MAU34201: Algebraic Topology I
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Disquisition II: An Example applying the Dog-Walking Lemma

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In what follows, the winding number of a loop γ in the plane $\mathbf{R}^2$ about some point $\mathbf{p}$ that does not lie on the loop is denoted by $n(\gamma, \mathbf{p})$.

Example Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be the closed curve in the complex plane defined such that

$$\gamma(t) = (3\cos 6\pi t + (\sin 16\pi t)(\sin 8\pi t), \ \ 4 \sin 6\pi t - 2e^{\cos 8\pi t - 1}\cos 8\pi t)$$

for all $t \in [0,1]$, where $i^2 = -1$. Let

$$\gamma_1(t) = (3\cos 6\pi t, \ \ 4\sin 6\pi t)$$

for all $t \in [0,1]$. Then $|\gamma_1(t)| \geq 3$ for all $t \in [0,1]$. Also $|\sin 16\pi t| \leq 1$ and $0 \leq e^{\cos 8\pi t - 1} \leq 1$ for all $t \in [0,1]$, and therefore

$$\bigl|\bigl((\sin 16\pi t)(\sin 8\pi t), \ \ -2e^{\cos 8\pi t - 1}\cos 8\pi t\bigr)\bigr|^2 \leq \sin^2 8\pi t + 4\cos^2 8\pi t \leq 4$$

for all $t \in [0,1]$. It follows that

$$\begin{aligned} |\gamma(t) - \gamma_1(t)| &= \bigl|\bigl((\sin 16\pi t)(\sin 8\pi t), \ \ -2e^{\cos 8\pi t - 1}\cos 8\pi t\bigr)\bigr| \\ &\leq 2 < |\gamma_1(t)| \end{aligned}$$

for all $t \in [0,1]$. The Dog-Walking Lemma (Lemma 7.3) then ensures that $n(\gamma, (0,0)) = n(\gamma_1, (0,0))$. Another application of the Dog-Walking Lemma then ensures that $n(\gamma_1, (0,0)) = n(\gamma_2, (0,0))$, where

$$\gamma_2(t) = (3\cos 6\pi t, \ \ 3\sin 6\pi t)$$

for all $t \in [0,1]$. Moreover

$$\gamma_2(t) = |\gamma_2(t)|(\cos \hat{\gamma}_2(t),\ \ \sin \hat{\gamma}_2(t))$$

where $\tilde{\gamma}_2\colon [0,1] \to \mathbb{R}$ is the real-valued function defined so that $\tilde{\gamma}_2(t) = 6\pi t$ for all $t \in [0,1]$. The definition of winding number ensures that

$$n(\gamma_2, (0,0)) = \frac{\hat{\gamma}_2(1) - \hat{\gamma}_2(0)}{2\pi} = 3.$$

Therefore $n(\gamma, (0,0)) = 3$.