MAU23302—Euclidean and Non-Euclidean Geometry
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Part II, Section 3:
The Disk Model of the Hyperbolic Plane

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2.1. Inversion of the Riemann Sphere in the Unit Circle

Let D denote the open unit disk in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let S denote the unit circle in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$S = \{z \in \mathbb{C} : |z| = 1\}$$

We define the *inversion* Ω of the Riemann sphere in the circle S bounding the open unit disk D to be the transformation of the Riemann sphere defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Then $\Omega(z) = z$ for all $z \in S$, and the composition $\Omega \circ \Omega$ of the inversion Ω with itself is the identity transformation of the Riemann sphere. Moreover Ω maps the open unit disk D into the region of the Riemann sphere that lies outside the unit circle S. The transformation $\Omega \colon \mathbb{P}^1 \to \mathbb{P}^1$ is characterized by the property that

$$\Omega\left(\frac{u}{v}\right) = \frac{\overline{v}}{\overline{u}}$$

for all complex numbers v and w that are not both zero.

Lemma 2.1

Let μ be a Möbius transformation of the Riemann sphere, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Also let a, b, c and d be complex coefficients determined so that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then $\Omega \circ \mu \circ \Omega$ is also a Möbius transformation, and moreover

$$\Omega(\mu(\Omega(z))) = \frac{\overline{d}z + \overline{c}}{\overline{b}z + \overline{a}}$$

for all complex numbers $z \in \mathbb{C}$ for which $\overline{b}z + \overline{a} \neq 0$.

Proof

The definition of Möbius transformations and of the inversion Ω of the Riemann sphere in the unit circle ensure that

$$\mu\left(\frac{u}{v}\right) = \frac{au+bv}{cu+dv} \quad \text{and} \quad \Omega\left(\frac{u}{v}\right) = \frac{\overline{v}}{\overline{u}}$$

for all complex numbers u and v that are not both zero. Consequently

$$\Omega\left(\mu\left(\Omega\left(\frac{u}{v}\right)\right)\right) = \Omega\left(\mu\left(\frac{\overline{v}}{\overline{u}}\right)\right) = \Omega\left(\frac{a\,\overline{v}+b\,\overline{u}}{c\,\overline{v}+d\,\overline{u}}\right) = \frac{\overline{d}\,u+\overline{c}\,v}{\overline{b}\,u+\overline{a}\,v}$$

for all complex numbers u and v that are not both zero. The result follows. $\blacksquare$

Proposition 2.2

Let μ be a Möbius transformation of the Riemann sphere, let D be the open unit disk in the complex plane, where

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let Ω be the inversion of the Riemann sphere in the unit circle that is defined so that

$$\Omega(0) = \infty, \quad \Omega(\infty) = 0 \quad \textit{and} \quad \Omega(z) = \frac{1}{\overline{z}} \textit{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Then the Möbius transformation μ maps the unit disk D onto itself if and only if both of the following two conditions are satisfied:

(i) $\Omega \circ \mu = \mu \circ \Omega$;

(ii) *there exists at least one $z \in D$ for which $\mu(z) \in D$.*

Proof

First suppose that the Möbius transformation μ maps the unit disk D onto itself. Let z be a complex number satisfying $|z| = 1$. If it were the case that $|\mu(z)| < 1$ then there would exist some complex number w for which $|w| < 1$ and $\mu(w) = \mu(z)$, because μ maps the open unit disk onto itself. But this is not possible because all Möbius transformations are invertible. Next we note that if it were the case that $|\mu(z)| > 1$ then, for real numbers t that are less than 1 but sufficiently close to 1, it would follow that $|tz| < 1$ but $|\mu(tz)| > 1$, contradicting the requirement that the Möbius transformation μ map the open unit disk onto itself. Consequently $|\mu(z)| = 1$. We conclude therefore that the Möbius transformation μ maps the unit circle bounding the open unit disk into itself. The same is true of the inverse of μ. Consequently the Möbius transformation μ must map the unit circle onto itself.

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1). Now $\Omega(z) = z$ and $|\mu(z)| = 1$ for all complex numbers z satisfying $|z| = 1$. It follows that $\hat{\mu}(z) = \mu(z)$ for all complex numbers z satisfying $|z| = 1$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Corollary 1.17). Consequently $\hat{\mu} = \mu$, and therefore $\Omega \circ \mu = \mu \circ \Omega$. It now follows directly that any Möbius transformation that maps the unit disk D onto itself must satisfy conditions (i) and (ii) in the statement of the proposition.

Conversely, suppose that Möbius transformation μ of the Riemann sphere satisfies conditions (i) and (ii) in the statement of the proposition. Then $\Omega \circ \mu = \mu \circ \Omega$. Let z be a complex number satisfying $|z| \neq 1$. Then $\Omega(z) \neq z$. It follows that $\mu(\Omega(z)) \neq \mu(z)$, because Möbius transformations are invertible transformations of the Riemann sphere, and therefore $\Omega(\mu(z)) \neq \mu(z)$, from which it follows that $|\mu(z)| \neq 1$. Consequently no complex number belonging to the open unit disk D is mapped by the Möbius transformation D to a point that lies on the unit circle. It follows that if one endpoint of a straight line segment or circular arc contained in the open disk D is mapped by μ into D, then the same must be true of the other endpoint of that straight line segment or circular arc.

Now the complex numbers belonging to the unit disk D can be joined to one another by straight line segments. Moreover condition (ii) in the statement of the proposition ensures that at least one complex number belonging to the unit disk D is mapped by the Möbius transformation μ into the unit disk D. Consequently the unit disk is mapped into itself by the Möbius transformation μ. Moreover if the Möbius transformation μ has the property that $\Omega \circ \mu = \mu \circ \Omega$ then

$$\Omega \circ \mu^{-1} = \mu^{-1} \circ \mu \circ \Omega \circ \mu^{-1} = \mu^{-1} \circ \Omega \circ \mu \circ \mu^{-1} = \mu^{-1} \circ \Omega,$$

and consequently the inverse μ^{-1} of the Möbius transformation μ also satisfies (i) and (ii) in the statement of the proposition, and therefore maps the open unit disk D into itself. It follows that if the Möbius transformation μ satisfies conditions (i) and (ii) then it must map the open unit disk D onto itself, as required. ∎

Corollary 2.3

Let μ be a Möbius transformation of the Riemann sphere, and let S be the unit circle consisting of all complex numbers z for which $|z| = 1$. Suppose that $\mu(S) \subset S$ and that $|\mu(0)| < 1$. Then the Möbius transformation μ maps the open unit disk onto itself. Moreover $\Omega \circ \mu = \mu \circ \Omega$, where Ω is the inversion of the Riemann sphere in the unit circle S, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Proof

Let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1), and moreover $\hat{\mu}(z) = \mu(z)$ for all $z \in S$, because $\mu(S) \subset S$ and $\Omega(z) = z$ for all $z \in S$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Corollary 1.17). It follows that $\hat{\mu} = \mu$, and therefore $\Omega \circ \mu = \mu \circ \Omega$. The required result now follows on applying Proposition 2.2. ∎

Lemma 2.4

Given distinct complex numbers z_1 and z_2, where $|z_1| = |z_2| = 1$, there exists a Möbius transformation μ of the Riemann sphere mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$.

Proof

Choose a complex number z_3 distinct from z_1 and z_2 for which $|z_3| = 1$. Then there exists a unique Möbius transformation μ_1 with the properties that $\mu_1(z_1) = -1$, $\mu_1(z_2) = 1$ and $\mu_1(z_3) = i$. Möbius transformations map circles to circles, and, given any three distinct complex numbers that are not collinear, there exists exactly one circle in the complex plane passing through all three of these complex numbers. Consequently the Möbius transformation μ_1 must map the unit circle onto itself.

If $|\mu_1(0)| < 0$ let the Möbius transformation μ be identical to μ_1; if $|\mu_1(0)| > 1$ or $\mu_1(0) = \infty$ let the Möbius transformation μ be defined so that $\mu(z) = 1/\mu_1(z)$ for all complex numbers z for which $\mu_1(z) \neq 0$. Then μ maps the unit circle onto itself, $\mu(z_1) = -1$, $\mu(z_2) = 1$ and $|\mu(0)| < 1$. Then $\mu(D)$ must map the open unit disk onto itself (see Corollary 2.3). The Möbius transformation μ then has the required properties. ∎

Proposition 2.5

Let a and b be complex numbers satisfying $|b| < |a|$, and let μ be the Möbius transformation of the Riemann sphere defined so that

$$\mu(z) = \frac{az + b}{\overline{b}\,z + \overline{a}} \quad \textit{whenever } \overline{b}\,z + \overline{a} \neq 0,$$

$\mu(-\overline{a}/\overline{b}) = \infty$ and $\mu(\infty) = a/\overline{b}$ in cases where $b \neq 0$ and $\mu(\infty) = \infty$ in cases where $b = 0$. Then $|\mu(z)| < 1$ whenever $|z| < 1$, $|\mu(z)| = 1$ whenever $|z| = 1$, and $|\mu(z)| > 1$ whenever $|z| > 1$ and $\overline{b}z + \overline{a} \neq 0$. Moreover the Möbius transformation μ maps the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself.

Proof

Calculating, we find that

$$\begin{aligned}
|\overline{b}z+\overline{a}|^2 - |az+b|^2 &= (\overline{b}z+\overline{a})(b\overline{z}+a) - (az+b)(\overline{a}\,\overline{z}+\overline{b}) \\
&= |b|^2|z|^2 + |a|^2 + a\,\overline{b}z + \overline{a}b\,\overline{z} \\
&\qquad - |a|^2|z|^2 - |b|^2 - a\,\overline{b}z - \overline{a}b\,\overline{z} \\
&= (|a|^2 - |b|^2)(1-|z|^2).
\end{aligned}$$

Consequently $|\mu(z)| < 1$ whenever $|z| < 1$, $|\mu(z)| = 1$ whenever $|z| = 1$ and $|\mu(z)| > 1$ whenever $|z| > 1$ and $\overline{b}z + \overline{a} \neq 0$.

Now the inverse μ^{-1} of the Möbius transformation μ is characterized by the property that

$$\mu^{-1}(z) = \frac{\overline{a}z - b}{-\overline{b}z + a}$$

for all complex numbers z for which $-\overline{b}z + a \neq 0$ (see Corollary 1.8). Because the coefficients of this Möbius transformation μ^{-1} have properties analogous to those of the Möbius transformation μ, we can conclude that μ^{-1} maps the open unit disk into itself, and therefore μ maps the open unit disk onto itself, as required. $\blacksquare$

Corollary 2.6

Let w be a complex number satisfying $|w| < 1$, and let μ_w be the Möbius transformation of the Riemann sphere that is defined so that $\mu_w(-1/\overline{w}) = \infty$, $\mu(\infty) = 1/\overline{w}$ and

$$\mu_w(z) = \frac{z + w}{1 + \overline{w}\, z}$$

for all complex numbers z distinct from $-1/\overline{w}$. Then the Möbius transformation μ_w maps the open unit disk onto itself. Moreover

$$\mu_w(tw) = \frac{t + 1}{1 + |w|^2 t}\, w$$

for all real numbers t distinct from $-1/|w|^2$, and consequently the diameter of the unit circle passing through 0 and w is mapped onto itself by the Möbius transformation μ_w. In particular $\mu_w(0) = w$ and $\mu_w(-w) = 0$.

Proposition 2.7

Let μ be a Möbius transformation of the Riemann sphere that maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ into itself and satisfies the condition $|\mu(0)| < 1$. Then there exist complex numbers a and b, where $|b| < |a|$, such that

$$\mu(z) = \frac{az + b}{\overline{b}z + \overline{a}} \quad \text{for all } z \in \mathbb{C} \text{ for which } \overline{a}z + \overline{b} \neq 0.$$

Proof

The Möbius transformation μ maps the unit circle into itself, and moreover $|\mu(0)| < 1$. It follows from Corollary 2.3 that $\Omega \circ \mu = \mu \circ \Omega$, where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Consequently $\mu = \Omega \circ \Omega \circ \mu = \Omega \circ \mu \circ \Omega$ because the composition of the inversion Ω with itself is the identity transformation of the Riemann sphere. Let a_0, b_0, c_0 and d_0 be complex coefficients determined so that

$$\mu(z) = \frac{a_0 z + b_0}{c_0 z + d_0} \quad \text{whenever } c_0 z + d_0 \neq 0.$$

Then the identity $\mu = \Omega \circ \mu \circ \Omega$ ensures that

$$\frac{a_0 z + b_0}{c_0 z + d_0} = \frac{\overline{d}_0 z + \overline{c}_0}{\overline{b}_0 z + \overline{a}_0}$$

for all complex numbers z for which $a_0 z + b_0 \neq 0$, $\overline{a}_0 + \overline{b}_0 z \neq 0$, $c_0 z + d_0 \neq 0$, and $\overline{c}_0 + \overline{d}_0 z \neq 0$ (see Lemma 2.1).

Consequently there exists some non-zero complex number ω with the property that $\overline{a}_0 = \omega d_0$, $\overline{b}_0 = \omega c_0$, $\overline{c}_0 = \omega b_0$ and $\overline{d}_0 = \omega a_0$ (see Proposition 1.9). It then follows that

$$\overline{a}_0\, \overline{d}_0 = \omega^2 a_0 d_0.$$

But

$$|\overline{a}_0\, \overline{d}_0| = |a_0 d_0|.$$

It follows that $|\omega^2| = 1$, and therefore $|\omega| = 1$. Accordingly a real number θ can be found so that

$$\omega = \cos 2\theta + \sqrt{-1}\, \sin 2\theta.$$

Let

$$\eta = \cos\theta + \sqrt{-1}\, \sin\theta.$$

It then follows from De Moivre's Theorem that $\eta^2 = \omega$. Now $\overline{\eta}^2\, \eta^2 = |\eta|^4 = 1$. It follows that $\overline{\eta}^2 \omega = 1$.

Let $a = \eta a_0$ and $b = \eta b_0$, $c = \eta c_0$ and $d = \eta d_0$. Then

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{whenever } cz + d \neq 0.$$

Also $a_0 = \overline{\eta} a$, $b_0 = \overline{\eta} b$, $c_0 = \overline{\eta} c$ and $d_0 = \overline{\eta} d$. Consequently

$$\overline{d} = \overline{\eta}\,\overline{d}_0 = \overline{\eta}\omega a_0 = \overline{\eta}^2 \omega a = a$$

and

$$\overline{c} = \overline{\eta}\,\overline{c}_0 = \overline{\eta}\omega b_0 = \overline{\eta}^2 \omega b = b.$$

Accordingly

$$\mu(z) = \frac{az + b}{\overline{b}\,z + \overline{a}} \quad \text{whenever } \overline{b}\,z + \overline{a} \neq 0.$$

Moreover $|\mu(0)| < 1$, and consequently $|b| < |a|$, as required. $\blacksquare$

2.2. The Poincaré Distance Function on the Unit Disk

Definition

Let D be the open unit disk in the complex plane $\mathbb{C}$, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The *Poincaré distance function* ρ on D is defined so that

$$\rho(z, w) = \log \left(\frac{|1 - \overline{w}\, z| + |z - w|}{|1 - \overline{w}\, z| - |z - w|} \right)$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$.

Note that

$$\frac{|z - w|}{|1 - \overline{w}\, z|} < 1$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$. (This follows directly from Corollary 2.6). Consequently the Poincaré distance $\rho(z, w)$ between any two points z and w of the unit disk is a well-defined positive real number.

Proposition 2.8

Let s and t be real numbers satisfying $-1 < s < t < 1$. Then the Poincaré distance, in the unit disk, between s and t is given by the formula

$$\rho(s,t) = \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right).$$

Proof

Evaluating, and noting that $1 - st > 0$ (because $|s| < 1$ and $|t| < 1$) and $|t - s| = t - s$ (since $s < t$ by assumption), we find that

$$\begin{aligned}
\rho(s,t) &= \log\left(\frac{|1-st| + |t-s|}{|1-st| - |t-s|}\right) \\
&= \log\left(\frac{1 - st + t - s}{1 - st + s - t}\right) \\
&= \log\left(\frac{(1-s)(1+t)}{(1+s)(1-t)}\right) \\
&= \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right),
\end{aligned}$$

as required. ∎

Proposition 2.9

Let ρ be the Poincaré distance function on the open unit disk D, and let δ be a positive real number. Then

$$\{z \in D : \rho(z, 0) = \delta\} = \{z \in D : |z| = R\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof

It follows from the definition of Poincaré distance function that all complex numbers z satisfying $\rho(z,0) = \delta$ are equidistant from zero. They therefore constitute a circle centred on zero. It remains to determine the radius of that circle. Now it follows, on applying Proposition 2.8, that

$$\delta = \log\left(\frac{1+R}{1-R}\right).$$

Consequently

$$e^\delta - 1 = \frac{2R}{1-R}, \quad e^\delta + 1 = \frac{2}{1-R},$$

and therefore

$$R = \frac{e^\delta - 1}{e^\delta + 1},$$

as required. ∎

The Poincaré distance function ρ on the unit disk D has the property that $\rho(z, w) = \rho(w, z)$ for all $z, w \in D$. It therefore follows immediately from Proposition 2.8 that

$$\rho(s, t) = \left| \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right) \right|$$

for all real numbers s and t satisfying $-1 < s < 1$ and $-1 < t < 1$.

Lemma 2.10

Let z and w be complex numbers, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Then

$$(z, \Omega(z); w, \Omega(w)) = \left| \frac{z - w}{1 - \overline{w}z} \right|^2$$

for all complex numbers z and w with the exception of those pairs z, w for which $|z| = 1$ and $z = w$.

Proof

Let z and w be complex numbers. Suppose that it is not the case that $|z| = 1$ and $z = w$. Examination of possible cases shows that it is not then possible for three of the complex numbers z, $\Omega(z)$, w and $\Omega(w)$ to coincide with one another. Indeed if $|z| \neq 1$ and $|w| \neq 1$ then exactly two of the points $z, \Omega(z), w, \Omega(w)$ will lie in the unit disk consisting of those complex numbers whose modulus is less than one, and therefore it is not possible for any three of the four points to coincide with one another. If $|z| = 1$, it would only be possible for three of the points $z, \Omega(z), w, \Omega(w)$ to coincide with one another if it were also the case that $w = z$. Consequently the cross-ratio $(z, \Omega(z); w, \Omega(w))$ is defined in all cases with the exception of those where $|z| = 1$ and $w = z$.

Now let $u_1 = z$, $v_1 = 1$, $u_2 = 1$, $v_2 = \overline{z}$, $u_3 = w$, $v_3 = 1$, $u_4 = 1$, $v_4 = \overline{w}$. Then $u_1/v_1 = z$, $u_2/v_2 = \Omega(z)$, $u_3/v_3 = w$ and $u_4/v_4 = \Omega(w)$. The definition of cross-ratio then ensures that

$$\begin{aligned}
(z, \Omega(z); w, \Omega(w)) &= \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&= \frac{(z - w)(\overline{w} - \overline{z})}{(1 - w\overline{z})(z\overline{w} - 1)} \\
&= \left| \frac{z - w}{1 - \overline{w} z} \right|^2,
\end{aligned}$$

as required. ∎

Proposition 2.11

Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z, w)$ denote the Poincaré distance between z and w. Then

$$\rho(z, w) = \log\left(\frac{1 + \sqrt{(z, \Omega(z); w, \Omega(w))}}{1 - \sqrt{(z, \Omega(z); w, \Omega(w))}}\right),$$

where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Proof

Evaluating, and applying the result of Lemma 2.10, we find that

$$\begin{aligned}
\rho(z,w) &= \log\left(\frac{|1-\overline{w}\,z| + |z-w|}{|1-\overline{w}\,z| - |z-w|}\right) \\
&= \log\left(\frac{1+\dfrac{|z-w|}{|1-\overline{w}\,z|}}{1-\dfrac{|z-w|}{|1-\overline{w}\,z|}}\right) \\
&= \log\left(\frac{1+\sqrt{(z,\Omega(z);w,\Omega(w))}}{1-\sqrt{(z,\Omega(z);w,\Omega(w))}}\right),
\end{aligned}$$

as required. ∎

Corollary 2.12

Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z, w)$ denote the Poincaré distance between z and w. Then the cross-ratio $(z, \Omega(z); w, \Omega(w))$ is expressed in terms of the Poincaré distance according to the formula

$$(z, \Omega(z); w, \Omega(w)) = \left(\frac{e^{\rho(z,w)} - 1}{e^{\rho(z,w)} + 1} \right)^2 .$$

Proof

Let $q = (z, \Omega(z); w, \Omega(w))$ and $s = \rho(z, w)$. It follows from Proposition 2.11 that

$$s = \log\left(\frac{1+\sqrt{q}}{1-\sqrt{q}}\right).$$

Consequently

$$e^s - 1 = \frac{2\sqrt{q}}{1-\sqrt{q}}, \quad e^s + 1 = \frac{2}{1-\sqrt{q}},$$

and thus

$$q = \left(\frac{e^s-1}{e^s+1}\right)^2.$$

The result follows. ∎

Definition

A transformation φ that maps the open unit disk D in the complex plane onto itself is said to be an *isometry* (with respect to Poincaré distance) if

$$\rho\Big(\varphi(z), \varphi(w)\Big) = \rho(z, w)$$

for all complex numbers z and w in the open unit disk D, where ρ denotes the Poincaré distance function on D.

Proposition 2.13

Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$. Then every Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself is an isometry with respect to the Poincaré distance function on D.

Proof

The Möbius transformation μ has the property that $\mu \circ \Omega = \Omega \circ \mu$, because it maps the unit disk onto itself (see Proposition 2.2). Moreover the values of cross-ratios are preserved under the action of Möbius transformations (Proposition 1.18). Consequently

$$\begin{aligned}&\Big(\mu(z), \Omega(\mu(z)); \mu(w), \Omega(\mu(w))\Big)\\ &\qquad = \Big(\mu(z), \mu(\Omega(z)); \mu(w), \mu(\Omega(w))\Big)\\ &\qquad = \Big(z, \Omega(z); w, \Omega(w)\Big).\end{aligned}$$

The required result therefore follows immediately from an identity previously established (Proposition 2.11) expressing the Poincaré distance $\rho(z, w)$ in terms of the cross-ratio $(z, \Omega(z); w, \Omega(w))$. ∎

Proposition 2.14

Let z_1, w_1, z_2 and w_2 be elements of the open unit disk D, where

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, where ρ denotes the Poincaré distance function on D. Then there exists a Möbius transformation μ mapping the open unit disk D onto itself with the property that $\mu(z_1) = z_2$ and $\mu(w_1) = w_2$.

Proof

The values of the cross-ratios

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) \quad \text{and} \quad (z_2, \Omega(z_2); w_2, \Omega(w_2))$$

are determined by the values of the Poincaré distances $\rho(z_1, w_1)$ and $\rho(z_2, w_2)$ respectively (see Corollary 2.12). Consequently

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) = (z_2, \Omega(z_2); w_2, \Omega(w_2)).$$

It follows from this that there exists a unique Möbius transformation μ with the properties that $\mu(z_1) = z_2$, $\mu(\Omega(z_1)) = \Omega(z_2)$, $\mu(w_1) = w_2$ and $\mu(\Omega(w_1)) = \Omega(w_2)$, (see Proposition 1.18).

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is itself a Möbius transformation (Lemma 2.1) Then

$$\begin{aligned}
\hat{\mu}(z_1) &= \Omega(\mu(\Omega(z_1))) = \Omega(\Omega(z_2)) = z_2, \\
\hat{\mu}(\Omega(z_1)) &= \Omega(\mu(\Omega(\Omega(z_1)))) = \Omega(\mu(z_1)) = \Omega(z_2), \\
\hat{\mu}(w_1) &= \Omega(\mu(\Omega(w_1))) = \Omega(\Omega(w_2)) = w_2, \\
\hat{\mu}(\Omega(w_1)) &= \Omega(\mu(\Omega(\Omega(w_1)))) = \Omega(\mu(w_1)) = \Omega(w_2).
\end{aligned}$$

Consequently the Möbius transformations μ and $\hat{\mu}$ both map z_1, $\Omega(z_1)$, w_1 and $\Omega(w_1)$ to z_2, $\Omega(z_2)$, w_2 and $\Omega(w_2)$ respectively. But two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Corollary 1.17). Consequently $\hat{\mu} = \mu$, and thus $\Omega \circ \mu = \mu \circ \Omega$. Moreover elements z_1 and z_2 of the open unit disk D are mapped into D. Applying Proposition 2.2, we conclude that the Möbius transformation μ maps the open unit disk D onto itself. This completes the proof. $\blacksquare$

Proposition 2.15

Let D be the open unit disk in the complex plane, let w_0 be a complex number lying in D, let δ be a positive real number, and let

$$\Gamma = \{z \in D : \rho(z, w_0) = \delta\}.$$

Then Γ is a circle contained within the open unit disk D. Moreover if w_0 lies on the real line then the centre of the circle Γ also lies on the real line.

Proof

Let

$$\Gamma_0 = \{z \in D : \rho(z, 0) = \delta\}.$$

Then Γ_0 is a circle in the complex plane (see Proposition 2.9). Now there exists a Möbius transformation μ mapping the open unit disk D onto itself with the property that $\mu(0) = w_0$ (see Corollary 2.6). Now the image $\mu(\Gamma_0)$ of the circle Γ_0 must itself be a circle contained within the unit disk. Indeed Möbius transformations map circles and straight lines to circles and straight lines (Proposition 1.10), and obviously $\mu(\Gamma_0)$ cannot be a straight line. Moreover $\mu(\Gamma_0) = \Gamma$, because Möbius transformations mapping the open unit disk D onto itself are isometries with respect to the Poincaré distance function ρ on the open unit disk (Proposition 2.13). The result follows. ∎

Proposition 2.16

Let ρ be the Poincaré distance function on the open unit disk D in the complex plane, let t be a real number satisfying $0 < t < 1$, and let w be a complex number distinct from 0 and t for which $|w| < 1$. Then

$$\rho(0, w) \leq \rho(0, t) + \rho(t, w).$$

Moreover $\rho(0, w) = \rho(0, t) + \rho(t, w)$ if and only if the complex number w is a positive real number for which $t < w < 1$.

Proof

We first note that

$$\rho(0, t) = \log\left(\frac{1+t}{1-t}\right)$$

(see Proposition 2.8).

Given a complex number w in the unit disk that is distinct from 0 and t, let real numbers s and u between -1 and 1 be determined so that

$$\log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right) = \rho(t,w)$$

and

$$\log\left(\frac{1+u}{1-u}\right) - \log\left(\frac{1+t}{1-t}\right) = \rho(t,w).$$

Then $-1 < s < t < u < 1$ and

$$\rho(s,t) = \rho(t,u) = \rho(t,w)$$

and consequently

$$\rho(s,u) = \rho(s,t) + \rho(t,u) = 2 \times \rho(t,u) < 2 \times \rho(0,u) = \rho(-u,u)$$

(again applying Proposition 2.8). It follows that $-u < s < t < u$.

Let

$$\Gamma_1 = \{z \in D : \rho(z, 0) = \rho(u, 0)\}$$

and

$$\Gamma_2 = \{z \in D : \rho(z, t) = \rho(u, t)\}.$$

It follows from Proposition 2.15 that Γ_1 and Γ_2 are circles in the complex plane, containined in the open unit disk D, whose centres lie on the real line. The circle Γ_1 passes through $-u$ and u, and the circle Γ_2 passes through s and u. Now $-u < s < u$. It follows from elementary geometry that all points of the circle Γ_2 with the exception of the point u lie within the circle Γ_1. Now the point w lies on the circle Γ_2. Therefore

$$\rho(0, w) \leq \rho(0, u) = \rho(0, t) + \rho(t, u) = \rho(0, t) + \rho(t, w).$$

Moreover $\rho(0, w) = \rho(0, t) + \rho(t, w)$ if and only if $w = u$, in which case w lies on the real line and $t < w < 1$. The result follows. ∎

Proposition 2.17 (Triangle Inequality for Poincaré Distance)

The Poincaré distance function ρ on the open unit disk D has the property that

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3)$$

for all complex numbers z_1, z_2 and z_3 belonging to the disk D.

Proof

This inequality follows directly in cases where any two of z_1, z_2 and z_3 coincide with one another. Accordingly it remains to prove that the inequality holds in cases where these three complex numbers are distinct.

Accordingly let z_1, z_2 and z_3 be any three distinct points of the unit disk D. There exists a real number t satisfying $0 < t < 1$ determined so that $\rho(0, t) = \rho(z_1, z_2)$. There then exists a Möbius transformation μ that maps the open unit disk onto itself and satisfies $\mu(0) = z_1$ and $\mu(t) = z_2$ (see Proposition 2.14). Let w be the unique point of the open unit disk for which $\mu(w) = z_3$. Then

$$\rho(0, w) \leq \rho(0, t) + \rho(t, w).$$

(see Proposition 2.16). But the Möbius transformation μ is an isometry of the Poincaré distance function (Proposition 2.13). Consequently

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3).$$

as required. ∎

Lemma 2.18

Let u be a real number satisfying $0 < u < 1$ and let z be a point of the open unit disk that does not lie on the real line between 0 and u. Then

$$\rho(0, u) < \rho(0, z) + \rho(z, u),$$

where ρ denotes the Poincaré distance function on the open unit disk.

Proof

A positive real number θ can be chosen for which t is a positive real number, where

$$t = (\cos\theta + \sqrt{-1}\,\sin\theta)z.$$

Let

$$w = (\cos\theta + \sqrt{-1}\,\sin\theta)u.$$

The condition in the statement of the lemma regarding the location of z ensures that the complex number w is not a real number lying between t and 1. It follows from Proposition 2.16 that

$$\rho(0, w) < \rho(0, t) + \rho(t, w).$$

Now rotations of the open unit disk about zero are isometries of the Poincaré distance function defined on the unit disk. Consequently

$$\rho(0, u) < \rho(0, z) + \rho(z, u),$$

as required. $\blacksquare$

2.3. Hyperbolic Length

Definition

Let Γ be a straight line segment or circular arc contained in the open unit disk, and let p and q be points lying on Γ. We define the *hyperbolic length* of Γ between the points p and q to be the smallest non-negative real number L with the property that

$$\rho(z_0, z_1) + \rho(z_1, z_2) + \cdots + \rho(z_{m-1}, z_m) \leq L$$

for all choices of distinct points $z_0, z_1, z_2, \ldots, z_{m-1}, z_m$ lying in order along the line or curve Γ with $z_0 = p$ and $z_m = q$.

Remark

Those familiar with the concept of *least upper bounds* will note that the hyperbolic length of Γ is, according to this definition, the least upper bound of the values of the sums of the prescribed form.

Now a basic principle of real analysis asserts that if a non-empty set of real numbers is bounded above, then that set has a least upper bound. Accordingly, in order to prove that any straight line segment or circular arc contained within the open unit disk in the complex plane has a well-defined hyperbolic length, provided that the endpoints of that segment or arc lie within the open disk, it would be necessary to show that there exists some positive real number M that is large enough to ensure that, whenever points $z_0, z_1, \ldots, z_m$ are taken in order along that segment or arc, then

$$\sum_{j=1}^{m} \rho(z_j, z_{j-1}) \leq M.$$

Now suppose that the straight line segment or circular arc is contained within a disk of radius R centred on zero in the complex plane, where $0 < R < 1$. One can then establish the existence of a real constant K, determined by R, such that $\rho(z, z') \leq K|z - z'|$ for all complex numbers z and z' satisfying $|z| \leq R$ and $|z'| \leq R$. One can then show that

$$\sum_{j=1}^{m} \rho(z_j, z_{j-1}) \leq KN,$$

where N is the Euclidean length of the straight line segment or arc in question. Consequently the basic principle of real analysis described above guarantees that the segment or circular arc has a well-defined hyperbolic length.

Remark

The definition given is applicable also to certain other curves besides straight line segments and circular arcs, provided that those curves are sufficiently well-behaved.

In particular, if the curve is parametrized by a real variable t so that the the points of the curve are of the form $x(t) + \sqrt{-1}\, y(t)$, where $x(t)$ and $y(t)$ are continuously differentiable real-valued functions of t as t increases from t_0 to t_1, then the hyperbolic length of the curve may be defined in the manner described. Its value can be shown to be equal to the value of the integral

$$\int_{t_0}^{t_1} \frac{2}{1 - x^2 - y^2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Given points p and q that lie on some straight line segment or circular arc Γ in the open unit disk, let us denote by

$$L_{\mathrm{hyp}}(\Gamma; p, q)$$

the hyperbolic length of Γ between the points p and q.

Lemma 2.19

Let p, q be points lying on straight line segment or circular arc Γ in the open unit disk. Then

$$L_{\mathrm{hyp}}(\Gamma; p, q) \geq \rho(p, q),$$

where $L_{\mathrm{hyp}}(\Gamma; p, q)$ denotes respectively the hyperbolic length of Γ between the points p and q and $\rho(p, q)$ denotes the Poincaré distance between p and q.

Proof

This result follows directly from the definition of hyperbolic length. (The criterion in that definition applies in particular to the case where the collection of points along Γ between p and q just consists of the two points p and q, with $m = 1$, $z_0 = p$ and $z_1 = q$, employing the notation employed in the definition of hyperbolic length given above.) ∎

Proposition 2.20

Let p, q and r be points lying in order along a straight line segment or circular arc Γ in the open unit disk. Then

$$L_{\mathrm{hyp}}(\Gamma; p, r) = L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r).$$

Proof

Let $z_0, z_1, z_2, \ldots, z_n$ be points in order along Γ with $z_0 = p$ and $z_n = r$. Then either $q = z_k$ for some integer k between 1 and $n - 1$ or else q lies between z_{k-1} and z_k for some integer k between 1 and n. In the case where $q = z_k$ for some integer k between 1 and $n - 1$, we find that

$$\begin{aligned}\sum_{j=0}^{n} \rho(z_{j-1}, z_j) &= \sum_{j=0}^{k} \rho(z_{j-1}, z_j) + \sum_{j=k+1}^{n} \rho(z_{j-1}, z_j) \\ &\leq L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r).\end{aligned}$$

In the case where q lies between z_{k-1} and z_k for some integer k between 1 and n, the Triangle Inequality satisfied by the Poincaré distance function (Proposition 2.17) ensures that

$$\begin{aligned}
\sum_{j=0}^{n} \rho(z_{j-1}, z_j) &= \sum_{j=0}^{k-1} \rho(z_{j-1}, z_j) + \rho(z_{k-1}, z_k) \\
&\qquad + \sum_{j=k+1}^{n} \rho(z_{j-1}, z_j) \\
&\leq \sum_{j=0}^{k-1} \rho(z_{j-1}, z_j) + \rho(z_{k-1}, q) \\
&\qquad + \rho(q, z_k) + \sum_{j=k+1}^{n} \rho(z_{j-1}, z_j) \\
&\leq L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r).
\end{aligned}$$

It follows from these observations that

$$L_{\mathrm{hyp}}(\Gamma; p, r) \leq L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r).$$

Now let some positive real number ε be given. Then there exist points $z_0, z_1, \ldots, z_m$ in order along Γ with $z_0 = p$ and $z_m = q$ such that

$$\sum_{j=1}^{m} \rho(z_{j-1}, z_j) > L_{\mathrm{hyp}}(\Gamma; p, q) - \varepsilon.$$

There also exist points $z_m, z_{m+1}, \ldots, z_n$ in order along Γ with $z_m = q$ and $z_n = r$ such that

$$\sum_{j=m+1}^{n} \rho(z_{j-1}, z_j) > L_{\mathrm{hyp}}(\Gamma; q, r) - \varepsilon.$$

Consequently

$$\sum_{j=1}^{n} \rho(z_{j-1}, z_j) > L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r) - 2\varepsilon.$$

It follows that

$$L_{\mathrm{hyp}}(\Gamma; p, r) > L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r) - 2\varepsilon$$

for all positive real numbers ε, and therefore

$$L_{\mathrm{hyp}}(\Gamma; p, r) \geq L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r).$$

The inequalities established within the proof now enable us to conclude that

$$L_{\mathrm{hyp}}(\Gamma; p, r) = L_{\mathrm{hyp}}(\Gamma; p, q) + L_{\mathrm{hyp}}(\Gamma; q, r),$$

as required. ∎

Proposition 2.21

Let Γ be the straight line segment in the open unit disk with endpoints p and q, where p and q are real numbers satisfying $-1 < p < q < 1$. Then the hyperbolic length of Γ is equal to the Poincaré distance $\rho(p, q)$ between p and q.

Proof

Let $t_0, t_1, \ldots, t_m$ be real numbers for which

$$p = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = q.$$

Applying Proposition 2.8 we find that

$$\begin{aligned}\sum_{j=1}^{m} \rho(t_{j-1}, t_j) &= \sum_{j=1}^{m} \left(\log\left(\frac{1+t_j}{1-t_j}\right) - \log\left(\frac{1+t_{j-1}}{1-t_{j-1}}\right)\right) \\ &= \log\left(\frac{1+q}{1-q}\right) - \log\left(\frac{1+p}{1-p}\right) = \rho(p, q).\end{aligned}$$

The result follows. ∎

Proposition 2.22

Let μ be a Möbius transformation mapping the open unit disk in the complex plane onto itself, and let Γ be a straight line segment or circular arc contained within the open unit disk. Then the hyperbolic length of the image $\mu(\Gamma)$ of Γ under the Möbius transformation μ is equal to the hyperbolic length of Γ itself.

Proof

This result follows from the definition of hyperbolic length, in view of the fact that Möbius transformations that map the open unit disk onto itself are isometries with respect to the Poincaré distance function (Proposition 2.13). ∎

2.4. Geodesics in the Open Unit Disk

Definition

We say that a straight line segment or circular arc contained within the open unit disk in the complex plane is a *geodesic* if the hyperbolic length of the segment or arc between any two points lying on it is equal to the Poincaré distance between those two points.

Proposition 2.23

Möbius transformations mapping the open unit disk onto itself map geodesics onto geodesics.

Proof

Möbius transformations mapping the open unit disk onto itself are isometries with respect to the Poincaré distance function (Proposition 2.13) and they preserve hyperbolic distance (Proposition 2.22) The result therefore follows immediately from these observations and the definition of geodesics in the open unit disk. ■

Theorem 2.24

Let Γ be a straight line segment or circular arc contained within the open unit disk in the complex plane. Then Γ is a geodesic if and only if the straight line or circle of which it forms part intersects the unit circle orthogonally.

Proof

First suppose that the straight line or circle of which Γ forms part intersects the unit circle orthogonlly at points z_1 and z_2. It follows from Lemma 2.4 that there exists a Möbius transformation μ of the Riemann sphere mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$. Now Möbius transformations map circles and straight lines to circles and straight lines (Proposition 1.10). Moreover they preserve the angles between circles and straight line segments at their points of intersection (see Proposition 1.27). Therefore the straight line or circle of which the image $\mu(\Gamma)$ under the the Möbius transformation μ forms part must intersect the unit circle orthogonally at -1 and 1, and consequently it must coincide with the real line. We conclude therefore that $\mu(\Gamma)$ must be contained within the real line.

It then follows from Proposition 2.21 $\mu(\Gamma)$ must be a geodesic. Now Möbius transformations that map the open unit disk onto itself map geodesics to geodesics (Proposition 2.23). Consequently Γ, being the image of geodesic under the inverse of the Möbius transformation μ, must itself be a geodesic.

Now suppose that Γ is a geodesic. Let p and q be points lying on Γ, and let u be the positive real number for which $\rho(0, u) = \rho(p, q)$, where ρ denotes the Poincaré distance function on the open unit disk. Then there exists a Möbius transformation μ, mapping the open unit disk onto itself, which is such as to ensure that $\mu(p) = 0$ and $\mu(q) = u$. Now Möbius transformations map circles and straight lines to circles and straight lines (Proposition 1.10). Consequently $\mu(\Gamma)$ is a straight line or circular arc on which lie the real numbers 0 and u.

Suppose that $\mu(\Gamma)$ were to pass through some point z of the unit disk that did not lie on the real line between 0 and u. Then, applying Lemma 2.18 and Proposition 2.20 it would follow that

$$\begin{aligned} L_{\mathrm{hyp}}(\mu(\Gamma); 0, u) &= L_{\mathrm{hyp}}(\mu(\Gamma); 0, z) + L_{\mathrm{hyp}}(\mu(\Gamma); z, u) \\ &\geq \rho(0, z) + \rho(z, u) > \rho(0, u). \end{aligned}$$

Consequently $\mu(\Gamma)$ would not be a geodesic. It follows that Γ would not be a geodesic, because Möbius transformations that map the open unit disk onto itself map geodesics to geodesics (Proposition 2.23).

2. The Disk Model of the Hyperbolic Plane (continued)

We conclude therefore that if Γ is a geodesic, and if μ is a Möbius transformation mapping the points p and q of Γ to 0 and u respectively, where $0 < u < 1$ and $\rho(0, u) = \rho(p, q)$, then all points of $\mu(\Gamma)$ must lie on the real line.

Now the real line cuts the unit circle orthogonally at the points of intersection. Also Möbius transformations preserve the angles between circles and straight line segments at their points of intersection (see Proposition 1.27). Therefore the straight line or circle of which Γ forms part must also intersect the unit circle orthogonally, as required. $\blacksquare$

2.5. Complete Geodesics

Definition

A geodesic contained within the open unit disk is said to be *complete* if it is the intersection of the open unit disk with a straight line or circle in the complex plane.

Proposition 2.25

Given two complete geodesics in the open unit disk D, there exists a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and maps one complete geodesic onto the other.

Proof

Let Γ_1 and Γ_2 be complete geodesics in the open unit disk D, and let l be the geodesic joining -1 and 1 that is the intersection of the disk D with the real axis of the complex plane. Then, given distinct points p_1 and q_1 lying on Γ_1, there exists a Möbius transformation μ_1 that maps the segment of Γ_1 with endpoints p_1 and q_1 into the real line. Then μ_1 maps the complete geodesic Γ_1 onto the complete geodesic l. Similarly there exists a Möbius transformation that maps the complete geodesic Γ_2 onto the complete geodesic l. Then $\mu_2^{-1} \circ \mu_1$ is a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and also maps the complete geodesic Γ_1 onto the complete geodesic Γ_2, as required. $\blacksquare$

2.6. Geodesic Rays and Segments

Definition

A *geodesic segment* is a geodesic that is a straight line segment or circular arc whose endpoints both lie within the open unit disk.

Definition

A *geodesic ray* is a geodesic that has an endpoint within the open unit disk and which includes that endpoint together with all points of a complete geodesic that lie between the endpoint and some point at which the straight line or circle of which the geodesic ray forms part crossses the unit circle that bounds the open unit disk.

2.7. The Group of Hyperbolic Motions of the Disk

Definition

Let X be a subset of the complex plane. A collection of invertible transformations of the set X is said to be a *transformation group* acting on the set X if the following conditions are satisfied:

(i) the identity transformation belongs to the collection;

(ii) any composition of transformations belonging to the collection must itself belong to the collection;

(iii) the inverse of any transformation belonging to the collection must itself belong to the collection.

The collection of all Möbius transformations of the Riemann sphere that map the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself is a transformation group acting on the open unit disk. Indeed the identity transformation is a Möbius transformation mapping the open unit disk onto itself, the composition of any two Möbius transformations that each map the open unit disk onto itself must also map the open unit disk onto itself, and the inverse of any Möbius transformation that maps the open unit disk onto itself must also map the open unit disk onto itself.

Definition

Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $\kappa\colon D \to D$ be the transformation of the open unit disk defined so that $\kappa(z) = \overline{z}$ for all $z \in D$, where $\overline{z}$ denotes the complex conjugate of the complex number z. A transformation of the open unit disk is said to be a *hyperbolic motion* of the unit disk if either it is a Möbius transformation mapping the unit disk D onto itself or else it expressible as a composition of transformations of the form $\mu \circ \kappa$, where μ is a Möbius transformation mapping the open unit disk onto itself.

Möbius transformations give rise to orientation-preserving transformations of the complex plane (see Proposition 1.28 and the discussion of orientation-preserving and orientation-reversing transformations of the complex plane that follows the proof of that proposition). Also the transformation $\kappa\colon D \to D$ that maps each complex number z in D to its complex conjugate $\overline{z}$ is orientation-reversing. Consequently a composition of two transformations in which some Möbius transformation follows the complex conjugation transformation κ is orientation-reversing.

Orientation-preserving hyperbolic motions are the analogues, in hyperbolic geometry, of transformations of the flat Euclidean plane that can be represented as the composition of a rotation followed by a translation.

Orientation-reversing hyperbolic motions are the analogues, in hyperbolic geometry, of reflections and glide reflections of the flat Euclidean plane.

Proposition 2.26

Let D be the open unit disk in the complex plane, consisting of those complex numbers z that satisfy $|z| < 1$. Then, given any orientation-preserving hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$, such that

$$\varphi(z) = \frac{az + b}{\overline{b}\,z + \overline{a}} \quad \text{for all } z \in D.$$

Similarly, given any orientation-reversing hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$ such that

$$\varphi(z) = \frac{a\,\overline{z} + b}{\overline{b}\,\overline{z} + \overline{a}} \quad \text{for all } z \in D.$$

Proof

This result follows directly on applying Proposition 2.7. ∎

Proposition 2.27

The collection of all hyperbolic motions of the open unit disk is a transformation group acting on the open unit disk.

Proof

The identity transformation is a Möbius transformation that maps the open unit disk onto itself and is thus a hyperbolic motion. Next let μ_1 and μ_2 be Möbius transformations that map the open unit disk onto itself, Then $\kappa \circ \mu_2 \circ \kappa$ is also a Möbius transformation that maps the open unit disk onto itself.

Indeed there exist complex numbers a_2 and b_2, where $|b_2| < |a_2|$, such that

$$\mu_2(z) = \frac{a_2 z + b_2}{\overline{b}_2\, z + \overline{a}_2}$$

for all complex numbers z for which $\overline{b}_2\, z + \overline{a}_2 \neq 0$ (see Proposition 2.7). Then

$$\kappa(\mu_2(\kappa(z))) = \frac{\overline{a}_2 z + \overline{b}_2}{b_2\, z + a_2},$$

and therefore $\kappa \circ \mu \circ \kappa$ is also a Möbius transformation that maps the open unit disk D onto itself. Now

$$\mu_1 \circ (\mu_2 \circ \kappa) = (\mu_1 \circ \mu_2) \circ \kappa, \quad (\mu_1 \circ \kappa) \circ \mu_2 = (\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)) \circ \kappa$$

and

$$(\mu_1 \circ \kappa) \circ (\mu_2 \circ \kappa) = \mu_1 \circ (\kappa \circ \mu_2 \circ \kappa).$$

Moreover $\mu_1 \circ \mu_2$ and $\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)$, being compositions of Möbius transformations that map the open unit disk onto itself, are themselves Möbius transformations that map the open unit disk onto itself. It follows from this observation that any composition of hyperbolic motions of the open unit disk is itself a hyperbolic motion of the open unit disk. Also

$$(\mu_2 \circ \kappa)^{-1} = \kappa \circ \mu_2^{-1} = (\kappa \circ \mu_2^{-1} \circ \kappa) \circ \kappa,$$

and the inverse of any Möbius transformation that maps the open unit disk onto itself must itself be a Möbius transformation that maps the open unit disk onto itself. Consequently the inverse of any hyperbolic motion is itself a hyperbolic motion. It follows that the collection of all hyperbolic motions of the open unit disk is indeed a transformation group acting on the open unit disk. ∎

Proposition 2.28

Let Γ be a complete geodesic in the open unit disk D. Then there exists an orientation-reversing hyperbolic motion φ with the property that $\varphi(z) = z$ for all complex numbers z that lie on the geodesic Γ and also those points of the open unit disk D that lie on one side of the geodesic Γ are mapped by φ to points that lie on the other side of Γ.

Proof

Let I be the set of real numbers t that satisfy the inequalities $-1 < t < 1$. Then I is a complete geodesic in the open unit disk D. There then exists a Möbius transformation μ that maps the geodesic I onto the geodesic Γ. (see Proposition 2.25). Let $\varphi = \mu \circ \kappa \circ \mu^{-1}$, where $\kappa(z) = \overline{z}$ for all $z \in D$. Then the orientation-reversing hyperbolic motion Γ has the required properties. ∎

Proposition 2.29

Let z_1, w_1, z_2 and w_2 be complex numbers belonging to the open unit disk D. Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, and suppose also that one of the sides of the geodesic Γ_1 in D passing through z_1 and w_1 has been chosen, and that one of the sides of the geodesic Γ_2 in D passing through z_2 and w_2 has also been chosen. Then there exists a hyperbolic motion φ with the following properties: $\varphi(z_1) = z_2$; $\varphi(w_1) = w_2$; φ maps complex numbers on the chosen side of the geodesic Γ_1 to complex numbers on the chosen side of the geodesic Γ_2.

Proof

It follows from Proposition 2.14 that there exists a Möbius transformation that maps the open unit disk onto itself and also maps z_1 and w_1 to z_2 and w_2 respectively. If this Möbius transformation does not itself map the chosen side of Γ_1 to the chosen side of Γ_2, then it may be composed with an orientation-reversing hyperbolic motion that fixes all complex numbers of the geodesic Γ_2 whilst mapping complex numbers on one side of Γ_2 to complex numbers on the other side. The result follows. ∎

Proposition 2.30

Let w be a complex number belonging to the open unit disk D in the complex plane, and let ρ denote the Poincaré distance function on D. Let δ be a positive real number. Then

$$\{z \in D : \rho(z, w) < \delta\} = \left\{ z \in D : \left| \frac{z - w}{1 - \overline{w}\, z} \right| < R \right\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof

Let

$$\mu_w(z) = \frac{z + w}{1 + \overline{w}z}$$

for all complex numbers z. Then μ_w is a Möbius transformation mapping the open unit disk onto itself for which $\mu_w(0) = w$ (see Corollary 2.6). Now Möbius transformations mapping the open unit disk onto itself are isometries with regard to the Poincaré distance function (see Proposition 2.13). Consequently

$$\{z \in D : \rho(z, w) < \delta\} = \{z \in D : \rho(\mu_w^{-1}(z), 0) < \delta\}.$$

The required result now follows on applying Proposition 2.9. ∎

Definition

Let D be the open unit disk in the complex plane that consists of those complex numbers z satisfying $|z| < 1$, and let C be a circle in the complex plane that is contained within D. A complex number w is said to be the *hyperbolic centre* of the circle C if the Poincaré distance between z and w is the same for all points z that lie on the circle C.

Proposition 2.31

Let C be a circle in the complex plane that is contained within the open unit disk D. Suppose that the circle C intersects the real axis at real numbers u and v, where $-1 < u < v < 1$. Suppose also that the hyperbolic centre of the circle C lies on the real axis, and is located at t, where $u < t < v$. Then

$$\left(\frac{1+t}{1-t}\right)^2 = \frac{(1+u)(1+v)}{(1-u)(1-v)}.$$

Proof

Applying Proposition 2.8, we find that t, u and v must satisfy the identity

$$\log\left(\frac{1+v}{1-v}\right) - \log\left(\frac{1+t}{1-t}\right) = \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+u}{1-u}\right).$$

Consequently

$$2\log\left(\frac{1+t}{1-t}\right) = \log\left(\frac{1+u}{1-u}\right) + \log\left(\frac{1+v}{1-v}\right).$$

The required result then follows on taking the exponential of both sides of this identity. ∎