Suppose that X is connected. Let $f: X \to \mathbb{Z}$ be a continuous integer-valued function on X, and let the integer m belong to the range $f(X)$ of the function f. Now the set $\mathbb{Z}$ of integers, with the usual topology, is a discrete topological space. It follows that the preimage, under the continuous function f, of any subset of the set $\mathbb{Z}$ of integers is an open subset of X. Let $A = f^{-1}(\{m\})$. Then $X \setminus A = f^{-1}(\mathbb{Z} \setminus \{m\})$. Consequently the sets A and $X \setminus A$ are open subsets of X. The set A must therefore be both open and closed in X. Moreover this set is non-empty, because the integer m belongs to the range of the function f. It therefore follows from the connectedness of X that $A = X$. Thus the function f is constant on X. We have now established that (i) implies (ii).

Clearly (ii) implies (iii). It therefore only remains to show that (iii) implies (i).

Now suppose that condition (iii) is satisfied. Let A be a non-empty subset of X that is both open and closed, and let $f: X \to \{0, 1\}$ be the function on X defined such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in X \setminus A$. Now the preimages of the subsets $\emptyset$, $\{0\}$, $\{1\}$ and $\{0, 1\}$ of $\{0, 1\}$ are the sets $\emptyset$, $X \setminus A$, A and X respectively. These subsets of X are open in X. It follows that the function f is continuous. Condition (iii) then ensures that the function f is constant, and therefore $f(x) = 1$ for all $x \in X$. We conclude therefore that $A = X$. Thus the sets $\emptyset$ and X are the only subsets of X that are both open and closed, and consequently the topological space X is connected. Thus (iii) implies (i).

We can now conclude that conditions (i), (ii) and (iii) are equivalent, as required. $\blacksquare$

Corollary G *All intervals in the real line $\mathbb{R}$ are connected.*

Proof It follows directly from the Intermediate Value Theorem of real analysis that any continuous function mapping an interval in the real line to the set $\{0, 1\}$ must be constant over that interval. Consequently the interval is connected. $\blacksquare$

Example Let $X = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$. The topological space X is not connected. Indeed let $f: X \to \mathbb{Z}$ be defined such that

$$f(x, y) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

Then the function f is continuous on X but is not constant.

Lemma H *Let X be a topological space and let A be a connected subset of X. Then the closure $\overline{A}$ of A is connected.*

Proof Let $f: \overline{A} \to \{0, 1\}$ be a continuous function mapping the closure of $\overline{A}$ into the discrete topological space $\{0, 1\}$. The connnectedness of the set A ensures that the function f is constant over the set A (see Proposition F). It then follows that the function f must be continuous throughout the closure of A (see Corollary D). We conclude therefore that every continuous function mapping the closure of $\overline{A}$ of the connected set A into the discrete topological space $\{0, 1\}$ must be a constant function, and consequently the closure $\overline{A}$ of A is connected, as required. $\blacksquare$

Lemma I *Let $\varphi: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a connected subset of X. Then the image $\varphi(A)$ of the connected set A is connected.*

Proof Let $g: \varphi(A) \to \{0, 1\}$ be a continuous function mapping the image $\varphi(A)$ of A into the discrete topological space $\{0, 1\}$. Then the composition function $g \circ \varphi: A \to \{0, 1\}$ is a continuous function on the connected set A. This continuous function must be constant over the set A (see Proposition F), and consequently the continuous function $g: \varphi(A) \to \{0, 1\}$ must be constant over the set $\varphi(A)$. We conclude therefore (applying Proposition F) that the set $\varphi(A)$ must be connected, as required. $\blacksquare$

Lemma J *Let X be a topological space, and let A and B be connected subsets of X. Suppose that the intersection of the sets A and B is non-empty. Then the union $A \cup B$ of the sets A and B is connected.*

Proof Let $f\colon A \cup B \to \{0,1\}$ be a continuous function mapping $A \cup B$ into the discrete topological space $\{0,1\}$. The restriction of the function f to the set A is continuous on A (see Lemma B), and consequently (applying Proposition F) the function f must be continuous over the set A. It must also be continuous over the set B. Moreover the value taken by the function f on A must be the same as that taken by the function f on B, because $A \cap B$ is non-empty. Consequently the function f must be continuous throughout the whole of $A \cup B$. We conclude therefore that every continuous function from $A \cup B$ to $\{0,1\}$ must be a constant function, and consequently the set $A \cup B$ is connected. ∎

Connected Components of Topological Spaces

Proposition K *Let X be a topological space. For each $p \in X$, let S_p be the union of all connected subsets of X that contain p. Then*

(i) S_p is connected,

(ii) S_p is closed,

(iii) if $p, q \in X$, then either $S_p = S_q$, or else $S_p \cap S_q = \emptyset$.

Proof Let p be a point of the topological space X, and let $f\colon S_p \to \{0,1\}$ be a continuous function mapping the set S_p into the discrete topological space $\{0,1\}$. Let A be a connected subset of X to which the point p belongs. Then $f(x) = f(p)$ for all $x \in A$. Now the set S_p is a union of such connected subsets. It follows that $f(x) = f(p)$ for all $x \in S_p$. We conclude therefore that every continuous function mapping the set S_p into $\{0,1\}$ must be constant, and therefore the set S_p must be connected (see Proposition F). This establishes (i).

Now the closure $\overline{S_p}$ of S_p is connected (see Lemma H). It follows from the definition of the set S_p that $\overline{S_p} \subset S_p$, and therefore $\overline{S_p} = S_p$. Consequently the set S_p is closed. This establishes (ii).

Finally, suppose that p and q are points of X for which $S_p \cap S_q \neq \emptyset$. The sets S_p and S_q are connected, and their intersection is non-empty. It follows that $S_p \cup S_q$ is connected (see Lemma J). It then follows from the definition of the sets S_p and S_q that $S_p \cup S_q \subset S_p$ and $S_p \cup S_q \subset S_q$, and consequently $S_p = S_q$. This establishes (iii), completing the proof. ∎

Given any topological space X, the connected subsets S_p of X defined as in the statement of Proposition K are referred to as the *connected components* of X. Now a point p of X belongs to at least one connected component because it belongs to the connected component S_p that it determines. Also we see from Proposition K, part (iii) that the point p cannot belong to more than one distinct connected component, because two distinct connected components cannot have non-empty intersection. It follows that the topological space X is the disjoint union of its connected components.

Example Let X be the subset of $\mathbb{R}^2$ defined so that $X = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ Then the connected components of X are the sets

$$\{(x, y) \in \mathbb{R}^2 : x > 0\} \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Example Let Y be the open subset of the real line defined so that

$$Y = \{x \in \mathbb{R} : |x - n| < \tfrac{1}{2} \text{ for some integer } n\}.$$

Then the connected components of the set Y are the sets J_n for all integers n, where, for each integer n, J_n is the open interval with endpoints $n - \frac{1}{2}$ and $n + \frac{1}{2}$.

Products of Connected Topological Spaces

Lemma L *A Cartesian product $X \times Y$ of two connected topological spaces X and Y is itself connected.*

Proof Let $f: X \times Y \to \{0, 1\}$ be a continuous function mapping the Cartesian product $X \times Y$ into the discrete topological space $\{0, 1\}$, and let (p, q) and (r, s) be points of $X \times Y$. Then the function mapping points x of X to $f(x, q)$ is a continuous function mapping the connected topological space X into the discrete topological space $\{0, 1\}$. This continuous function must be constant (see Proposition F). Consequently $f(p, q) = f(r, q)$. Similarly $f(r, q) = f(r, s)$. It follows that $f(p, q) = f(r, s)$. We conclude therefore that every continuous function mapping the Cartesian product space $X \times Y$ into the discrete topological space $\{0, 1\}$ must be a constant function. It follows (applying Proposition F) that $X \times Y$ must be a connected topological space, as required.