MAU34201: Algebraic Topology I
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Disquisition III: The Brouwer Fixed Point Theorem in Two Dimensions

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Proposition. *Let E^2 be the closed unit disk, defined so that*

$$E^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

and let S^1 be the unit circle bounding the unit disk. Then there does not exist a continuous map $\rho\colon E^2 \to S^1$ with the property that $\rho(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^1$, where S^1 denotes the boundary circle of the closed unit disk E^2.

Proof Let $\sigma\colon [0,1] \to \mathbb{R}^2$ be the loop in $\mathbb{R}^2$, traversing the unit circle once in the anticlockwise direction, that is defined so that

$$\sigma(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all $t \in [0,1]$. Then the winding number of the loop σ about the origin $(0,0)$ is equal to one.

Now suppose that a continuous map $\rho\colon E^2 \to S^1$ were to exist with the property that $\rho(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^1$. Let

$$\gamma_\tau(t) = \rho(\tau\ \cos 2\pi t, \tau\ \sin 2\pi t)$$

for all real numbers t and τ in the closed unit interval $[0,1]$. Then the function sending each ordered pair (t, τ) in the closed unit square $[0,1] \times [0,1]$ to $\gamma_\tau(t)$ would be a continuous function on the closed unit square. Moreover the values of this function would all belong to the closed unit circle. It would follow that the loops γ_0 and γ_1 would have the same winding number about the origin $(0,0)$ (see Proposition 7.6). Consequently the winding number of the loop γ_1 about the origin would be equal to zero. But $\gamma_1(t) = \rho(\sigma(t)) = \sigma(t)$ for all $t \in [0,1]$, and the winding number of the loop σ about the origin is equal to one. Consequently there cannot exist any continuous map $\rho \colon E^2 \to S^1$ with the property that $\rho(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^1$, because the existence of such a map would lead to a logical contradiction. The required result has thus been proved. ∎

The Brouwer Fixed Point Theorem in Two Dimensions. *Let E^2 be the closed unit disk, defined so that*

$$E^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\},$$

and let $\varphi \colon E^2 \to E^2$ be a continuous map which maps the closed unit disk E^2 into itself. Then there exists some $\mathbf{p} \in E^2$ such that $\varphi(\mathbf{p}) = \mathbf{p}$.

Proof Suppose that there did not exist any fixed point $\mathbf{p}$ of $\varphi \colon E^2 \to E^2$. Then one could define a continuous map $\rho \colon E^2 \to S^1$ as follows: for each $\mathbf{x} \in E^2$, let $\rho(\mathbf{x})$ be the point on the boundary S^1 of E^2 obtained by continuing the line segment joining $\varphi(\mathbf{x})$ to $\mathbf{x}$ beyond $\mathbf{x}$ until it intersects S^1 at the point $\rho(\mathbf{x})$. Then $\rho \colon E^2 \to S^1$ would be a continuous map, and moreover $\rho(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^1$. But we have previously shown that there does not exist any continuous map $\rho \colon E^2 \to S^1$ with this property. We conclude that $\varphi \colon E^2 \to E^2$ must have at least one fixed point. ∎

Remark The Brouwer Fixed Point Theorem is also valid in higher dimensions. This theorem states that any continuous map from the closed n-dimensional ball into itself must have at least one fixed point. The proof of the theorem for $n > 2$ is analogous to the proof for $n = 2$, once one has shown that there is no continuous map from the closed n-dimensional ball to its boundary which is the identity map on the boundary. However winding numbers cannot be used to prove this result, and thus more powerful topological techniques need to be employed.