MAU34201: Algebraic Topology I
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Disquisition VI: On the Helicoid Surface

David R. Wilkins

Proposition A. *Let M be the surface in three-dimensional space $\mathbb{R}^3$ consisting of all points (x, y, z) of $\mathbb{R}^3$ that satisfy all of the following conditions:—*

- $(x, y) \neq (0, 0)$,
- $x = \sqrt{x^2 + y^2}\, \cos 2\pi z$,
- $y = \sqrt{x^2 + y^2}\, \sin 2\pi z$.

and let $\chi\colon M \to \mathbb{R}^2 \setminus \{(0, 0)\}$ be the function defined so that $\chi(x, y, z) = (x, y)$ for all $(x, y, z) \in M$. Then the map χ is a covering map.

Proof For each real number θ, let Ω_θ be the open set in $\mathbb{R}^2$ that is the complement of the ray

$$\{(t \cos\theta, t \sin\theta) : t \in \mathbb{R} \text{ and } t \leq 0\}.$$

Then, for each real number θ, there exists a corresponding continuous function $\omega_\theta\colon \Omega_\theta \to \mathbb{R}$ characterized by the properties that $\theta - \pi < \omega_\theta(x, y) < \theta + \pi$,

$$x = \sqrt{x^2 + y^2}\, \cos\omega_\theta(x, y) \quad \text{and} \quad y = \sqrt{x^2 + y^2}\, \sin\omega_\theta(x, y)$$

for all $(x, y) \in \Omega_\theta$ (see Proposition 7.1).

For each real number θ and for each integer m, let $\sigma_{\theta,m}\colon \Omega_\theta \to M$ be the continuous function defined so that

$$\sigma_{\theta,m}(x, y) = (x, y, (2\pi)^{-1}\omega_\theta(x, y) + m)$$

for all $(x, y) \in \Omega_\theta$. The continuity of the function ω_θ ensures the continuity of the function $\sigma_{\theta,m}$. For all real numbers θ, the preimage $\chi^{-1}(\Omega_\theta)$ of the open set Ω_θ is the disjoint union of the sets $\sigma_{\theta,m}(\Omega_\theta)$ as m ranges over the set $\mathbb{Z}$ of integers. Each of these sets $\sigma_{\theta,m}(\Omega_\theta)$ is an open subset of M. Indeed

$$\sigma_{\theta,m}(\Omega_\theta) = \{(x,y,z) \in M : (x,y) \in \Omega_\theta \text{ and } \frac{\theta}{2\pi} + m - \tfrac{1}{2} < z < \frac{\theta}{2\pi} + m + \tfrac{1}{2}\}.$$

Moreover the restriction of the function χ to each of these sets $\sigma_{\theta,m}(\Omega_\theta)$ maps that set homeomorhically onto Ω_θ. Indeed the functions χ and $\sigma_{\theta,m}$ are continuous, and $\chi(\sigma_{\theta,m}(x,y)) = (x,y)$ for all $(x,y) \in \Omega_\theta$. Suppose that $(x,y,z) \in \sigma_{\theta,m}(\Omega_\theta)$. Then $(x,y,z) = \sigma_{\theta,m}(x,y)$, and therefore

$$\sigma_{\theta,m}(\chi(x,y,z)) = \sigma_{\theta,m}(\chi(\sigma_{\theta,m}(x,y)) = \sigma_{\theta,m}(x,y) = (x,y,z).$$

Thus $\sigma_{\theta,m}(\chi(x,y,z)) = (x,y,z)$ for all $(x,y,z) \in \sigma_{\theta,m}(\Omega_\theta)$. It follows that the function from Ω_θ to $\sigma_{\theta,m}(\Omega_\theta)$ that sends each point (x,y) of Ω_θ to $\sigma_{\theta,m}(x,y)$ is a continuous function that is the inverse of the function from $\sigma_{\theta,m}(\Omega_\theta)$ to Ω_θ that sends each point (x,y,z) of $\sigma_{\theta,m}(\Omega_\theta)$ to $\chi(x,y,z)$, where $\chi(x,y,z) = (x,y)$. Consequently the subset $\sigma_{\theta,m}(\Omega_\theta)$ of M is mapped by χ homeomorphically onto Ω_θ.

We have now shown that, for all real numbers θ, the preimage $\chi^{-1}(\Omega_\theta)$ of the open subset Ω_θ of $\mathbb{R}^2 \setminus \{(0,0)\}$ is a disjoint union of the sets $\sigma_{\theta,m}(\Omega_\theta)$ as m ranges over the set of all integers. Moreover each of the subsets $\sigma_{\theta,m}(\Omega_\theta)$ of the surface M is open in M and is mapped by χ homeomorphically onto the open set Ω_θ. Thus the open set Ω_θ is evenly covered by the map χ. It follows that the surjective continuous function $\chi\colon M \to \mathbb{R}^2 \setminus \{(0,0)\}$ is a covering map, which is what we were required to prove. ∎

The surface M described in the statement of Proposition A is a *helicoid* in $\mathbb{R}^3$.

In what follows, we refer to the set $\mathbb{R}^2 \setminus \{(0,0)\}$ consisting of all points of the plane $\mathbb{R}^2$ distinct from the origin $(0,0)$ as the *punctured plane*.

Let $\gamma\colon [0,1] \to \mathbb{R}^2 \setminus \{0\}$ be a loop in the punctured plane. Then γ is a continuous function mapping the closed unit interval $[0,1]$ into the punctured plane, and $\gamma(0) = \gamma(1)$. The Path-Lifting Theorem (Theorem 4.13) ensures the existence of a path $\tilde{\gamma}\colon [0,1] \to M$ with the property that $\chi \circ \tilde{\gamma} = \gamma$, where $\chi\colon M \to \mathbb{R}^2 \setminus \{(0,0)\}$ is the covering map that sends a point (x,y,z) of the helicoid surface M to the point (x,y) of the punctured plane. Let $\tilde{\gamma}(t) = (f(t), g(t), h(t))$ for all $t \in [0,1]$, where f, g and h are real-valued function on the unit interval $[0,1]$. Then $\gamma(t) = (f(t), g(t))$ for all $t \in [0,1]$, and moreover

$$f(t) = \sqrt{f(t)^2 + g(t)^2}\, \cos(2\pi h(t))$$

and

$$g(t) = \sqrt{f(t)^2 + g(t)^2}\, \cos(2\pi h(t)).$$

The definition of winding number adopted in Section 7 of the lecture notes therefore ensures that the winding number of the loop γ about the origin is equal to $h(1) - h(0)$.

The covering map $\chi\colon M \to \mathbb{R}^2 \setminus \{(0,0)\}$ from the helicoid to the punctured plane is closely related to the exponential map of complex variable theory.

Given a complex number $u + iv$, where u and v are real numbers and $i = \sqrt{-1}$, the exponential $\exp(u + iv)$ of the complex number $u + iv$ takes the form $x + iy$ where $x = e^u \cos v$ and $y = e^u \sin v$. Thus, if we represent points of the complex plane $\mathbb{C}^2$ by ordered pairs of real numbers, where the components of those ordered pairs are the real and imaginary parts of the complex number, then the exponential map corresponds to a function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, where

$$\varphi(u, v) = (e^u \cos v,\ e^u \sin v).$$

for all $(u, v) \in \mathbb{R}^2$. Let the continuous function $\tilde{\varphi}\colon \mathbb{R}^2 \to M$ from the plane $\mathbb{R}^2$ to the helicoid surface M be defined so that

$$\tilde{\varphi}(u, v) = \left(e^u \cos v,\ \ e^u \sin v,\ \ \frac{v}{2\pi}\right)$$

for all $(u, v) \in \mathbb{R}^2$. Then the continuous function $\tilde{\varphi}\colon \mathbb{R}^2 \to M$ is a homeomorphism whose inverse maps a point (x, y, z) of the helicoid surface M to the point $(\frac{1}{2}\log(x^2 + y^2), 2\pi v)$ of the plane $\mathbb{R}^2$. Now the composition of a homeomorphism and a covering map is obviously a covering map. We conclude therefore that the function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ that corresponds to the exponential function of complex analysis is a covering map.

We have thus established the following result.

Proposition B. *The function from the complex plane $\mathbb{C}$ to the punctured complex plane $\mathbb{C}\setminus\{0\}$ that maps each complex number w to $\exp w$ is a covering map.*

Indeed, for any real number θ the open subset $\mathbb{C} \setminus L_\theta$ of the punctured complex plane $\mathbb{C} \setminus \{0\}$ is evenly covered by the exponential map, where

$$L_\theta = \{te^{i\theta} \in \mathbb{C} : t \in \mathbb{R} \text{ and } t \leq 0\}.$$

Let $\gamma\colon [0, 1] \to \mathbb{C}$ be a loop in the complex plane and let w be a complex number that does not lie on the loop γ. There then exists a path $\tilde{\gamma}\colon [0, 1] \to \mathbb{C}$ in the complex plane that is such as to ensure that

$$\exp \tilde{\gamma}(t) = \gamma(t) - w.$$

(This follows from a straightforward application of Proposition 7.2, taking into account the definition of the exponential map. The result also follows directly on applying the Path-Lifting Theorem, Theorem 4.13, to the exponential map, given that that map has been shown to be a covering map.) One can then easily establish from the relevant definitions that the winding number of the loop γ about the complex number w is equal to

$$\frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi\sqrt{-1}}.$$