MAU34201: Algebraic Topology I
Michaelmas Term 2020
Assignment 1

Due 11pm on Monday, 23 November, 2020

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You should not collaborate with other members of the class in completing this assignment.

The assignment has been designed with a view to providing some perspective on Section 5 of the lecture notes, concerned with the fundamental group of a topological space, and particularly on the proof strategy of Proposition 5.3. It might be useful to review subsection 2.5 of the lecture notes, which discuss homotopies between continuous maps.

For questions preceding the final question, your answers should take the form of lists of numbers, or specifications of functions as apppropriate, without supporting computations, rough working and explanations. It is of course advisable to ensure, e.g., through private calculations on the side, that the solutions supplied are effective. The final question asks for proofs or justifications of stated results related to Proposition 5.3 in the lecture notes, but it should be possible to answer the parts of that final question making use of the results of earlier questions on the assignment.

Module MAU34201—Algebraic Topology I.
Michaelmas Term 2020.
Assignment I.

Name (please print): . . . .

Student number: . . . .

Date submitted: . . . .

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Signed:

. . . .

Basic definitions and notation

Let X be a topological space. We consider paths in X that are represented as continuous functions mapping the closed unit interval $[0, 1]$ into the topological space X. Unless specifically stated otherwise, the domain of all paths in this assignment is the closed unit interval $[0, 1]$. A path $\gamma\colon [0, 1] \to X$ in the topological space X *starts* at the point $\gamma(0)$ and *ends* at the point $\gamma(1)$, and is thus a path *from* $\gamma(0)$ *to* $\gamma(1)$.

Given paths $\gamma_1, \gamma_2, \ldots, \gamma_m$ in X, where $\gamma_j(0) = \gamma_{j-1}(1)$ for all integers j between 2 and m, and given real numbers $u_0, u_1, \ldots, u_m$, where

$$u_0 < u_1 < u_2 < \cdots < u_m,$$

let

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1, \gamma_2, \ldots, \gamma_m)$$

denote the path $\gamma\colon [u_0, u_m] \to X$ defined such that

$$\gamma(t) = \gamma_j \left(\frac{t - u_{j-1}}{u_j - u_{j-1}} \right)$$

for all integers j between 1 and m and for all real numbers t satisfying $u_{j-1} \leq t \leq u_j$. Normally $u_0 = 0$ and $u_m = 1$.

Also let

$$\gamma_1 \,.\, \gamma_2 \,.\, \gamma_3 \,.\, \cdots \,.\, \gamma_m.$$

denote the path representable as

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m)$$

with $u_j = j/m$ for $j = 0, 1, \ldots, m$.

Thus for example if γ_1, γ_2 and γ_3 are paths, each defined over the unit interval $[0, 1]$, then

$$\begin{aligned} (\gamma_1 \,.\, \gamma_2 \,.\, \gamma_3)(t) &= \mathrm{Concat}_{0,\frac{1}{3},\frac{2}{3},1}(\gamma_1, \gamma_2, \gamma_3) \\ &= \begin{cases} \gamma_1(3t) & \text{if } 0 \leq t \leq \frac{1}{3}; \\ \gamma_2(3t - 1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ \gamma_3(3t - 2) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases} \end{aligned}$$

For each point p of the topological space X, let $\varepsilon_p\colon [0, 1] \to X$ denote the constant loop at the point p, defined such that $\varepsilon_p(t) = p$ for all $x \in [0, 1]$.

Given a path γ in X defined over the unit interval $[0, 1]$, let γ^{-1} denote the reversed path that is defined so that $\gamma^{-1}(t) = \gamma(1 - t)$ for all $t \in [0, 1]$.

1. Let X be a topological space, let α, β, and γ, γ_1 and γ_2 be paths in X, each represented as a continuous function defined on the closed unit interval, where $\alpha(1) = \beta(0)$ and

$$\beta(1) = \gamma(0) = \gamma(1) = \gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1).$$

Note that γ, γ_1 and γ_2 are then loops in X based at the point $\beta(1)$.

(a) List the values of real numbers $u_0, u_1, u_2, u_3, u_4, u_5, u_6$, where $0 = u_0 < u_1 < u_2 < u_3 < u_4 < u_5 < u_6 = 1$, for which

$$\begin{aligned}&(\beta \,.\, \gamma_1 \,.\, \beta^{-1}) \,.\, (\beta \,.\, \gamma_2 \,.\, \beta^{-1})\\ &\quad = \mathrm{Concat}_{u_0,u_1,u_2,u_3,u_4,u_5,u_6}(\beta, \gamma_1, \beta^{-1}, \beta, \gamma_2, \beta^{-1}).\end{aligned}$$

(b) List the values of real numbers v_0, v_1, v_2, v_3, v_4, where $0 = v_0 < v_1 < v_2 < v_3 < v_4 = 1$, for which

$$\begin{aligned}&\beta \,.\, (\gamma_1 \,.\, \gamma_2) \,.\, \beta^{-1}\\ &\quad = \mathrm{Concat}_{v_0,v_1,v_2,v_3,v_4}(\beta, \gamma_1, \gamma_2, \beta^{-1}).\end{aligned}$$

(c) List the values of real numbers $w_0, w_1, w_2, w_3, w_4, w_5$, where $0 = w_0 < w_1 < w_2 < w_3 < w_4 < w_5 = 1$, for which

$$\begin{aligned}&\alpha \,.\, (\beta \,.\, \gamma \,.\, \beta^{-1}) \,.\, \alpha^{-1}\\ &\quad = \mathrm{Concat}_{w_0,w_1,w_2,w_3,w_4,w_5}(\alpha, \beta, \gamma, \beta^{-1}, \alpha^{-1}).\end{aligned}$$

(d) List the values of real numbers $z_0, z_1, z_2, z_3, z_4, z_5$, where $0 = z_0 < z_1 < z_2 < z_3 < z_4 < z_5 = 1$, for which

$$\begin{aligned}&(\alpha \,.\, \beta) \,.\, \gamma \,.\, (\beta^{-1} \,.\, \alpha^{-1})\\ &\quad = \mathrm{Concat}_{z_0,z_1,z_2,z_3,z_4,z_5}(\alpha, \beta, \gamma, \beta^{-1}, \alpha^{-1}).\end{aligned}$$

[Note: in answering this question, simply give the lists of values that you are asked to provide. You should not include rough work, or computations, or explanations of how you determined the relevant list.]

2. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_m$ be paths in a topological space X, where $\alpha_1(0) = \beta_1(0)$, $\alpha_m(1) = \beta_m(1)$ and

$$\alpha_{j-1}(1) = \alpha_j(0) = \beta_j(0) = \beta_{j-1}(1)$$

for all integers j between 2 and m. Suppose that $\alpha_j \simeq \beta_j$ rel $\{0, 1\}$ for $j = 1, 2, \ldots, m$. Moreover, for each integer j between 1 and m, let $H_j\colon [0,1] \times [0,1] \to X$ be a homotopy between the paths α_j and β_j that satisfies the following properties: $H_j(t, 0) = \alpha_j(t)$ and $H_j(t, 1) = \beta_j(t)$ for all $t \in [0, 1]$; also $H_j(0, \tau) = \alpha_j(0) = \beta_j(0)$ and $H_j(1, \tau) = \alpha_j(1) = \beta_j(1)$ for all $\tau \in [0, 1]$. Now let $u_0, u_1, \ldots, u_m$ be real numbers satisfying

$$0 = u_0 < u_1 < u_2 < \cdots < u_m = 1,$$

and let

$$\alpha = \mathrm{Concat}_{u_0,u_1,u_2,\ldots,u_m}(\alpha_1, \alpha_2, \ldots, \alpha_m)$$

and

$$\beta = \mathrm{Concat}_{u_0,u_1,u_2,\ldots,u_m}(\beta_1, \beta_2, \ldots, \beta_m).$$

Write down an expression that specifies a homotopy H between the paths α and β which satisfies all the following properties: $H(t, 0) = \alpha(t)$ and $H(t, 1) = \beta(t)$ for all $t \in [0, 1]$; also $H(0, \tau) = \alpha(0) = \beta(0)$ and $H(1, \tau) = \alpha(1) = \beta(1)$ for all $\tau \in [0, 1]$. (Note that the existence of a homotopy H with these properties ensures that $\alpha \simeq \beta$ rel $\{0, 1\}$.)

[Note: in answering this question, you should write down an expression that, for each integer j betweeen 1 and m, determines the value of $H(t, \tau)$ when $0 \leq \tau \leq 1$ and $u_{j-1} \leq t \leq u_j$. You should not include more than the required expression specifying H in your answer. Computations, rough working and explanations are not required or expected, and should not be included.]

3. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be paths in a topological space X, where $\gamma_{j-1}(1) = \gamma_j(0)$ for each integer j between 2 and m. Let

$$u_0, u_1, u_2, \ldots, u_m \quad \text{and} \quad v_0, v_1, v_2, \ldots, v_m$$

be real numbers satisfying

$$0 = u_0 < u_1 < u_2 < \cdots < u_m = 1$$

and

$$0 = v_0 < v_1 < v_2 < \cdots < v_m = 1,$$

and let

$$\gamma = \text{Concat}_{u_0,u_1,u_2,\ldots,u_m}(\gamma_1, \gamma_2, \ldots, \gamma_m)$$

and

$$\eta = \text{Concat}_{v_0,v_1,v_2,\ldots,v_m}(\gamma_1, \gamma_2, \ldots, \gamma_m).$$

Write down an expression that specifies a homotopy K between the paths γ and η which satisfies all the following properties: $K(t, 0) = \gamma(t)$ and $K(t, 1) = \eta(t)$ for all $t \in [0, 1]$; also $K(0, \tau) = \gamma(0) = \eta(0)$ and $K(1, \tau) = \gamma(1) = \eta(1)$ for all $\tau \in [0, 1]$. (Note that the existence of a homotopy K with these properties ensures that $\gamma \simeq \eta$ rel $\{0, 1\}$.)

[Note: in answering this question, it suffices to write down an expression that, for each integer j betweeen 1 and m, determines the value of $K(t, \tau)$ when $0 \leq \tau \leq 1$ and

$$(1 - \tau)u_{j-1} + \tau v_{j-1} \leq t \leq (1 - \tau)u_j + \tau v_j.$$

You should not include in your answer more than the required expression specifying the value of K in the portion of the domain of K determined by j as specified above (or in other appropriate portions of the domain of K if the hint provided is not followed). Computations, rough working and explanations are not required or expected, and should not be included.]

4. Throughout this question let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be paths in a topological space X, where $\gamma_{j-1}(1) = \gamma_j(0)$ for each integer j between 2 and m, and let $u_0, u_1, u_2, \ldots, u_m$ be real numbers satisfying

$$0 = u_0 < u_1 < u_2 < \cdots < u_m = 1.$$

(Also, for each point p of X, let ε_p denote the constant path at the point p.)

(a) Let w be a real number satisfying $u_0 < w < u_1$, and let $p = \gamma_1(0)$. Write down an expression that specifies a homotopy L_0 between the paths

$$\mathrm{Concat}_{u_0,w,u_1,u_2,\ldots,u_m}(\varepsilon_p, \gamma_1, \gamma_2, \ldots, \gamma_m)$$

and

$$\mathrm{Concat}_{u_0,u_1,u_2,\ldots,u_m}(\gamma_1, \gamma_2, \ldots, \gamma_m).$$

(b) Again let w be a real number satisfying $u_0 < w < u_1$, and let $p = \gamma_1(1) = \gamma_2(0)$. Write down an expression that specifies a homotopy L_1 between the paths

$$\mathrm{Concat}_{u_0,w,u_1,\ldots,u_m}(\gamma_1, \varepsilon_p, \gamma_2, \ldots, \gamma_m)$$

and

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1, \gamma_2, \ldots, \gamma_m).$$

(c) Now let w be a real number satisfying $u_{m-1} < w < u_m$, and let $p = \gamma_m(1)$. Write down an expression that specifies a homotopy L_m between the paths

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_{m-1},w,u_m}(\gamma_1, \gamma_2, \ldots, \gamma_m, \varepsilon_p)$$

and

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_{m-1},u_m}(\gamma_1, \gamma_2, \ldots, \gamma_m).$$

(d) Now let k be an integer between 2 and $m-1$, let w be a real number satisfying $u_{k-1} < w < u_k$, and let $p = \gamma_k(1) = \gamma_{k+1}(0)$. Write down an expression that specifies a homotopy L_k between the paths

$$\mathrm{Concat}_{u_0,\ldots,u_{k-1},w,u_k,\ldots,u_m}(\gamma_1, \ldots, \gamma_k, \varepsilon_p, \gamma_{k+1}, \ldots, \gamma_m)$$

and

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1, \gamma_2, \ldots, \gamma_m).$$

[Note: computations, rough working and explanations are not required or expected, and should not be included.]

5. Throughout this question let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be paths in a topological space X, where $\gamma_{j-1}(1) = \gamma_j(0)$ for each integer j between 2 and m, and let $u_0, u_1, u_2, \ldots, u_m$ be real numbers satisfying

$$0 = u_0 < u_1 < u_2 < \cdots < u_m = 1.$$

Also, for each point p of X, let ε_p denote the constant path at the point p.

(a) Let w be a real number satisfying $u_0 < w < u_1$, and let $\eta\colon [0,1] \to X$ be a path that starts at the point p where $p = \gamma_2(0)$. Write down an expression that specifies a homotopy M_1 between the paths

$$\mathrm{Concat}_{u_0,w,u_1,u_2,\ldots,u_m}(\eta, \eta^{-1}, \gamma_2, \ldots, \gamma_m)$$

and

$$\mathrm{Concat}_{u_0,u_1,u_2,\ldots,u_m}(\varepsilon_p, \gamma_2, \ldots, \gamma_m).$$

(b) Now let w be a real number satisfying $u_{m-1} < w < u_m$, and let $\eta\colon [0,1] \to X$ be a path that starts at the point p where $p = \gamma_{m-1}(1)$. Write down an expression that specifies a homotopy M_m between the paths

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_{m-1},w,u_m}(\gamma_1, \ldots, \gamma_{m-1}, \eta, \eta^{-1})$$

and

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_{m-1},u_m}(\gamma_1, \gamma_2, \ldots, \varepsilon_p).$$

(c) Now let k be an integer between 2 and $m-1$ for which $\gamma_{k-1}(1) = \gamma_{k+1}(0)$, let w be a real number satisfying $u_{k-1} < w < u_k$, and let $\eta\colon [0,1] \to X$ be a path that starts at the point p, where $p = \gamma_{k-1}(1) = \gamma_{k+1}(0)$. Write down an expression that specifies a homotopy M_k between the paths

$$\mathrm{Concat}_{u_0,\ldots,u_{k-1},w,u_k,\ldots,u_m}(\gamma_1, \ldots, \gamma_{k-1}, \eta, \eta^{-1}, \gamma_{k+1}, \ldots, \gamma_m)$$

and

$$\mathrm{Concat}_{u_0,u_1,\ldots,u_m}(\gamma_1, \ldots, \gamma_{k-1}, \varepsilon_p, \gamma_{k+1}, \ldots, \gamma_m).$$

[Note: computations, rough working and explanations are not required or expected, and should not be included.]

6. Let X be a topological space and, given any path $\alpha\colon [0,1] \to X$, let $\Theta_\alpha\colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ be the well-defined homomorphism of fundamental groups which sends the homotopy class $[\gamma]$ of any loop γ based at $\alpha(1)$ to the homotopy class $[\alpha.\gamma.\alpha^{-1}]$ of the loop $\alpha.\gamma.\alpha^{-1}$, where

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ \gamma(3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \alpha(3-3t) & \text{if } \frac{2}{3} \le t \le 1 \end{cases}$$

(i.e., $\alpha.\gamma.\alpha^{-1}$ represents 'α followed by γ followed by α reversed').

In answering the following, you should apply the results of previous problems included in this assignment if you can see your way to doing so. Some previous problems, or parts of problems, are relevant, and some have indeed been devised with the tasks of this question in mind.

(a) Show that $\Theta_{\alpha.\beta} = \Theta_\alpha \circ \Theta_\beta$ for all paths α and β in X satisfying $\beta(0) = \alpha(1)$.

(b) Show that Θ_α is the identity homomorphism whenever α is a constant path.

(c) Let α and η be paths in X satisfying $\alpha(0) = \eta(0)$ and $\alpha(1) = \eta(1)$. Suppose that $\alpha \simeq \eta$ rel $\{0, 1\}$. Show that $\Theta_\alpha = \Theta_\eta$.

(d) It has been proved (in course notes) that

$$\Theta_\alpha\colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$$

is an isomorphism for all paths α in X. (This shows that, up to isomorphism, the fundamental group of a path-connected topological space does not depend on the choice of basepoint.) Explain briefly how this follows from the results of earlier parts of this question and/or other problems included in this assignment.