MAU34201: Algebraic Topology I
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Disquisition VII: A Riemann Surface covering an Open Set in the Complex Plane

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Example Let $Z = \mathbb{C} \setminus \{1, -1\}$, let

$$\tilde{Z} = \{(z, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } w^2 = z^2 - 1\},$$

and let $\psi \colon \tilde{Z} \to Z$ be defined so that $\psi(z, w) = z$ for all $(z, w) \in \tilde{Z}$. Let $(z_0, w_0) \in \tilde{Z}$, and let $z = z_0 + \zeta$. Then $w_0^2 = z_0^2 - 1$ and

$$\begin{aligned} z^2 - 1 &= z_0^2 - 1 + 2z_0\zeta + \zeta^2 = w_0^2 + 2z_0\zeta + \zeta^2 \\ &= w_0^2 \left(1 + \frac{2z_0\zeta + \zeta^2}{w_0^2}\right). \end{aligned}$$

Now the continuity at zero of the function sending each complex number ζ to $(2z_0\zeta + \zeta^2)/w_0^2$ ensures that there exists some positive real number δ such that

$$\left| \frac{2z_0\zeta + \zeta^2}{w_0^2} \right| < 1$$

whenever $|\zeta| < \delta$. Let $D(z_0, \delta)$ be the open disk of radius δ about z_0 in the complex plane. Then

$$\operatorname{Re}\left[1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2}\right] > 0$$

for all $z \in D(z_0, \delta)$.

Let

$$F(z) = \frac{1}{2} \log \left(1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2}\right)$$

for all $z \in D(z_0, \delta)$, where $\log(re^{i\theta}) = \log r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $-\pi < \theta < \pi$. Then $F(z)$ is a continuous function of z on $D(z_0, \delta)$, and

$$(\exp(F(z)))^2 = 1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2} = \frac{z^2 - 1}{w_0^2}$$

for all $z \in D(z_0, \delta)$.

Let $(z, w) \in \psi^{-1}(D(z_0, \delta))$. Then $z \in D(z_0, \delta)$ and

$$w^2 = z^2 - 1 = \Big(w_0 \exp(F(z))\Big)^2,$$

and therefore $w = \pm w_0 \exp(F(z))$. It follows that $\psi^{-1}(D(z_0, \delta)) = W_+ \cup W_-$ where

$$\begin{aligned} W_+ &= \{(z, w) \in \mathbb{C}^2 : z \in D(z_0, \delta) \text{ and } w = w_0 \exp(F(z))\}, \\ W_- &= \{(z, w) \in \mathbb{C}^2 : z \in D(z_0, \delta) \text{ and } w = -w_0 \exp(F(z))\}, \end{aligned}$$

Now

$$\mathrm{Re}\left[1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2}\right] > 0$$

for all $z \in D(z_0, \delta)$. It follows from the definition of $F(z)$ that

$$-\tfrac{1}{4}\pi < \mathrm{Im}[F(z)] < \tfrac{1}{4}\pi$$

for all $z \in D(z_0, \delta)$, and therefore

$$\mathrm{Re}[\exp(F(z))] = \exp(\mathrm{Re}[F(z)]) \cos(\mathrm{Im}[F(z)]) > 0$$

for all $z \in D(z_0, \delta)$. It follows that

$$\begin{aligned} W_+ &= \left\{(z, w) \in \tilde{Z} : z \in D(z_0, \delta) \text{ and } \mathrm{Re}\left[\frac{w}{w_0}\right] > 0\right\}, \\ &= \left\{(z, w) \in \psi^{-1}\left(D(z_0, \delta)\right) : \mathrm{Re}\left[\frac{w}{w_0}\right] > 0\right\}, \\ W_- &= \left\{(z, w) \in \tilde{Z} : z \in D(z_0, \delta) \text{ and } \mathrm{Re}\left[\frac{w}{w_0}\right] < 0\right\}, \\ &= \left\{(z, w) \in \psi^{-1}\left(D(z_0, \delta)\right) : \mathrm{Re}\left[\frac{w}{w_0}\right] < 0\right\}. \end{aligned}$$

Now $\psi^{-1}(D(z_0, \delta))$ is open in $\tilde{Z}$, because the it is the preimage of the open subset $D(z_0, \delta)$ of Z under the continuous map $\psi\colon \tilde{Z} \to Z$. Moreover the function mapping (z, w) to the real part of w/w_0 is continuous on $\psi^{-1}(D(z_0, \delta))$.

It follows that W_+ and W_- are open in $\tilde{Z}$. Also $W_+ \cap W_- = \emptyset$, and the map $\psi: \tilde{Z} \to Z$ maps each of the sets W_+ and W_- homeomorphically onto $D(z_0, \delta)$. It follows that the open disk $D(z_0, \delta)$ is evenly covered by the map $\psi: \tilde{Z} \to Z$. We have thus shown that any complex number z_0 distinct from 1 and -1 belongs to some open disk of positive radius contained in the set Z that is evenly covered by the map $\psi: \tilde{Z} \to Z$. It follows from tthat that the map $\psi: \tilde{Z} \to Z$ is a covering map.

Let $\hat{f}(z, w) = w$ for all $(z, w) \in \tilde{Z}$. Then

$$\hat{f}(\tilde{z})^2 = \psi(\tilde{z})^2 - 1$$

for all $\tilde{z} \in \tilde{Z}$. It follows that the function $\hat{f}: \tilde{Z} \to \mathbb{C}$ represents in some sense the many-valued 'function' $\sqrt{z^2 - 1}$. However this function $\tilde{z}$ is not defined on the open subset Z of the complex plane, but is instead defined over a covering space $\tilde{Z}$ of this open set. This covering space is the *Riemann surface* for the 'function' $\sqrt{z^2 - 1}$. This method of representing many-valued 'functions' of a complex variable using single-valued functions defined over a covering space was initiated and extensively developed by Bernhard Riemann (1826–1866) in his doctoral thesis.