MAU34201: Algebraic Topology I
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Disquisition IV: The Borsuk-Ulam Theorem

David R. Wilkins

Lemma A. *Let $\psi: S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a continuous function defined on S^1, where*

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Suppose that $\psi(-\mathbf{x}) = -\psi(\mathbf{x})$ for all $\mathbf{x} \in S^1$. Also let $\sigma: [0,1] \to S^1$ be the loop, traversing the unit circle once in the anticlockwise direction, defined so that

$$\sigma(t) = (\cos 2\pi t, \ \sin 2\pi t)$$

for all $t \in [0,1]$. Then the winding number $n(\psi \circ \sigma, (0,0))$ of $\psi \circ \sigma$ about the origin $(0,0)$ is odd.

Proof It follows from Proposition 7.2 that there exists a continuous real-valued function $\hat{\gamma}: [0,1] \to \mathbb{R}$ with the property that

$$\psi(\sigma(t)) = |\psi(\sigma(t))|(\cos \hat{\gamma}(t), \ \sin \hat{\gamma}(t))$$

for all $t \in [0,1]$. Now $\psi(\sigma(t + \frac{1}{2})) = -\psi(\sigma(t))$ for all $t \in [0, \frac{1}{2}]$, because $\sigma(t + \frac{1}{2}) = -\sigma(t)$ and $\psi(-\mathbf{x}) = -\psi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$. It follows that $\hat{\gamma}(t + \frac{1}{2}) = \hat{\gamma}(t) + (2m+1)\pi$ for some integer m. (The value of m for which this identity is valid does not depend on t, since every continuous function from $[0, \frac{1}{2}]$ to the set of integers is necessarily constant.) Hence

$$n(\psi \circ \sigma, (0,0)) = \frac{\hat{\gamma}(1) - \hat{\gamma}(0)}{2\pi} = \frac{\hat{\gamma}(1) - \hat{\gamma}(\frac{1}{2})}{2\pi} + \frac{\hat{\gamma}(\frac{1}{2}) - \hat{\gamma}(0)}{2\pi} = 2m + 1.$$

Thus $n(\psi \circ \sigma, (0,0))$ is an odd integer, as required. ∎

We denote by E and S^2 the closed unit disk and the unit sphere in $\mathbb{R}^3$ respectively, where

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

and

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Proposition B. *Let $\varphi\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $\varphi(-\mathbf{n}) = -\varphi(\mathbf{n})$ for all $\mathbf{n} \in S^2$. Then there exists some point $\mathbf{n}$ of S^2 with the property that $f(\mathbf{n}) = 0$.*

Proof Let $\mu\colon E \to S^2$ be the map defined by

$$\mu(x, y) = (x, y, \sqrt{1 - x^2 - y^2}).$$

(Thus the map μ maps the closed disk E homeomorphically onto the upper hemisphere in $\mathbb{R}^3$.) Let $\sigma\colon [0, 1] \to S^2$ be the parameterization of the equator in S^2 defined by

$$\sigma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

for all $t \in [0, 1]$, and let $\varphi\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $\varphi(-\mathbf{n}) = -\varphi(\mathbf{n})$ for all $\mathbf{n} \in S^2$. We show that there exists some point $\mathbf{p}$ of the closed unit disk E for which $\varphi(\mu(\mathbf{p})) = (0, 0)$.

For each real number τ satisfying $0 \leq \tau \leq 1$, let γ_τ be the loop in $\mathbb{R}^2$ defined so that

$$\gamma_\tau(t) = \varphi(\mu(\tau\sigma(t)))$$

for all $t \in [0, 1]$. If the origin $(0, 0)$ did not belong to the image $\varphi(\mu(E))$ of the closed unit disk E under the composition function $\varphi \circ \mu$, the origin would not lie on any of the loops γ_τ for which $0 \leq \tau \leq 1$ and therefore the winding numbers of the loops γ_0 and γ_1 about zero would be equal to one another. But the loop γ_0 is a constant loop. Consequently the winding number of the loop γ_1 about the origin would be zero. But $\gamma_1(t) = \varphi(\mu(\sigma(t)))$ for all $t \in [0, 1]$. It would therefore follow that the winding number of the loop $\varphi \circ \mu \circ \sigma$ about the origin would be equal to zero. But it follows from Lemma A that, assuming that the origin does not belong to the image of the unit circle under the map $\varphi \circ \mu$, this winding number must be an odd integer. Thus the supposition that the origin did not belong to the image $\varphi(\mu(E))$ of the closed unit disk under the composition map $\varphi \circ \mu$ leads to a contradiction. consequently there must exists some point $\mathbf{n}$ of $\mu(E)$ for which $\varphi(\mathbf{n}) = 0$. This completes the proof. ∎

Borsuk-Ulam Theorem. *Let $\varphi\colon S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists some point $\mathbf{n}$ of S^2 with the property that $\varphi(-\mathbf{n}) = \varphi(\mathbf{n})$.*

Proof This result follows immediately on applying Proposition B to the continuous function $\psi\colon S^2 \to \mathbb{R}^2$ defined by $\psi(\mathbf{n}) = \varphi(\mathbf{n}) - \varphi(-\mathbf{n})$. ∎

Remark If we say that two points of the surface of the Earth are *antipodal* if the line joining those points passes through the centre of the Earth, and if we represent the surface of the earth by the 2-dimensional sphere S^2, with antipodal points on the surface of the Earth corresponding to antipodal points on the sphere, and if we think of, for example, temperature and pressure as being continuous functions of position on the surface of the Earth, then the Borsuk-Ulam Theorem guarantees the existence of at least one pair of antipodal points on the surface of the Earth where the temperature and pressure at one point are equal to the temperature and pressure at the other.

Remark It is possible to generalize the Borsuk-Ulam Theorem to n dimensions. Let S^n be the unit n-sphere centered on the origin in $\mathbb{R}^n$. The Borsuk-Ulam Theorem in n-dimensions states that if $\varphi\colon S^n \to \mathbb{R}^n$ is a continuous map then there exists some point $\mathbf{x}$ of S^n with the property that $\varphi(\mathbf{x}) = \varphi(-\mathbf{x})$.