MAU34201: Algebraic Topology I
Michaelmas Term 2020
Assignment 2

Due 11pm on Monday, 7 December, 2020

Please upload submissions to Blackboard by the due date in the form of a **single PDF document.** Please do not submit as multiple files, or photographs of pages in image formats.

Please complete and sign the standard plagiarism declaration, reproduced on the following page. It suffices to copy it out by hand, if that facilitates submission.

You should not collaborate with other members of the class in completing this assignment.

The first question on this assignment relates to subsection 6.4, concerning *Fundamental Groups of Orbit Spaces*, in the particular example where a group acts freely and properly discontinuously on the plane, giving rise to an orbit space that can be identified with the Klein Bottle. This group action on the plane is discussed in Disquisition I.

The second question relates to the section 7, concerning winding numbers of loops in the plane.

When asked to justify results below, your justifications should address the specifics of the scenario presented. You should not quote verbatim general theory from the lecture notes, or make extensive paraphrases of such material. Appropriate citations to the lecture notes and disquisitions should suffice in relation to any matter discussed there.

Module MAU34201—Algebraic Topology I.
Michaelmas Term 2020.
Assignment 2.

Name (please print): . . . .

Student number: . . . .

Date submitted: . . . .

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at

`http://www.tcd.ie/calendar`

I have also completed the Online Tutorial on avoiding plagiarism *Ready Steady Write*, located at

`http://tcd-ie.libguides.com/plagiarism/ready-steady-write`

Signed:

. . . .

1. Let Γ be the group whose elements are represents as pairs of integers, with the group operation $\#$ on G defined so that

$$(m_1, n_1)\#(m_2, n_2) = (m_1 + m_2, n_1 + (-1)^{m_1} n_2)$$

for all integers m_1, n_1, m_2 and n_2. This group acts freely and properly discontinuously on the plane $\mathbb{R}^2$, where an element of the group Γ represented by an ordered pair (m, n) acts on the plane $\mathbb{R}^2$ by means of the homeomorphism $\theta_{m,n}$, mapping $\mathbb{R}^2$ onto itself, that is defined so that

$$\theta_{m,n}(x, y) = (x + m, (-1)^m y + n)$$

for all $(x, y) \in \mathbb{R}^2$.

[Note that, in answering this assignment question, you are not expected to prove that the group Γ acts freely and properly discontinuously on $\mathbb{R}^2$: that matter is discussed as one of the examples presented in Disquition I.]

The orbit space $\mathbb{R}^2/\Gamma$ then represents a topological surface commonly known as the *Klein Bottle*. Let K denote the orbit space $\mathbb{R}^2/\Gamma$, and let $\rho\colon \mathbb{R}^2 \to K$ be the quotient map that sends each point of the plane to its orbit under the action of the group Γ. The general results proved in lecture notes establish that the quotient $\rho\colon \mathbb{R}^2 \to K$ is a covering map.

The specifics of the question follow. Let α, β and γ be paths in $\mathbb{R}^2$, represented as continuous functions mapping the closed unit interval $[0, 1]$ into $\mathbb{R}^2$. Suppose that

$$\alpha(0) = (1, 1), \quad \alpha(1) = (5, 3), \quad \beta(0) = (2, 4), \quad \beta(1) = (3, -1)$$

and

$$\gamma(0) = (0, -2).$$

Suppose also that $\rho \circ \gamma$ is the concatenation $(\rho \circ \alpha) . (\rho \circ \beta)$ of the loops $\rho \circ \alpha$ and $\rho \circ \beta$. Determine $\gamma(1)$.

[Note that the starting and finishing points of the paths α and β and the starting point of γ all belong to the orbit of the point $(0, 0)$ under the action of the group Γ, because that orbit consists of those points of $\mathbb{R}^2$ whose components are integers. Otherwise the choice of values is fairly arbitrary. These paths in $\mathbb{R}^2$ are mapped by the quotient map ρ to loops in the Klein Bottle K.]

2. Determine the value of the winding number about the origin $(0, 0)$ of the loop $\gamma\colon [0, 1] \to \mathbb{R}^2$, where, for all real number t, the first and second components of $\gamma(t)$ are

$$5\,\cos(8\pi t) + 3\,\cos(6\pi t)\,\cos(8\pi t) - 2\,\sin(10\pi t) - \cos(6\pi t)\,\sin(10\pi t)$$

and

$$5\,\sin(8\pi t) + 3\,\cos(6\pi t)\,\sin(8\pi t) + 2\,\cos(10\pi t) + \cos(6\pi t)\,\cos(10\pi t)$$

respectively.