# MAU34201: Algebraic Topology I
# Michaelmas Term 2020
# Solutions to Assignment 1

December 21, 2020

1. (a)

$$u_0 = 0, \quad u_1 = \tfrac{1}{6}, \quad u_2 = \tfrac{1}{3}, \quad u_3 = \tfrac{1}{2},$$
$$u_4 = \tfrac{2}{3}, \quad u_5 = \tfrac{5}{6}, \quad u_6 = 1.$$

(b)

$$v_0 = 0, \quad v_1 = \tfrac{1}{3}, \quad v_2 = \tfrac{1}{2}, \quad v_3 = \tfrac{2}{3}, \quad v_4 = 1.$$

(c)

$$w_0 = 0, \quad w_1 = \tfrac{1}{3}, \quad w_2 = \tfrac{4}{9}, \quad w_3 = \tfrac{5}{9}, \quad w_4 = \tfrac{2}{3}, \quad w_5 = 1.$$

(d)

$$z_0 = 0, \quad z_1 = \tfrac{1}{6}, \quad z_2 = \tfrac{1}{3}, \quad z_3 = \tfrac{2}{3}, \quad z_4 = \tfrac{5}{6}, \quad z_5 = 1.$$

2.

$$H(t,\tau) = H_j\left(\frac{t - u_{j-1}}{u_j - u_{j-1}}, \tau\right)$$

whenever $u_{j-1} \leq t \leq u_j$ and $0 \leq \tau \leq 1$.

3.

$$K(t,\tau) = \gamma_j\left(\frac{t - (1-\tau)u_{j-1} - \tau v_{j-1}}{(1-\tau)(u_j - u_{j-1}) + \tau(v_j - v_{j-1})}, \tau\right)$$

whenever

$$(1-\tau)u_{j-1} + \tau v_{j-1} \leq t \leq (1-\tau)u_j + \tau v_j.$$

**Alternatively**, for a different, more complicated, but valid homotopy, we can let

$$K(t,\tau) = \gamma((1-\tau)t + \tau\theta(t))$$

where

$$\theta(t) = \frac{v_j - t}{v_j - v_{j-1}}\, u_{j-1} + \frac{t - v_{j-1}}{v_j - v_{j-1}}\, u_j$$

when $u_{j-1} \le t \le u_j$. Note that in order to express $K(t,\tau)$ in the form $K(t,\tau) = \gamma_k(m_j(k))$ for $w_{k-1,\tau} \le t \le w_{k,\tau}$, one would need to solve equations such as

$$(1-\tau)w_{k,\tau} + \tau\theta(w_{k,\tau}) = u_k$$

for $w_{k,\tau}$ in terms of $\tau$ and the numbers $u_0, u_1, \ldots, u_m$ and $v_0, v_1, \ldots, v_m$ Whilst this might be feasible in concrete situations when $m$ is small, this approach does not appear to scale well when $m$ is large.

4. (a)

$$\begin{aligned} L_0(t,\tau) &= \begin{cases} p & \text{if } 0 \le t \le (1-\tau)w \\ \gamma_1\left(\dfrac{t-(1-\tau)w}{u_1-(1-\tau)w}\right) & \text{if } (1-\tau)w \le t \le u_1 \\ \gamma(t) & \text{if } u_1 \le t \le 1 \end{cases} \\ &= \begin{cases} \gamma(0) & \text{if } 0 \le t \le (1-\tau)w, \\ \gamma\left(\dfrac{tu_1-(1-\tau)wu_1}{u_1-(1-\tau)w}\right) & \text{if } (1-\tau)w \le t \le u_1, \\ \gamma(t) & \text{if } u_1 \le t \le 1, \end{cases} \end{aligned}$$

where

$$\gamma = \mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m).$$

**Alternatively**, for a different but valid homotopy,

$$L_0(t,\tau) = \gamma((1-\tau)\theta_0(t) + \tau t)$$

for all $t \in [0,1]$ and $\tau \in [0,1]$, where $\gamma$ is defined as above and

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \le t \le w; \\ \dfrac{(t-w)u_1}{u_1 - w} & \text{if } w \le t \le u_1; \\ t & \text{if } u_1 \le t \le 1. \end{cases}$$

(b)

$$L_1(t,\tau) = \begin{cases} \gamma_1\left(\dfrac{t}{(1-\tau)w+\tau u_1}\right) & \text{if } 0 \le t \le (1-\tau)w+\tau u_1 \\ p & \text{if } (1-\tau)w+\tau u_1 \le t \le u_1 \\ \gamma(t) & \text{if } u_1 \le t \le 1 \end{cases}$$
$$= \begin{cases} \gamma\left(\dfrac{u_1 t}{(1-\tau)w+\tau u_1}\right) & \text{if } 0 \le t \le (1-\tau)w+\tau u_1, \\ \gamma(u_1) & \text{if } (1-\tau)w+\tau u_1 \le t \le u_1, \\ \gamma(t) & \text{if } u_1 \le t \le 1, \end{cases}$$

where

$$\gamma = \mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1,\gamma_2,\gamma_3,\ldots,\gamma_m).$$

**Alternatively**, for a different but valid homotopy,

$$L_1(t,\tau) = \gamma((1-\tau)\theta_1(t)+\tau t)$$

for all $t \in [0,1]$ and $\tau \in [0,1]$, where $\gamma$ is defined as above and

$$\theta_1(t) = \begin{cases} \dfrac{tu_1}{w} & \text{if } 0 \le t \le w; \\ u_1 & \text{if } w \le t \le u_1; \\ t & \text{if } u_1 \le t \le 1. \end{cases}$$

(c)

$$L_m(t,\tau) = \begin{cases} \gamma(t) & \text{if } 0 \le t \le u_{m-1} \\ \gamma_m\left(\dfrac{t-u_{m-1}}{(1-\tau)w+\tau-u_{m-1}}\right) & \text{if } u_{m-1} \le t \le (1-\tau)w+\tau \\ p & \text{if } (1-\tau)w+\tau \le t \le 1 \end{cases}$$
$$= \begin{cases} \gamma(t) & \text{if } 0 \le t \le u_{m-1}, \\ \gamma\left(\dfrac{(1-\tau)(w-1)u_{m-1}+(1-u_{m-1})t}{(1-\tau)w+\tau-u_{m-1}}\right) & \text{if } u_{m-1} \le t \le (1-\tau)w+\tau, \\ \gamma(1) & \text{if } (1-\tau)w+\tau \le t \le 1, \end{cases}$$

where

$$\gamma = \mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1,\gamma_2,\gamma_3,\ldots,\gamma_m).$$

**Alternatively**, for a different but valid homotopy,

$$L_m(t,\tau) = \gamma((1-\tau)\theta_m(t) + \tau t)$$

for all $t \in [0,1]$ and $\tau \in [0,1]$, where $\gamma$ is defined as above and

$$\theta_m(t) = \begin{cases} t & \text{if } 0 \le t \le u_{m-1}; \\ \dfrac{(1-u_{m-1})t - (1-w)u_{m-1}}{w - u_{m-1}} & \text{if } u_{m-1} \le t \le w; \\ 1 & \text{if } w \le t \le 1. \end{cases}$$

(d)

$$\begin{aligned} L_k(t,\tau) &= \begin{cases} \gamma(t) & \text{if } 0 \le t \le u_{k-1} \\ \gamma_k\left(\dfrac{t - u_{k-1}}{(1-\tau)w + \tau u_k - u_{k-1}}\right) & \text{if } u_{k-1} \le t \le (1-\tau)w + \tau u_k \\ p & \text{if } (1-\tau)w + \tau u_k \le t \le u_k \\ \gamma(t) & \text{if } u_k \le t \le 1 \end{cases} \\ &= \begin{cases} \gamma(t) & \text{if } 0 \le t \le u_{k-1}, \\ \gamma\left(\dfrac{(1-\tau)(w-u_k)u_{k-1} + (u_k - u_{k-1})t}{(1-\tau)w + \tau u_k - u_{k-1}}\right) & \\ & \text{if } u_{k-1} \le t \le (1-\tau)w + \tau u_k, \\ \gamma(u_k) & \text{if } (1-\tau)w + \tau u_k \le t \le u_k, \\ \gamma(t) & \text{if } u_k \le t \le 1, \end{cases} \end{aligned}$$

where

$$\gamma = \mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m).$$

**Alternatively**, for a different but valid homotopy,

$$L_k(t,\tau) = \gamma((1-\tau)\theta_k(t) + \tau t)$$

for all $t \in [0,1]$ and $\tau \in [0,1]$, where $\gamma$ is defined as above and

$$\theta_k(t) = \begin{cases} t & \text{if } 0 \le t \le u_{k-1}; \\ \dfrac{(u_k - u_{k-1})t - (u_k - w)u_{k-1}}{w - u_{k-1}} & \text{if } u_{k-1} \le t \le w; \\ u_k & \text{if } w \le t \le u_k; \\ t & \text{if } u_k \le t \le 1. \end{cases}$$

5. (a)

$$M_1(t,\tau) = \begin{cases} \eta\left(\dfrac{(1-\tau)t}{w}\right) & \text{if } 0 \le t \le w; \\ \eta\left(\dfrac{(1-\tau)(u_1-t)}{u_1-w}\right) & \text{if } w \le t \le u_1; \\ \gamma(t) & \text{if } u_1 \le t \le 1. \end{cases}$$

(b)

$$M_m(t,\tau) = \begin{cases} \gamma(t) & \text{if } 0 \le t \le u_{m-1}. \\ \eta\left(\dfrac{(1-\tau)(t-u_{m-1}}{w-u_{m-1}}\right) & \text{if } u_{m-1} \le t \le w; \\ \eta\left(\dfrac{(1-\tau)(1-t)}{1-w}\right) & \text{if } w \le t \le 1. \end{cases}$$

(c)

$$M_k(t,\tau) = \begin{cases} \gamma(t) & \text{if } 0 \le t \le u_{k-1}. \\ \eta\left(\dfrac{(1-\tau)(t-u_{k-1}}{w-u_{k-1}}\right) & \text{if } u_{k-1} \le t \le w; \\ \eta\left(\dfrac{(1-\tau)(u_k-t)}{u_k-w}\right) & \text{if } w \le t \le u_k; \\ \gamma(t) & \text{if } u_k \le t \le 1. \end{cases}$$

6. (a) Let $\gamma$ be a loop in $X$ based at $\beta(1)$. Then $\Theta_\beta([\gamma]) = [\beta \,.\, \gamma \,.\, \beta^{-1}]$ and hence, applying results of previous questions,

$$\begin{aligned} (\Theta_\alpha \circ \Theta_\beta)([\gamma]) &= \Theta_\alpha(\Theta_\beta([\gamma])) = [\alpha \,.\, (\beta \,.\, \gamma \,.\, \beta^{-1}) \,.\, \alpha^{-1}] \\ &= [\mathrm{Concat}_{0,\frac{1}{3},\frac{4}{9},\frac{5}{9},\frac{2}{3},1}(\alpha,\beta,\gamma,\beta^{-1},\alpha^{-1})] \\ &= [\mathrm{Concat}_{0,\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6},1}(\alpha,\beta,\gamma,\beta^{-1},\alpha^{-1})] \\ &= [(\alpha \,.\, \beta) \,.\, \gamma \,.\, (\beta^{-1} \,.\, \alpha^{-1})] \\ &= \Theta_{\alpha.\beta}([\gamma]) \end{aligned}$$

(applying question parts 1(c), 1(d) and question 3). Consequently $\Theta_\alpha \circ \Theta_\beta = \Theta_{\alpha.\beta}$.

(b) Let $\alpha(t) = p$ for all $t \in [0,1]$. Then $\alpha = \varepsilon_p$. Let $\gamma$ be a loop in $X$ based at the point $p$. Then

$$\begin{aligned} \Theta_\alpha([\gamma]) &= [\varepsilon_p \,.\, \gamma \,.\, \varepsilon_p] = [\mathrm{Concat}_{0,\frac{1}{3},\frac{2}{3},1}(\varepsilon_p,\gamma,\varepsilon_p)] \\ &= [\mathrm{Concat}_{0,\frac{2}{3},1}(\gamma,\varepsilon_p)] = [\mathrm{Concat}_{0,1}(\gamma)] = [\gamma] \end{aligned}$$

(applying questions parts 4(a) and 4(c)). Consequently $\Theta_\alpha$ is the identity isomorphism of the fundamental group of $X$ with basepoint $p$.

(c) Let $\gamma$ be a loop in $X$ based at $\alpha(1)$. Then

$$\Theta_\alpha([\gamma]) = [\mathrm{Concat}_{0,\frac{1}{3},\frac{2}{3},1}(\alpha,\gamma,\alpha^{-1})] = [\mathrm{Concat}_{0,\frac{1}{3},\frac{2}{3},1}(\eta,\gamma,\eta^{-1})] = \Theta_\eta([\gamma])$$

(applying question 2). The result follows.

(d) Let $\gamma$ be a loop in $X$ based at $p$, where $p = \alpha(0)$. Then

$$\begin{aligned}(\Theta_\alpha \circ \Theta_{\alpha^{-1}})([\gamma]) &= [\mathrm{Concat}_{0,\frac{1}{3},\frac{4}{9},\frac{5}{9},\frac{2}{3},1}(\alpha,\alpha^{-1},\gamma,\alpha,\alpha^{-1})] \\ &= [\mathrm{Concat}_{0,\frac{4}{9},\frac{5}{9},\frac{2}{3},1}(\varepsilon_p,\gamma,\alpha,\alpha^{-1})] \\ &= [\mathrm{Concat}_{0,\frac{4}{9},\frac{5}{9},1}(\varepsilon_p,\gamma,\varepsilon_p)] \\ &= [\mathrm{Concat}_{0,\frac{5}{9},1}(\gamma,\varepsilon_p)] = [\mathrm{Concat}_{0,1}(\gamma)] \\ &= [\gamma]\end{aligned}$$

(using in particular the results of questions 4 and 5). Consequently $\Theta_\alpha \circ \Theta_{\alpha^{-1}}$ is the identity isomorphism of $\pi_1(X, \alpha(0))$. Similarly $\Theta_{\alpha^{-1}} \circ \Theta_\alpha$ is the identity isomorphism of $\pi_1(X, \alpha(1))$. Consequently $\Theta_\alpha$ is an isomorphism with inverse $\Theta_{\alpha^{-1}}$. (There are various valid alternatives that apply previous results to prove the desired result.)