MAU34201: Algebraic Topology I
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Disquisition I: Examples of Orbit Spaces

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Example The group $\mathbb{Z}$ of integers under addition acts freely and properly discontinuously on the real line $\mathbb{R}$. Indeed each integer n determines a corresponding homeomorphism $\theta_n: \mathbb{R} \to \mathbb{R}$, where $\theta_n(x) = x + n$ for all $x \in \mathbb{R}$. Moreover $\theta_m \circ \theta_n = \theta_{m+n}$ for all $m, n \in \mathbb{Z}$, and θ_0 is the identity map of $\mathbb{R}$. If $U = (-\frac{1}{2}, \frac{1}{2})$ then $\theta_n(U) \cap U = \emptyset$ for all non-zero integers n.

The real line $\mathbb{R}$ is simply-connected. It therefore follows from Corollary 6.12 that $\pi_1(\mathbb{R}/\mathbb{Z}, b) \cong \mathbb{Z}$ for any point b of $\mathbb{R}/\mathbb{Z}$.

Let $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map from the real line $\mathbb{R}$ to the orbit space $\mathbb{R}/\mathbb{Z}$ that sends each real number to its orbit under the action of the group of integers, let $p: \mathbb{R} \to S^1$ be defined such that

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

for all $t \in \mathbb{R}$. Then $p(t_1) = p(t_2)$ for all real numbers t_1 and t_2 satisfying $q(t_1) = q(t_2)$. Thus there is a well-defined function $h: \mathbb{R}/\mathbb{Z} \to S^1$ characterized by the property that $h(q(t)) = p(t)$ for all real numbers t.

The continuous map $h: \mathbb{R}/\mathbb{Z} \to S^1$ is a homeomorphism (see Corollary 6.7). It follows that

$$\pi_1(S^1, h(b)) \cong \pi_1(\mathbb{R}/\mathbb{Z}, b) \cong \mathbb{Z}$$

for all $b \in \mathbb{R}/\mathbb{Z}$.

Example The group $\mathbb{Z}^n$ of ordered n-tuples of integers under addition acts freely and properly discontinuously on $\mathbb{R}^n$, where

$$\theta_{(m_1, m_2, \ldots, m_n)}(x_1, x_2, \ldots, x_n) = (x_1 + m_1, x_2 + m_2, \ldots, x_n + m_n)$$

for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional torus, homeomorphic to the product of n circles. It follows from Corollary 6.12 that the fundamental group of this n-dimensional torus is isomorphic to the group $\mathbb{Z}^n$.

Example Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ centred on the origin, and let C_2 denote the cyclic group of order 2. Then $C_2 = \{e, a\}$, where $e^2 = a^2 = e$ and $ea = ae = a$. The group C_2 acts freely and discontinuously on S^n, where e acts as the identity map of S^n and a acts as the antipodal map sending $\mathbf{x}$ to $-\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The orbit space S^n/C^2 is homeomorphic to real projective n-dimensional space $\mathbb{R}P^n$. Now the n-dimensional sphere is simply-connected if $n > 1$. It follows from Corollary 6.12 that the fundamental group of $\mathbb{R}P^n$ is isomorphic to the cyclic group C_2 when $n > 1$.

Note that S^0 is a pair of points, and $\mathbb{R}P^0$ is a single point. Also S^1 is a circle (which is not simply-connected) and $\mathbb{R}P^1$ is homeomorphic to a circle. Moreover, for any $b \in S^1$, the homomorphism $q_\#: \pi_1(S^1, b) \to \pi_1(\mathbb{R}P^1, q(b))$ corresponds to the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ that sends each integer n to $2n$. This is consistent with the conclusions of Corollary 6.11 in this example.

Example Given a pair (m, n) of integers, let $\theta_{m,n}: \mathbb{R}^2 \to \mathbb{R}^2$ be the homeomorphism of the plane $\mathbb{R}^2$ defined such that

$$\theta_{m,n}(x, y) = (x + m, (-1)^m y + n)$$

for all $(x, y) \in \mathbb{R}^2$. Let (m_1, n_1) and (m_2, n_2) be ordered pairs of integers. Then

$$\theta_{m_1,n_1} \circ \theta_{m_2,n_2} = \theta_{m_1+m_2, n_1+(-1)^{m_1} n_2}.$$

Let Γ be the group whose elements are represented as ordered pairs of integers, where the group operation $\#$ on Γ is defined such that

$$(m_1, n_1)\#(m_2, n_2) = (m_1 + m_2, n_1 + (-1)^{m_1} n_2)$$

for all $(m_1, n_1), (m_2, n_2) \in \Gamma$. The group Γ is non-Abelian, and its identity element is $(0, 0)$. This group acts on the plane $\mathbb{R}^2$: given $(m, n) \in \Gamma$ the corresponding symmetry $\theta_{m,n}$ is a translation if m is even, and is a glide reflection if m is odd.

Given a pair (m, n) of integers, the corresponding homeomorphism $\theta_{m,n}$ maps an open disk about the point (x, y) onto an open disk of the same radius about the point $\theta_{(m,n)}(x, y)$. It follows that if D is the open disk of radius $\frac{1}{2}$ about the point (x, y), and if $D \cap \theta_{m,n}(D)$ is non-empty, then $(m, n) = (0, 0)$. Thus the group Γ maps freely and properly discontinuously on the plane $\mathbb{R}^2$.

Now each orbit intersects the closed unit square S, where $S = [0,1] \times [0,1]$. If $0 < x < 1$ and $0 < y < 1$ then the orbit of (x, y) intersects the square S in one point, namely the point (x, y). If $0 < x < 1$, then the orbit of $(x, 0)$ intersects the square in two points $(x, 0)$ and $(x, 1)$. If $0 < y < 1$ then the orbit of $(0, y)$ intersects the square S in the two points $(0, y)$ and $(1, 1 - y)$. (Note that $(1, 1 - y) = \theta_{1,1}(0, y)$.) And the orbit of any corner of the square S intersects the square in the four corners of the square. The restriction $q|S$ of the quotient map $q: \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ to the square S is a continuous surjection defined on the square: one can readily verify that it is an identification map. It follows that the orbit space $\mathbb{R}^2/\Gamma$ is homeomorphic to the identification space obtained from the closed square S by identifying together the points $(x, 0)$ and $(x, 1)$ where the real number x satisfies $0 < x < 1$, identifying together the points $(0, y)$ and $(1, 1 - y)$ where the real number y satisfies $0 < y < 1$, and identifying together the four corners of the square. The identification space obtained in this fashion is a closed non-orientable surface, first described by Felix Klein in 1882, and now known as the *Klein bottle*. Apparently the surface was initially referred to as the *Kleinsche Fläche* (Klein's Surface), but this name was incorrectly translated into English, and, as a result the surface is now referred to as the Klein Bottle (*Kleinsche Flasche*).

The plane $\mathbb{R}^2$ is simply-connected. It follows from Corollary 6.12 that the fundamental group of the Klein bottle is isomorphic to the group Γ defined above.