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Section 5: The Fundamental Group of a Topological Space

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5 The Fundamental Group of a Topological Space

5.1 The Fundamental Group

Let X be a topological space, and let p and q be points of X. A *path* in X from p to q is represented as a continuous function $\gamma\colon [0,1] \to X$ for which $\gamma(0) = p$ and $\gamma(1) = q$. A *loop* in X based at p is represented as a continuous function $\gamma\colon [0,1] \to X$ for which $\gamma(0) = \gamma(1) = p$.

We can concatenate paths. Let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in some topological space X. Suppose that $\alpha(1) = \beta(0)$. We define the *product path* $\alpha\,.\,\beta\colon [0,1] \to X$ by

$$(\alpha\,.\,\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Let $\gamma\colon [0,1] \to X$ be a path in X. The *inverse path* $\gamma^{-1}\colon [0,1] \to X$ is defined so that $\gamma^{-1}(t) = \gamma(1-t)$ for all $t \in [0,1]$. (Thus if γ is a path from the point p to the point q then γ^{-1} is the path from q to p obtained by traversing γ in the reverse direction.)

Let X be a topological space, and let $p \in X$ be some chosen point of X. We define an equivalence relation on the set of all loops based at the basepoint p of X, where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0,1\}$. We denote the equivalence class of a loop $\gamma\colon [0,1] \to X$ based at p by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ. The set of equivalence classes of loops based at p is denoted by $\pi_1(X,p)$. Thus two loops γ_0 and γ_1 represent the same element of $\pi_1(X,p)$ if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0,1\}$ (i.e., if and only if there exists a homotopy $F\colon [0,1] \times [0,1] \to X$ that maps $(t,0)$ and $(t,1)$ to $\gamma_0(t)$ and $\gamma_1(t)$ for all $t \in [0,1]$ and also maps $(0,\tau)$ and $(1,\tau)$ to the basepoint p for all $\tau \in [0,1]$).

Theorem 5.1 *Let X be a topological space, let p be some chosen point of X, and let $\pi_1(X,p)$ be the set of all based homotopy classes of loops based at the point p. Then $\pi_1(X,p)$ is a group, the group operation on $\pi_1(X,p)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1\,.\,\gamma_2]$ for all loops γ_1 and γ_2 based at p.*

Proof First we show that the product operation on $\pi_1(X,p)$ is well-defined. Let γ_1, γ_1', γ_2 and γ_2' be loops in X based at the point p. Suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. Let $F_1\colon [0,1] \times [0,1] \to X$ be a homotopy between

γ_1 and γ_1', and let $F_2\colon [0,1] \times [0,1] \to X$ be a homotopy between γ_2 and γ_2', and where the homotopies F_1 and F_2 map $(0,\tau)$ and $(1,\tau)$ to p for all $\tau \in [0,1]$. Then let $F\colon [0,1] \times [0,1] \to X$ be defined so that

$$F(t,\tau) = \begin{cases} F_1(2t,\tau) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ F_2(2t-1,\tau) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then F is itself a homotopy from $\gamma_1 \,.\, \gamma_2$ to $\gamma_1' \,.\, \gamma_2'$. Moreover $F(0,\tau) = F(1,\tau) = p$ for all $\tau \in [0,1]$. Thus $[\gamma_1 \,.\, \gamma_2] = [\gamma_1' \,.\, \gamma_2']$. We conclude that the product operation on $\pi_1(X,p)$ is well-defined.

Next we show that the product operation on $\pi_1(X,p)$ is associative. Let γ_1, γ_2 and γ_3 be loops based at p, and let $\alpha = (\gamma_1 . \gamma_2) . \gamma_3$. Then $\gamma_1 . (\gamma_2 . \gamma_3) = \alpha \circ \theta$, where

$$\theta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Let $G\colon [0,1] \times [0,1] \to X$ be the continuous function defined so that $G(t,\tau) = \alpha((1-\tau)t + \tau\theta(t))$ for all $t, \tau \in [0,1]$. Then the continuous function G is a homotopy between $(\gamma_1 . \gamma_2) . \gamma_3$ and $\gamma_1 . (\gamma_2 . \gamma_3)$, and moreover this homotopy maps $(0,\tau)$ and $(1,\tau)$ to p for all $\tau \in [0,1]$. It follows that $(\gamma_1 \,.\, \gamma_2) \,.\, \gamma_3 \simeq \gamma_1 \,.\, (\gamma_2 \,.\, \gamma_3)$ rel $\{0,1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$. This shows that the product operation on $\pi_1(X,p)$ is associative.

Let $\varepsilon\colon [0,1] \to X$ denote the constant loop at p, defined by $\varepsilon(t) = p$ for all $t \in [0,1]$. Let functions θ_0 and θ_1 mapping the closed unit interval $[0,1]$ onto itself be defined so that

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2t - 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$\theta_1(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for all $t \in [0,1]$. Then $\varepsilon \,.\, \gamma = \gamma \circ \theta_0$ and $\gamma \,.\, \varepsilon = \gamma \circ \theta_1$ for any loop γ based at p. But the continuous map $(t,\tau) \mapsto \gamma((1-\tau)t + \tau\theta_j(t))$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0,1$ which sends $(0,\tau)$ and $(1,\tau)$ to p for all $\tau \in [0,1]$. Therefore $\varepsilon \,.\, \gamma \simeq \gamma \simeq \gamma \,.\, \varepsilon$ rel $\{0,1\}$. Consequently $[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ is the identity element of $\pi_1(X,p)$.

It only remains to verify the existence of inverses. Let the continuous function $K\colon [0,1] \times [0,1] \to X$ be defined so that

$$K(t,\tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma(2\tau(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then continuous function K is a homotopy between the loops ε and $\gamma \,.\, \gamma^{-1}$, and moreover this homotopy sends $(0, \tau)$ and $(1, \tau)$ to p for all $\tau \in [0, 1]$. Therefore $\varepsilon \simeq \gamma \,.\, \gamma^{-1} \operatorname{rel}\{0, 1\}$. It follows that $[\gamma][\gamma^{-1}] = [\gamma \,.\, \gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1}, we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required. ∎

Let p be a point of some topological space X. The group $\pi_1(X, p)$ is referred to as the *fundamental group* of X based at the point p.

Proposition 5.2 *Let $\varphi \colon X \to Y$ be a continuous map between topological spaces X and Y, and let p be a point of X. Then φ induces a homomorphism $\varphi_\# \colon \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ between the fundamental groups of X and Y with basepoints p and $\varphi(p)$ respectively, where $\varphi_\#([\gamma]) = [\varphi \circ \gamma]$ for all loops $\gamma \colon [0, 1] \to X$ based at p.*

Proof Let γ_0 and γ_1 be loops in X based at the point p that belong to the same based homotopy class, and thus represent the same element of the fundamental group of the topological space X with basepoint p. Then there exists a homotopy $H \colon [0, 1] \times [0, 1] \to X$ with the properties that $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1(t)$ for all $t \in [0, 1]$ and $H(0, \tau) = H(1, \tau) = p$ for all $\tau \in [0, 1]$.

Let $K = \varphi \circ H$. Then $K(t, 0) = \varphi(\gamma_0(t))$ and $K(t, 1) = \varphi(\gamma_1(t))$ for all $t \in [0, 1]$ and $K(0, \tau) = K(1, \tau) = \varphi(p)$ for all $\tau \in [0, 1]$. It follows that K is a based homotopy between the loops $\varphi \circ \gamma_0$ and $\varphi \circ \gamma_1$, and therefore those loops represent the same element of the fundamental group $\pi(Y, \varphi(p))$ of Y with basepoint $\varphi(p)$. Thus the map φ induces a well-defined function

$$\varphi_\# \colon \pi_1(X, p) \to \pi_1(Y, \varphi(p)).$$

Now if γ is the concatenation of loops α and β in X, where each of the loops α and β is based at the point p, then $\varphi \circ \gamma$ is the concatenation of the loops $\varphi \circ \alpha$ and $\varphi \circ \beta$. It follows that

$$\varphi_\#([\alpha]\,[\beta]) = \varphi_\#([\gamma]) = [\varphi \circ \gamma] = [\varphi \circ \alpha]\,[\varphi \circ \beta] = \varphi_\#([\alpha])\,\varphi_\#([\beta]).$$

Consequently the function $\varphi_\#$ is a homomorphism from the fundamental group $\pi_1(X, p)$ of X with basepoint p to the fundamental group $\pi_1(Y, \varphi(p))$ of Y with basepoint $\varphi(p)$. This completes the proof. ∎

If p, q and r are points belonging to topological spaces X, Y and Z, and if $\varphi \colon X \to Y$ and $\psi \colon Y \to Z$ are continuous maps satisfying $\varphi(p) = q$ and $\psi(q) = r$, then the induced homomorphisms $\varphi_\# \colon \pi_1(X, p) \to \pi_1(Y, q)$ and $\psi_\# \colon \pi_1(Y, q) \to \pi_1(Z, r)$ satisfy $\psi_\# \circ \varphi_\# = (\psi \circ \varphi)_\#$.

The property just described can in particular be applied in the case when $\varphi\colon X \to Y$ is a homeomorphism whose inverse is $\psi\colon Y \to X$. We can then conclude that $\varphi_\#\colon \pi_1(X,p) \to \pi_1(Y,\varphi(p))$ is an isomorphism of groups whose inverse is $\psi_\#\colon \pi_1(Y,\varphi(p)) \to \pi_1(X,p)$.

Proposition 5.3 *Let X be a topological space, and let α be a path in X. Then there is a well-defined isomorphism $\Theta_\alpha\colon \pi_1(X,\alpha(1)) \to \pi_1(X,\alpha(0))$ between the fundamental groups at the endpoints at that path which sends the based homotopy class $[\gamma]$ of any loop γ based at $\alpha(1)$ to the based homotopy class of the loop $\alpha.\gamma.\alpha^{-1}$ based at $\alpha(0)$, where*

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \textit{if } 0 \le t \le \frac{1}{3}; \\ \gamma(3t-1) & \textit{if } \frac{1}{3} \le t \le \frac{2}{3}; \\ \alpha(3-3t) & \textit{if } \frac{2}{3} \le t \le 1. \end{cases}$$

(Note that $\alpha.\gamma.\alpha^{-1}$ represents 'α followed by γ followed by α reversed').

Proof We first show that the function Θ_α between the fundamental groups of the topological space based at the points $\alpha(1)$ and $\alpha(0)$ is a homomorphism.

Let γ_1 and γ_2 be loops in X based at the point $\alpha(1)$. Then the product $\Theta_\alpha([\gamma_1])\Theta_\alpha([\gamma_2])$ in the fundamental group $\pi_1(X,\alpha(0))$ is represented by the loop $\eta_1\colon [0,1] \to X$ based at $\alpha(0)$ where

$$\eta_1(t) = \begin{cases} \alpha(6t) & \text{if } 0 \le t \le \frac{1}{6}, \\ \gamma_1(6t-1) & \text{if } \frac{1}{6} \le t \le \frac{1}{3}, \\ \alpha(3-6t) & \text{if } \frac{1}{3} \le t \le \frac{1}{2}, \\ \alpha(6t-3) & \text{if } \frac{1}{2} \le t \le \frac{2}{3}, \\ \gamma_2(6t-4) & \text{if } \frac{2}{3} \le t \le \frac{5}{6}, \\ \alpha(6-6t) & \text{if } \frac{5}{6} \le t \le 1. \end{cases}$$

Also the element $\Theta_\alpha([\gamma_1][\gamma_2])$ in the fundamental group is equal to $\Theta_\alpha([\gamma_1 \,.\, \gamma_2])$, where $\gamma_1 \,.\, \gamma_2$ is the concatenation of the loops γ_1 and γ_2, and therefore $\Theta_\alpha([\gamma_1][\gamma_2])$ is represented by the loop $\eta_2\colon [0,1] \to X$ in X based at $\alpha(0)$, where

$$\eta_2(t) = \begin{cases} \alpha(3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ \gamma_1(6t-2) & \text{if } \frac{1}{3} \le t \le \frac{1}{2}, \\ \gamma_2(6t-3) & \text{if } \frac{1}{2} \le t \le \frac{2}{3}, \\ \alpha(3-3t) & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Let $H_1\colon [0,1]\times[0,1]\to X$ be defined so that

$$H_1(t,\tau)=\begin{cases}\alpha(6t) & \text{if } 0\le t\le \frac{1}{6},\\ \gamma_1(6t-1) & \text{if } \frac{1}{6}\le t\le \frac{1}{3},\\ \alpha(1+2\tau-6\tau t) & \text{if } \frac{1}{3}\le t\le \frac{1}{2},\\ \alpha(1-4\tau+6\tau t) & \text{if } \frac{1}{2}\le t\le \frac{2}{3},\\ \gamma_2(6t-4) & \text{if } \frac{2}{3}\le t\le \frac{5}{6},\\ \alpha(6-6t) & \text{if } \frac{5}{6}\le t\le 1.\end{cases}$$

Note that $1+2\tau-6\tau t$ decreases from 1 to $1-\tau$ as t increases from $\frac{1}{3}$ to $\frac{1}{2}$, and $1-4\tau+6\tau t$ increases from $1-\tau$ to 1 as t increases from $\frac{1}{2}$ to $\frac{2}{3}$. Note also that $H_1(0,\tau)=H_1(1,\tau)=\alpha(0)$ for all $\tau\in[0,1]$.

Let η_0 be the loop in X based at $\alpha(0)$ defined so that

$$\eta_0(t)=\begin{cases}\alpha(6t) & \text{if } 0\le t\le \frac{1}{6},\\ \gamma_1(6t-1) & \text{if } \frac{1}{6}\le t\le \frac{1}{3},\\ \alpha(1) & \text{if } \frac{1}{3}\le t\le \frac{2}{3},\\ \gamma_2(6t-4) & \text{if } \frac{2}{3}\le t\le \frac{5}{6},\\ \alpha(6-6t) & \text{if } \frac{5}{6}\le t\le 1.\end{cases}$$

Then the function H_1 is a based homotopy between the loops η_1 and η_0, and thus the loops η_1 and η_0 represent the same element of the fundamental group $\pi_1(X,\alpha(0))$ of X based at the point $\alpha(0)$.

But $\eta_0=\eta_2\circ\kappa$, where κ is the monotonically increasing function mapping the unit interval $[0,1]$ onto itself defined so that

$$\kappa(t)=\begin{cases}2t & \text{if } 0\le t\le \frac{1}{6},\\ t+\frac{1}{6} & \text{if } \frac{1}{6}\le t\le \frac{1}{3},\\ \frac{1}{2} & \text{if } \frac{1}{3}\le t\le \frac{2}{3},\\ t-\frac{1}{6} & \text{if } \frac{2}{3}\le t\le \frac{5}{6},\\ 2t-1 & \text{if } \frac{5}{6}\le t\le 1.\end{cases}$$

Now $0\le t-\tau t+\tau\kappa(t)\le 1$ whenever $0\le t\le 1$ and $0\le\tau\le 1$. Let $H_2\colon[0,1]\times[0,1]\to X$ be the continuous map defined so that

$$H_2(t,\tau)=\eta_2(t-\tau t+\tau\kappa(t))$$

for all $t\in[0,1]$ and $\tau\in[0,1]$. Then $H_2(t,0)=\eta_2(t)$ and $H_2(t,1)=\eta_2(\kappa(t))=\eta_0(t)$ for all $t\in[0,1]$. Also $H_2(0,\tau)=\eta_2(0)=\alpha(0)$ and

$H_2(1,\tau) = \eta_2(1) = \alpha(0)$ for all $\tau \in [0,1]$. It follows that H_2 is a based homotopy between the loops η_2 and η_0. It follows that the loops η_0 and η_2 represent the same element of the fundamental group $\pi_1(X,\alpha(0))$ of X with basepoint $\alpha(0)$.

Combining the results obtained above, we see that the identity

$$\Theta_\alpha([\gamma_1])\Theta_\alpha([\gamma_2]) = [\eta_1] = [\eta_0] = [\eta_2] = \Theta_\alpha([\gamma_1][\gamma_2])$$

holds in the fundamental group $\pi_1(X,\alpha(0))$ of X based at the point $\alpha(0)$. Thus the function $\Theta_\alpha\colon \pi_1(X,\alpha(1)) \to \pi_1(X,\alpha(0))$ is a group homomorphism.

Now let $\Theta_{\alpha^{-1}}$ be the corresponding homomorphism from $\pi_1(X,\alpha(0))$ to $\pi_1(X,\alpha(1))$ determined by the inverse α^{-1} of the path α, where $\alpha^{-1}(t) = \alpha(1-t)$ for all $t \in [0,1]$, and let γ be a loop in X based at $\alpha(1)$. Then $\Theta_{\alpha^{-1}}(\Theta_\alpha([\gamma]))$ is represented by the loop $\zeta_0\colon [0,1] \to X$ based at $\alpha(1)$, where

$$\zeta_0(t) = \begin{cases} \alpha(1-3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ \alpha(9t-3) & \text{if } \frac{1}{3} \le t \le \frac{4}{9}, \\ \gamma(9t-4) & \text{if } \frac{4}{9} \le t \le \frac{5}{9}, \\ \alpha(6-9t) & \text{if } \frac{5}{9} \le t \le \frac{2}{3}, \\ \alpha(3t-2) & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Now let $K_1\colon [0,1]\times[0,1] \to X$ be defined so that

$$K_1(t,\tau) = \begin{cases} \alpha(1-3t+3t\tau) & \text{if } 0 \le t \le \frac{1}{3}, \\ \alpha(9t-3+4\tau-9t\tau) & \text{if } \frac{1}{3} \le t \le \frac{4}{9}, \\ \gamma(9t-4) & \text{if } \frac{4}{9} \le t \le \frac{5}{9}, \\ \alpha(6-9t-5\tau+9t\tau) & \text{if } \frac{5}{9} \le t \le \frac{2}{3}, \\ \alpha(3t-2+3\tau-3t\tau) & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Now $0 \le 1-3t+3t\tau \le 1$ whenever $0 \le t \le \frac{1}{3}$ and $0 \le \tau \le 1$, $0 \le 9t-3+4\tau-9t\tau \le 1$ whenever $\frac{1}{3} \le t \le \frac{4}{9}$ and $0 \le \tau \le 1$, $0 \le 6-9t-5\tau+9t\tau$ whenever $\frac{5}{9} \le t \le \frac{2}{3}$ and $0 \le \tau \le 1$, and $0 \le 3t-2+3\tau-3t\tau \le 1$ whenever $\frac{2}{3} \le t \le 1$ and $0 \le \tau \le 1$. Consequently the function K_1 is well-defined. Also $K_1(0,\tau) = \alpha(1)$ and $K_1(1,\tau) = \alpha(1)$ for all $\tau \in [0,1]$.

Let $\zeta_1\colon [0,1] \to X$ be the loop based at $\alpha(1)$ defined such that

$$\zeta_1 = \begin{cases} \alpha(1) & \text{if } 0 \le t \le \frac{4}{9}, \\ \gamma(9t-4) & \text{if } \frac{4}{9} \le t \le \frac{5}{9}, \\ \alpha(1) & \text{if } \frac{5}{9} \le t \le 1. \end{cases}$$

Then $K_1(t, 0) = \zeta_0(t)$ and $K_1(t, 1) = \zeta_1(t)$ for all $t \in [0, 1]$. We have previously noted that $K_1(0, \tau) = K_1(1, \tau) = \alpha(1)$. It follows that K_1 is a based homotopy between the loops ζ_0 and ζ_1, and thus those loops represent the same element of the fundamental group $\pi_1(X, \alpha(1))$.

Now let $\varphi\colon [0, 1] \to [0, 1]$ be defined such that

$$\varphi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{4}{9}, \\ 9t - 4 & \text{if } \frac{4}{9} \leq t \leq \frac{5}{9}, \\ 1 & \text{if } \frac{5}{9} \leq t \leq 1. \end{cases}$$

Then let

$$K_2(t, \tau) = \gamma(1 - \tau + \tau\varphi(t))$$

for all $t \in [0, 1]$ and $\tau \in [0, 1]$. Then $K_2(t, 0) = \gamma(t)$ and $K_2(t, 1) = \zeta_1(t)$ for all $t \in [0, 1]$, and $K_2(0, \tau) = K_2(1, \tau) = \alpha(1)$ for all $\tau \in [0, 1]$. It follows that K_2 is a based homotopy between the loops γ and ζ_1. Consequently

$$\Theta_{\alpha^{-1}}(\Theta_\alpha([\gamma])) = [\zeta_0] = [\zeta_1] = [\gamma].$$

This identity holds for all loops γ based at $\alpha(1)$. We conclude therefore that $\Theta_\alpha\colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ is an isomorphism of groups whose inverse is the homomorphism $\Theta_{\alpha^{-1}}\colon \pi_1(X, \alpha(0)) \to \pi_1(X, \alpha(1))$ determined by the inverse α^{-1} of the path α. The result follows. ∎

Corollary 5.4 *Let X be a path-connected topological space. Then the isomorphism class of the fundamental group of the space is independent of the choice of basepoint within the topological space.*

Proposition 5.5 *A path-connected topological space X is simply connected if and only if there exists some point p of X for which the fundamental group $\pi_1(X, p)$ is trivial.*

Proof It follows from Proposition 3.17 and the definition of the fundamental group that a path-connected topological space is simply connected if and only if the fundamental group $\pi_1(X, p)$ of X with basepoint p is the trivial group for all points p of X. It then follows from Proposition 5.3 that this is the case if and only if there exists at least one point p of the path-connected topological space for which $\pi_1(X, p)$ is the trivial group. The result follows. ∎