MAU34201: Algebraic Topology I
Michaelmas Term 2020
Solutions to Assignment 2

January 17, 2021

1. This problem may be solved as follows. First we determine the finishing points of paths $\hat{\alpha}\colon [0,1] \to \mathbb{R}^2$ and $\hat{\beta}\colon [0,1] \to \mathbb{R}^2$ for which $\hat{\alpha}(0) = (0,0)$, $\hat{\beta}(0) = (0,0)$, $\rho \circ \hat{\alpha} = \rho \circ \alpha$ and $\rho \circ \hat{\beta} = \rho \circ \beta$. We can then determine the finishing point of the path $\hat{\gamma}\colon [0,1] \to \mathbb{R}^2$ for which $\hat{\gamma}(0) = (0,0)$ and $\rho \circ \hat{\gamma} = (\rho \circ \alpha) \,.\, (\rho \circ \beta)$. And then, having determined the finishing point of $\hat{\gamma}$ we can determine the finishing point of γ.

   Now it follows from the definition of orbit spaces, free and properly discontinuous group actions and standard results on the uniqueness of lifts of paths with respect to covering maps that there exist integers m and n for which $\hat{\alpha} = \theta_{(m,n)} \circ \alpha$. Now $\alpha(0) = (1,1)$. Accordingly we require that $\theta_{(m,n)}(1,1) = (0,0)$. Consequently we require that

   $$(1 + m, (-1)^m \times 1 + n) = (0,0).$$

   Clearly $(m,n) = (-1,1)$. It follows that

   $$\hat{\alpha}(1) = \theta_{(-2,-4)}(\alpha(1)) = \theta_{(-1,1)}(5,3) = (4,-2).$$

   Next we determine integers m and n for which $\hat{\beta} = \theta_{(m,n)} \circ \beta$. Now $\beta(0) = (2,4)$. Accordingly we require that $\theta_{(m,n)}(2,4) = (0,0)$. Consequently we require that

   $$(2 + m, (-1)^m \times 4 + n) = (0,0).$$

   Clearly $(m,n) = (-2,-4)$. It follows that

   $$\hat{\beta}(1) = \theta_{(-2,-4)}(\beta(1)) = \theta_{(-2,-4)}(3,-1) = (1,-5).$$

Now the lift of $\rho \circ \beta$ that starts at $\hat{\alpha}(1)$ is $\theta_{(4,-2)} \circ \hat{\beta}$. The finishing point of this path is $\hat{\gamma}(1)$. Accordingly

$$\hat{\gamma}(1) = \theta_{(4,-2)}(\hat{\beta}(1)) = \theta_{(4,-2)}(1,-5) = (5,-7).$$

Now $\gamma = \theta_{(0,-2)} \circ \hat{\gamma}$. It follows that

$$\gamma(1) = \theta_{(0,-2)}(5,-7) = (5,-9).$$

2. Let $\gamma_1 \colon [0,1] \to \mathbb{R}^2$ be the loop defined so that, for all $t \in [0,1]$, the components of $\gamma_1(t)$ are

$$5\ \cos(8\pi t) + 3\ \cos(6\pi t)\ \cos(8\pi t)$$

and

$$5\ \sin(8\pi t) + 3\ \cos(6\pi t)\ \sin(8\pi t)$$

Then, for all $t \in [0,1]$, the components of $\gamma(t) - \gamma_1(t)$ are

$$-2\ \sin(10\pi t) - \cos(6\pi t)\ \sin(10\pi t)$$

and

$$2\ \cos(10\pi t) + \cos(6\pi t)\ \cos(10\pi t).$$

Consequently

$$|\gamma_1(t)| = 5 + 3\cos(6\pi t) \quad \text{and} \quad |\gamma(t) - \gamma_1(t)| = 2 + \cos(6\pi t)$$

Now

$$|\gamma(t) - \gamma_1(t)| < |\gamma(t)|$$

for all $t \in [0,1]$. It follows from the Dog-Walking Lemma that the the loops γ and γ_1 have the same winding number around the origin. Also

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_2(t)|$$

for all $t \in [0,1]$, where $\gamma_2(t)$ has components $5\cos(8\pi t)$ and $5\sin(8,\pi t)$. Consequently the winding numbers of γ_1 and γ_2 about the origin are equal. The winding number of the loop γ_2 about the origin is equal to 4. Consequently the winding number of the loop γ about the origin is also equal to 4.