MAU34201: Algebraic Topology I
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Disquisition VIII: Examples concerning Local Homeomorphisms and Covering Maps

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Example Consider the continuous function from the real line $\mathbb{R}$ to the closed unit interval $[-1, 1]$ that sends each real number θ to $\sin\theta$.

This function is not a local homeomorphism. Indeed there is no open set containing $\frac{1}{2}\pi$ that is mapped homeomorphically onto an open set in the unit interval $[-1, 1]$. Indeed, given an open set V to which the real number $\frac{1}{2}\pi$ belongs, there exists some positive real number u small enough to ensure that $\frac{1}{2}\pi - u \in V$ and $\frac{1}{2}\pi + u \in V$. Both these elements of $\mathbb{R}$ are then mapped by the sine function to the same value and consequently the open set V is not mapped bijectively onto any subset of $[-1, 1]$. We conclude therefore that the continuous function that maps each real number θ to $\sin\theta$ is not a local homeomorphism from $\mathbb{R}$ to $[-1, 1]$. All covering maps are local homeomorphisms. Consequently the continuous function in question is not a covering map.

Example Let n be a non-zero integer, and let S^1 denote the unit circle in $\mathbb{R}^2$. Consider the continuous function from S^1 to S^1 sending $(\cos\theta, \sin\theta)$ to $(\cos n\theta, \sin n\theta)$ for all real numbers θ. This function is a covering map. Indeed the function is surjective. Moreover both $S^1 \setminus \{(-1, 0)\}$ and $S^1 \setminus \{(1, 0)\}$ are evenly covered. Indeed the preimage of $S^1 \setminus \{(-1, 0)\}$ is the disjoint union of the sets

$$\left\{(\cos\varphi, \sin\varphi) : \frac{(2k-1)\pi}{n} < \varphi < \frac{(2k+1)\pi}{n}\right\}$$

for $k = 0, 1, \ldots, n-1$, these sets are disjoint, and each of these sets is open in S^1 and is mapped homeomorphically onto $S^1 \setminus \{(-1, 0)\}$ under the map in question. Thus $S^1 \setminus \{(-1, 0)\}$ is evenly covered. A similar argument shows that $S^1 \setminus \{(1,0)\}$ is evenly covered. Consequently every point of the circle S^1 belongs to some evenly covered subset of that circle, and therefore the function in question is a covering map. It must therefore be a local homeomorphism.

Example Consider the map $\mu\colon (-2,2) \to S^1$, where $\mu(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in (-2,2)$. It can easily be shown that there is no open set U containing the point $(1,0)$ that is evenly covered by the map μ. Indeed suppose that there were to exist such an open set U. Then there would exist some δ satisfying $0 < \delta < \frac{1}{2}$ such that $U_\delta \subset U$, where

$$U_\delta = \{(\cos 2\pi t, \sin 2\pi t) : -\delta < t < \delta\}.$$

The open set U_δ would then be evenly covered by the map μ. However the connected components of $\mu^{-1}(U_\delta)$ are $(-2, -2+\delta)$, $(-1-\delta, -1+\delta)$, $(-\delta, \delta)$, $(1-\delta, 1+\delta)$ and $(2-\delta, 2)$, and neither $(-2,-2+\delta)$ nor $(2-\delta, 2)$ is mapped homeomorphically onto U_δ by μ.

Example Let

$$H = \{z \in \mathbb{C} : \mathrm{Re}[z] < 0\}$$

and

$$D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Let $\psi\colon H \to D$ be the function defined so that $\psi(z) = \exp(z)$ for all $z \in H$. Given any real number α, let

$$V_\alpha = D^* \setminus \{t(\cos\alpha + i \,\sin\alpha) : -1 < t < 0\}.$$

Also let

$$W_{\alpha,m} = \{z \in \mathbb{C} : \mathrm{Re}[z] < 0 \text{ and } \alpha + (2m-1)\pi < \mathrm{Im}[z] < \alpha + (2m+1)\pi\}$$

for each integer m. Now

$$\psi(u+iv) = \exp(u+iv) = e^u(\cos v + i \,\sin v)$$

for all real numbers u and v, where $i = \sqrt{-1}$. It follows that if $z \in H$ and $\alpha \in \mathbb{R}$, and if $\exp(z) \in V_\alpha$ then $\mathrm{Im}[z]$ cannot equal $\alpha + (2m+1)\pi$ for any integer m, and consequently $z \in W_{\alpha,m}$ for some integer m. We conclude therefore that $\psi^{-1}(V_\alpha)$ is the disjoint union of the open subsets $W_{\alpha,m}$ of H as m ranges of the set $\mathbb{Z}$ of integers.

Let α be a real number, and let m be an integer. Then the set $W_{\alpha,m}$ consists of those complex numbers $u + iv$ where u and v are real numbers,

$u < 0$ and $\alpha + (2m-1)\pi < v < \alpha + (2m+1)\pi$. A complex number $u + iv$ satisfying these conditions is mapped to a complex number of the form $r(\cos\theta + i\sin\theta)$ where $0 < r < 1$ and $\alpha - \pi < \theta < \alpha + \pi$. Now the complex numbers that can be expressed in this form are the elements of the set V_α. Moreover if r and α are real numbers satisfying $0 < r < 1$ and $\alpha - \pi < \theta < \alpha + \pi$, and if m is an integer then $\log r + i(\theta + 2m\pi) \in W_{\alpha,m}$ and

$$r(\cos\theta + i\sin\theta) = \psi(\log r + i(\theta + 2m\pi)).$$

The continuity of the sine, cosine and logarithm functions ensures that the function ψ maps $W_{\alpha,m}$ homeomorphically onto V_α for each real number α and for each integer m. The open sets V_α cover D^*. We conclude therefore that $\psi\colon H \to D^*$ is a covering map.

Example Consider the continuous function from $\{z \in \mathbb{C} : -4\pi < \operatorname{Im} z < 4\pi\}$ to $\{z \in \mathbb{C} : |z| > 0\}$ sending z to $\exp(z)$. This map is not a covering map. One can show that open sets in $\mathbb{C} \setminus \{0\}$ are not evenly covered if they contain positive real numbers. An alternative approach is to note that if $S = \{z \in \mathbb{C} : -4\pi < \operatorname{Im} z < 4\pi\}$ and if $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ is defined by $\gamma(t) = \exp(5\pi i t)$ for all $t \in [0,1]$, where $i = \sqrt{-1}$, then there is no lift $\tilde{\gamma}\colon [0,1] \to S$ with $\exp \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = 1$. Indeed there is a unique path in the complex plane satisfying these conditions, this path sends $t \in [0,1]$ to $5\pi i t$, and it leaves the domain S of the map in question when $t > \frac{4}{5}$.

The function in question is a local homeomorphism. Indeed the function from the entire complex plane $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$ that sends each complex number z to $\exp(z)$ is a covering map, and the restriction of any covering map to an open subset of its domain is a local homeomorphism.