MAU34201: Algebraic Topology I
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Disquisition V: The Fundamental Group of the Circle

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The following result follows from a direct application of a result that establishes that a continuous real-valued function can be defined along any path in the plane $\mathbb{R}^2$ that does not pass through the origin, where the value of this function at any point of the path represents the angle through which the vector $(0,1)$ needs to be rotated in the anticlockwise direction so as to point in the direction of the displacement vector from the origin to the point on the path in question. (The result follows directly on applying (see Proposition 7.2).

Proposition A. *Let S^1 be the unit circle in the Euclidean plane, defined so that*

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $\kappa: \mathbb{R} \to S^1$ be the continuous function mapping the real line $\mathbb{R}$ into the circle S^1 defined so that

$$\kappa(x) = (\cos 2\pi x, \ \sin 2\pi x)$$

for all real numbers x. Then, given any path $\gamma: [0,1] \to S^1$ in the circle S^1, represented as a continuous function mapping the closed unit interval $[0,1]$ into the circle, there exists a continuous real-valued function $\tilde{\gamma}: [0,1] \to \mathbb{R}$ defined over the closed unit interval $[0,1]$ with the property that $\kappa(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [0,1]$.

Remark The continuous function $\kappa: \mathbb{R} \to S^1$ is a covering map. Consequently the result stated in Proposition A above is an immediate consequence of the Path-Lifting Theorem for covering maps (Theorem 4.13), when this theorem is applied to the covering map κ wrapping the real line around the unit circle.

If the path γ is a loop in S^1 (which is the case if and only if $\gamma(0) = \gamma(1)$, then $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an integer. Moreover if $\eta\colon [0,1] \to \mathbb{R}$ is a continuous real-valued function on $[0,1]$ with the property that $\kappa(\eta(t)) = \gamma(t)$ for all $t \in [0,1]$ then there exists some integer m, independent of the value of t, determined so that

$$\eta(t) = \tilde{\gamma}(t) + m$$

for all $t \in [0,1]$. This follows from the fact that the function sending each real number t between 0 and 1 to $\eta(t) - \tilde{\gamma}(t)$ is a continuous integer-valued function on the closed unit interval $[0,1]$, and is thus a constant function on that interval. Consequently each loop γ in S^1 (represented as a continuous function mapping $[0,1]$ into S^1) determines an associated integer $n(\gamma)$ characterized by the property that

$$n(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$$

for any continuous real-valued function $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ on $[0,1]$ for which $\kappa \circ \tilde{\gamma} = \gamma$. This integer $n(\gamma)$ is the *winding number* of the loop γ. (This winding number is the winding number of the loop γ about the origin, when one considers the circle S^1 as being embedded in $\mathbb{R}^2$ as the unit circle consisting of all points whose Euclidean distance from the origin is equal to one.)

Now let $H\colon [0,1] \times [0,1] \to S^1$ be a continuous map that satisfies $H(0,\tau) = H(1,\tau)$ for all $\tau \in [0,1]$. Also, for each $\tau \in [0,1]$, let $n(\gamma_\tau)$ be the winding number of the loop γ_τ in S^1 defined such that $\gamma_\tau(t) = H(t,\tau)$ for all $t \in [0,1]$. Then $n(\gamma_0) = n(\gamma_1)$. (This follows directly as a special case of Proposition 7.6.)

The following result follows immediately.

Proposition B. *Let S^1 be the unit circle in the Euclidean plane, defined so that*

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $\mathbf{b}$ be a point of S^1. Let α and β be loops in S^1 based at $\mathbf{b}$. Suppose that $\alpha \simeq \beta$ rel $\{0,1\}$. Then $n(\alpha) = n(\beta)$, where $n(\alpha)$ and $n(\beta)$ denote the winding numbers of the loops α and β respectively.

Remark The result stated in Proposition B above can be deduced directly, within the context of the general theory of covering maps, on applying the Homotopy-Lifting Theorem (see Theorem 4.14 and Proposition 6.1).

The following theorem identifies the fundamental group of the circle. It is a special case of the result stated in Corollary 6.12. A proof is given below, making use of the results concerning winding numbers of loops in the circle, that does not explicitly assume general results concerning covering maps or concerning group actions in which groups act freely and properly discontinuously on topological spaces.

Theorem C. *Let S^1 be the unit circle in the Euclidean plane, defined so that*

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $\mathbf{b}$ be a point of S^1. Then the function sending each loop γ in S^1 based at $\mathbf{b}$ to its winding number $n(\gamma)$ induces an isomorphism from the fundamental group $\pi_1(S^1, \mathbf{b})$ of the circle S^1 to the group $\mathbb{Z}$ of integers.

Proof Let $\kappa: \mathbb{R} \to S^1$ denote the function from $\mathbb{R}$ to S^1 defined so that

$$\kappa(t) = (\cos 2\pi t, \, \sin 2\pi t)$$

for all real numbers t. Also, for each loop $\gamma: [0, 1] \to S^1$ in S^1 based at $\mathbf{b}$ let $[\gamma]$ denote the element of the fundamental group $\pi_1(S^1, \mathbf{b})$ determined by γ, and let $n(\gamma)$ denote the winding number of γ. Every element of $\pi_1(S^1, \mathbf{b})$ is the based homotopy class $[\gamma]$ of some loop γ in S^1 based at $\mathbf{b}$. If $\tilde{\gamma}: [0, 1] \to \mathbb{R}$ is a real-valued function for which $\kappa \circ \tilde{\gamma} = \gamma$ then $n(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$.

Let α and β be loops in S^1 based at $\mathbf{b}$. Suppose that $[\alpha] = [\beta]$. Then $\alpha \simeq \beta$ rel $\{0, 1\}$. It then follows from Proposition B above that $n(\alpha) = n(\beta)$. It follows from this that there is a well-defined function $\lambda: \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ characterized by the property that $\lambda([\gamma]) = n(\gamma)$ for all loops γ in S^1 based at $\mathbf{b}$.

Next we show that the function $\lambda: \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is a homomorphism. Let $\alpha: [0, 1] \to S^1$ and $\beta: [0, 1] \to S^1$ be loops in S^1 based at $\mathbf{b}$. Then there exists a continuous real-valued function $\eta: [0, 1] \to \mathbb{R}$ with the property that

$$\kappa(\eta(t)) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $\kappa(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in \mathbb{R}$ (see Proposition A above). Then $\alpha(t) = \kappa(\eta(\frac{1}{2}t))$ for all $t \in [0, 1]$. It follows from the definition of winding numbers that $n(\alpha) = \eta(\frac{1}{2}) - \eta(0)$. Also $\beta(t) = \kappa(\eta(\frac{1}{2}(t+1)))$ for all $t \in [0, 1]$, and therefore $n(\beta) = \eta(1) - \eta(\frac{1}{2})$. It follows that

$$n(\alpha) + n(\beta) = \eta(1) - \eta(0) = n(\kappa \circ \eta) = n(\alpha \, . \, \beta),$$

where $\alpha \,.\, \beta$ is the concatenation of the loops α and β. It follows that

$$\lambda([\alpha]) + \lambda([\beta]) = n(\alpha) + n(\beta) = n(\alpha \,.\, \beta) = \lambda([\alpha \,.\, \beta]) = \lambda([\alpha][\beta]).$$

We conclude that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is a homomorphism.

Next we show that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is injective. Let α and β be loops in S^1 for which $n(\alpha) = n(\beta)$. Then there exist real-valued functions $\tilde{\alpha}\colon [0,1] \to \mathbb{R}$ and $\tilde{\beta}\colon [0,1] \to \mathbb{R}$ for which $\alpha = \kappa \circ \tilde{\alpha}$ and $\beta = \kappa \circ \tilde{\beta}$ (see Proposition A above). Moreover

$$\tilde{\alpha}(1) - \tilde{\alpha}(0) = n(\alpha) = n(\beta) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

Also $\kappa(\tilde{\alpha}(0)) = \mathbf{b} = \kappa(\tilde{\beta}(0))$, and therefore there exists some integer m for which $\tilde{\beta}(0) = \tilde{\alpha}(0) + m$. Then

$$\tilde{\beta}(1) = \tilde{\beta}(1) - \tilde{\beta}(0) + \tilde{\alpha}(0) + m = \tilde{\alpha}(1) + m.$$

Let

$$F(t, \tau) = (1 - \tau)\tilde{\alpha}(t) + \tau(\tilde{\beta}(t) - m).$$

Then $F(t, 0) = \tilde{\alpha}(t)$ and $F(t, 1) = \tilde{\beta}(t) - m$ for all $t \in [0, 1]$. Also $F(0, \tau) = \tilde{\alpha}(0)$ and $F(1, \tau) = \tilde{\alpha}(1)$ for all $\tau \in [0, 1]$. Let $H\colon [0, 1] \times [0, 1] \to S^1$ be defined so that $H(t, \tau) = \kappa(F(t, \tau))$ for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Then $H(t, 0) = \alpha(t)$ and $H(t, 1) = \beta(t)$ for all $t \in [0, 1]$. Also $H(0, \tau) = H(1, \tau) = \mathbf{b}$ for all $\tau \in [0, 1]$. It follows that $\alpha \simeq \beta$ rel $\{0, 1\}$ and therefore $[\alpha] = [\beta]$ in $\pi_1(X, \mathbf{b})$. We conclude therefore that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is injective.

Let m be an integer, let t_0 be a real number for which $\kappa(t_0) = \mathbf{b}$, and let $\gamma(t) = \kappa(t_0 + mt)$ for all $t \in [0, 1]$. Then $\gamma\colon [0, 1] \to S^1$ is a loop in S^1 based at $\mathbf{b}$, and $\lambda([\gamma]) = n(\gamma) = m$. We conclude that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is surjective. We have now shown that the function λ is a homomorphism that is both injective and surjective. It follows that $\lambda\colon \pi_1(S^1, \mathbf{b}) \to \mathbb{Z}$ is an isomorphism. This completes the proof. ∎

Proposition D. *Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$. Then $\pi_1(X, (1, 0)) \cong \mathbb{Z}$.*

Proof Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

let $i\colon S^1 \to X$ be the inclusion map, and let $r\colon X \to S^1$ be the radial projection map, defined such that

$$r(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

for all $(x, y) \in X$. Now the composition map $r \circ i$ is the identity map of S^1. Let

$$u(x, y, \tau) = \frac{1 - \tau}{\sqrt{x^2 + y^2}} + \tau$$

for all $(x, y) \in X$ and $\tau \in [0, 1]$. Then the function $F: X \times [0, 1] \to X$ that sends $((x, y), \tau) \in X \times [0, 1]$ to $(u(x, y, \tau)x, u(x, y, \tau)y)$ is a homotopy between the composition map $i \circ r$ and the identity map of the punctured plane X. Moreover $F((x, y), \tau) = (x, y)$ for all $(x, y) \in S^1$ and $\tau \in [0, 1]$.

Let $\gamma: [0, 1] \to X$ be a loop in X based at $(1, 0)$ and let $H: [0, 1] \times [0, 1] \to X$ be defined so that $H(t, \tau) = F(\gamma(t), \tau)$ for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Then $H(t, 0) = r(\gamma(t))$ and $H(t, 1) = \gamma(t)$ for all $t \in [0, 1]$, and $H(0, \tau) = H(1, \tau) = (1, 0)$ for all $\tau \in [0, 1]$, and therefore $i \circ r \circ \gamma \simeq \gamma \text{ rel } \{0, 1\}$.

Now the continuous maps $i: S^1 \to X$ and $r: X \to S^1$ induce well-defined homomorphisms $i_\#: \pi_1(S^1, (1, 0)) \to \pi_1(X, (1, 0))$ and $r_\#: \pi_1(X, (1, 0)) \to \pi_1(S^1, (1, 0))$, where $i_\#[\eta] = [i \circ \eta]$ for all loops η in S^1 based at $(1, 0)$ and $r_\#[\gamma] = [r \circ \gamma]$ for all loops γ in X based at $(1, 0)$. Moreover

$$i_\#(r_\#([\gamma]) = i_\#([r \circ \gamma]) = [i \circ r \circ \gamma] = [\gamma]$$

for all loops γ in X based at $(1, 0)$, and

$$r_\#(i_\#([\eta]) = r_\#([i \circ \eta]) = [r \circ i \circ \eta] = [\eta]$$

for all loops η in S^1 based at $(1, 0)$. It follows that the homomorphism $i_\#: \pi_1(S^1, (1, 0)) \to \pi_1(X, (1, 0))$ is an isomorphism whose inverse is the homomorphism $r_\#: \pi_1(X, (1, 0)) \to \pi_1(S^1, (1, 0))$, and therefore

$$\pi_1(X, (1, 0)) \cong \pi_1(S^1, (1, 0)) \cong \mathbb{Z},$$

as required. ∎

Example Let E^2 be the closed unit disk in $\mathbb{R}^2$ and let S^1 be its boundary circle, where

$$\begin{aligned} E^2 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, \\ S^1 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \end{aligned}$$

let $\iota: S^1 \to E^2$ be the inclusion map, and let $\mathbf{b} = (1, 0)$. Suppose there were to exist a continuous map $\rho: E^2 \to S^1$ with the property that $\rho(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^1$. Then $\rho \circ \iota: S^1 \to S^1$ would be the identity map of the unit circle S^1. It would then follow that $\rho_\# \circ \iota_\#$ would be the identity isomorphism of $\pi_1(S^1, \mathbf{b})$, where $\iota_\#: \pi_1(S^1, \mathbf{b}) \to \pi_1(E^2, \mathbf{b})$ and $\rho_\#: \pi_1(E^2, \mathbf{b}) \to \pi_1(S^1, \mathbf{b})$ denote the homomorphisms of fundamental groups induced by $\iota: S^1 \to E^2$ and $\rho: E^2 \to S^1$ respectively.

But $\pi_1(E^2, \mathbf{b})$ is the trivial group, because E^2 is a convex set in $\mathbb{R}^2$, and $\pi_1(S^1, \mathbf{b}) \cong \mathbb{Z}$ (see Theorem C above). It follows that the identity homomorphism of $\pi_1(S^1, \mathbf{b})$ cannot be expressed as a composition of two homomorphisms $\theta \circ \varphi$ where θ is a homomorphism from $\pi_1(S^1, \mathbf{b})$ to $\pi_1(E^2, \mathbf{b})$ and φ is a homomorphism from $\pi_1(E^2, \mathbf{b})$ to $\pi_1(S^1, \mathbf{b})$. Therefore there cannot exist any continous map $\rho: E^2 \to S^1$ with the property that $\rho(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^1$. (This results are discussed, using the theory of winding numbers in a more direct fashion, in the Disquisition on the Brouwer Fixed Point Theorem in two dimensions. Moreover the result is used to establish the Brouwer Fixed Point Theorem in the two-dimensional case which ensures that every continuous map from the two-dimensional closed disk E^2 to itself has a fixed point.