Module MAU23302: Euclidean and Non-Euclidean Geometry
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Part II, (Sections 1 and 2)
Introduction to Non-Euclidean Geometry

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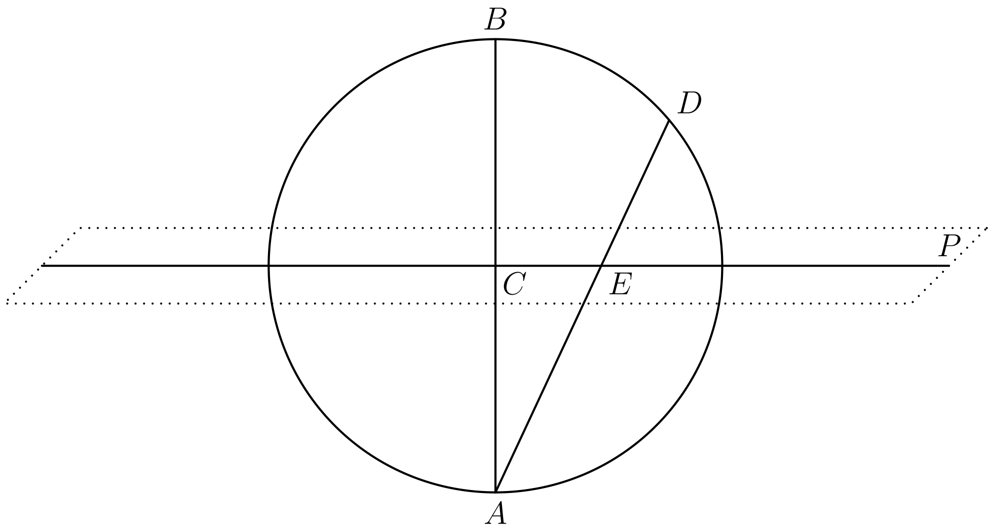
1 Möbius Transformations and Cross-Ratios

1.1 Stereographic Projection

Let a sphere in three-dimensional space be given, let C be the centre of that sphere, let AB be a diameter of that sphere with endpoints A and B, and let P be the plane through the centre of the sphere that is perpendicular to the diameter AB. Given a point D of the sphere distinct from the point A, the image of D under *stereographic projection* from the point A is defined to be the point E at which the line passing through the points A and D intersects the plane P.



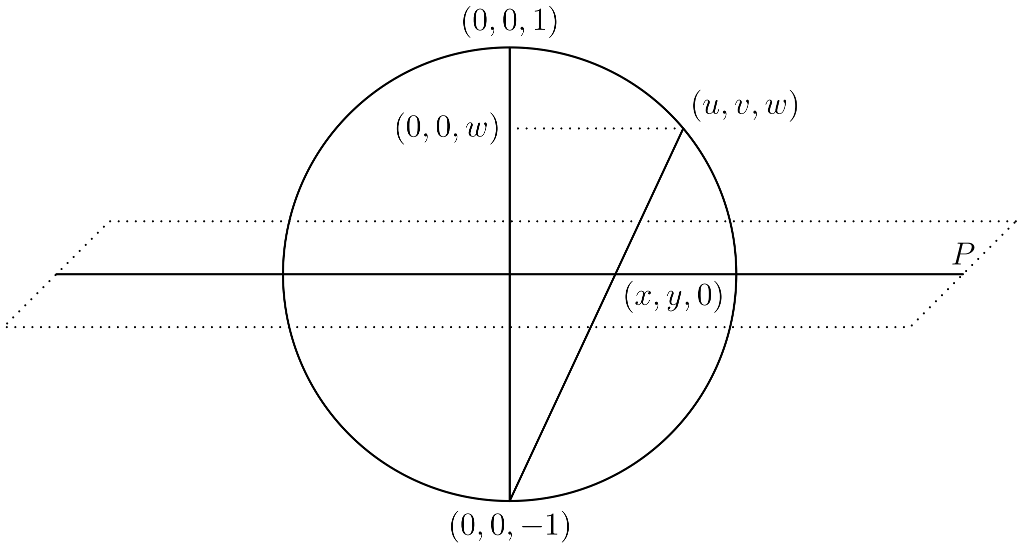
Proposition 1.1 *Let S^2 be the unit sphere in $\mathbb{R}^3$, consisting of those points (u, v, w) of $\mathbb{R}^3$ that satisfy the equation $u^2 + v^2 + w^2 = 1$, and let P be the plane consisting of those points (u, v, w) of $\mathbb{R}^3$ for which $w = 0$. Then, for each point (u, v, w) of S^2 distinct from the point $(0, 0, -1)$, the straight line passing through the points (u, v, w) and $(0, 0, -1)$ intersects the plane P at the point $(x, y, 0)$ at which*

$$x = \frac{u}{w+1} \quad \textit{and} \quad y = \frac{v}{w+1}.$$

Proof Let $A = (0, 0, -1)$, $D = (u, v, w)$ and $E = (x, y, 0)$. Then the displacements of the points D and E from the point A are represented by the vectors $(u, v, w + 1)$ and $(x, y, 1)$ respectively. These vectors are parallel because the points A, D and E are collinear. Consequently

$$\frac{x}{u} = \frac{y}{v} = \frac{1}{w+1}.$$

The result follows. ∎



Definition Let (u, v, w) be a point on the unit sphere distinct from the point $(0, 0, -1)$, where $u^2 + v^2 + w^2 = 1$, and let (x, y) be a point of the plane $\mathbb{R}^2$. We say that the point (x, y) is the *image* of the point (u, v, w) under *stereographic projection* from the point $(0, 0, -1)$ if

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Proposition 1.2 *Each point (x, y) of $\mathbb{R}^2$ is the image, under stereographic projection from the point $(0, 0, -1)$, of the point (u, v, w) of the unit sphere for which*

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \textit{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

This point (u, v, w) is distinct from the point $(0, 0, -1)$.

Proof Given a point (x, y) of $\mathbb{R}^2$, the straight line passing through the points $(0, 0, -1)$ and $(x, y, 0)$ is not tangent to the unit sphere, and therefore intersects the unit sphere at some point distinct from $(0, 0, -1)$. It follows that every point of $\mathbb{R}^2$ is the image, under stereographic projection from $(0, 0, -1)$, of some point of the unit sphere distinct from the point $(0, 0, -1)$.

Let (x, y) be the image, under stereographical projection from the point $(0, 0, -1)$, of a point (u, v, w), where $u^2 + v^2 + w^2 = 1$ and $w \neq -1$. Then

$$x = \frac{u}{w+1}, \quad y = \frac{v}{w+1}.$$

It follows that

$$x^2 + y^2 = \frac{u^2 + v^2}{(w+1)^2} = \frac{1 - w^2}{(w+1)^2} = \frac{1-w}{w+1}.$$

It follows that

$$w(x^2 + y^2) + x^2 + y^2 = 1 - w,$$

and therefore

$$w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

But then

$$1 + w = 1 + \frac{1 - x^2 - y^2}{1 + x^2 + y^2} = \frac{2}{1 + x^2 + y^2},$$

and therefore

$$\begin{aligned} u &= (1+w)x = \frac{2x}{1 + x^2 + y^2}, \\ v &= (1+w)y = \frac{2y}{1 + x^2 + y^2}. \end{aligned}$$

Conversely if

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

then

$$u^2 + v^2 + w^2 = \frac{4(x^2 + y^2) + (1 - x^2 - y^2)^2}{(1 + x^2 + y^2)^2} = 1,$$

because

$$\begin{aligned} &4(x^2 + y^2) + (1 - x^2 - y^2)^2 \\ &\quad = 4(x^2 + y^2) + 1 - 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &\quad = 1 + 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &\quad = (1 + x^2 + y^2)^2. \end{aligned}$$

Also $w > -1$ and

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

The result follows. ∎

1.2 The Riemann Sphere

The *Riemann sphere* $\mathbb{P}^1$ may be defined as the set $\mathbb{C} \cup \{\infty\}$ obtained by augmenting the system $\mathbb{C}$ of complex numbers with an additional element, denoted by ∞, where ∞ is not itself a complex number, but is an additional element added to the set, with the additional conventions that

$$z + \infty = \infty, \quad \infty \times \infty = \infty, \quad \frac{z}{\infty} = 0 \quad \text{and} \quad \frac{\infty}{z} = \infty$$

for all complex numbers z, and

$$z \times \infty = \infty, \qquad \text{and} \quad \frac{z}{0} = \infty$$

for all non-zero complex numbers z. The symbol ∞ cannot be added to, or subtracted from, itself. Also 0 and ∞ cannot be divided by themselves.

Note that, because the sum of two elements of $\mathbb{P}^1$ is not defined for every single pair of elements of $\mathbb{P}^1$, this set cannot be regarded as constituting a group under the operation of addition. Similarly its non-zero elements cannot be regarded as constituting a group under multiplication. In particular, the Riemann sphere cannot be regarded as constituting a field.

Note that any element of the Riemann sphere can be represented in the form $\dfrac{u}{v}$, where u and v are complex numbers that are not both equal to zero. Moreover the values of this fraction are determined as follows:

- $\dfrac{u}{v} = z$ for some non-zero complex number z if and only if $u \neq 0$, $v \neq 0$ and $u = zv$;
- $\dfrac{u}{v} = 0$ if and only if $u = 0$ and $v \neq 0$;
- $\dfrac{u}{v} = \infty$ if and only if $u \neq 0$ and $v = 0$.

Lemma 1.3 *Let u, v, u' and v' be complex numbers, where u and v are not both zero and also u' and v' are not both zero. Then the following are true:*

(i) $\dfrac{u}{v} = \dfrac{u'}{v'}$ *if and only if* $v'u = u'v$;

(ii) $\dfrac{u}{v} = \dfrac{u'}{v'}$ *if and only if there exists some non-zero complex number w for which $u' = wu$ and $v' = wv$;*

(iii) in cases where $\dfrac{u}{v} = \dfrac{u'}{v'}$ *it follows that $u = 0$ if and only if $u' = 0$;*

(iv) in cases where $\dfrac{u}{v} = \dfrac{u'}{v'}$ *it follows that* $v = 0$ *if and only if* $v' = 0$.

Proof First suppose that the complex numbers u, v, u' and v' are all non-zero. Then all four properties follow directly.

Next suppose that $u = 0$. Then $v \neq 0$ and $\dfrac{u}{v} = 0$. It follows in this case that $\dfrac{u}{v} = \dfrac{u'}{v'}$ if and only if $\dfrac{u'}{v'} = 0$, in which case $u' = 0$. Thus in cases where $\dfrac{u}{v} = \dfrac{u'}{v'}$ we find that $u = 0$ implies that $u' = 0$. Similarly $u' = 0$ if and only if $u = 0$, and thus $u = 0$ if and only if $u' = 0$. Note also that in cases where $u = 0$ and $\dfrac{u}{v} = \dfrac{u'}{v'}$, the complex numbers v and v' are both non-zero, and consequently the identities $u' = wu$ and $v' = wv$ hold simultaneously on taking $w = \dfrac{v'}{v}$.

Next suppose that $v = 0$. Then $u \neq 0$ and $\dfrac{u}{v} = \infty$. It follows in this case that $\dfrac{u}{v} = \dfrac{u'}{v'}$ if and only if $\dfrac{u'}{v'} = \infty$, in which case $v' = 0$. Thus in cases where $\dfrac{u}{v} = \dfrac{u'}{v'}$ we find that $v = 0$ implies that $v' = 0$. Similarly $v' = 0$ if and only if $v = 0$, and thus $v = 0$ if and only if $v' = 0$. Note also that in cases where $v = 0$ and $\dfrac{u}{v} = \dfrac{u'}{v'}$, the complex numbers u and u' are both non-zero, and consequently the identities $u' = wu$ and $v' = wv$ hold simultaneously on taking $w = \dfrac{u'}{u}$.

Consequently all four properties (i), (ii), (iii) and (iv) have been established, as required. ∎

Lemma 1.4 *Let* p_1 *and* p_2 *be elements of the Riemann sphere that are not both equal to* ∞*, and let* u_1*,* u_2*,* v_1 *and* v_2 *be complex numbers, where* u_1 *and* v_1 *are not both zero,* u_2 *and* v_2 *are not both zero, and* v_1 *and* v_2 *are not both zero, such that*

$$p_1 = \frac{u_1}{v_1} \quad \textit{and} \quad p_2 = \frac{u_2}{v_2}.$$

Then the sum $p_1 + p_2$ *of the elements* p_1 *and* p_2 *of the Riemann sphere is defined so as to ensure that*

$$p_1 + p_2 = \frac{v_1 u_2 + v_2 u_1}{v_1 v_2}.$$

Proof If $v_1 = 0$ then $v_2 u_1 \neq 0$, and consequently $v_1 u_2 + v_2 u_1 \neq 0$. Similarly if $v_2 = 0$ then $v_1 u_2 \neq 0$, and consequently $v_1 u_2 + v_2 u_1 \neq 0$. It follows that, in all cases, the complex numbers $v_1 u_2 + v_2 u_1$ and $v_1 v_2$ are not both zero, and consequently there is a well-defined element of the Riemann sphere that is determined by the fraction

$$\frac{v_1 u_2 + v_2 u_1}{v_1 v_2}.$$

If neither of p_1 and p_2 is the element ∞ of the Riemann sphere, then both p_1 and p_2 are complex numbers, and the above fraction represents the sum of those complex numbers, determined in the usual fashion within the algebra of complex numbers. On the other hand, if exactly one of the elements p_1 and p_2 of the Riemann sphere concides with ∞ then exactly one of the complex numbers v_1 and v_2 is equal to zero, and the above fraction represents the element ∞ of the Riemann sphere. The result follows. ∎

The following proposition follows directly from Proposition 1.2.

Proposition 1.5 *Let $\varphi: \mathbb{P}^1 \to \mathbb{R}^3$ be the mapping from the Riemann sphere $\mathbb{P}^1$ to $\mathbb{R}^3$ defined such that $\varphi(\infty) = (0, 0, -1)$ and*

$$\varphi(x + y\sqrt{-1}) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right)$$

for all real numbers x and y. Then the map φ sets up a one-to-one correspondence between points of the Riemann sphere $\mathbb{P}^1$ and points of the unit sphere S^2 in $\mathbb{R}^3$. To each point of the Riemann sphere $\mathbb{P}^1$ there corresponds exactly one point of the unit sphere S^2 in three-dimensional Euclidean space, and vice versa. Moreover if (u, v, w) is a point of the unit sphere S^2 distinct from $(0, 0, -1)$ then $(u, v, w) = \varphi(x + y\sqrt{-1})$, where

$$x = \frac{u}{w + 1} \quad \textit{and} \quad y = \frac{v}{w + 1}.$$

1.3 Möbius Transformations

Lemma 1.6 *Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$. Then these complex numbers determine a well-defined function $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ mapping the Riemann sphere $\mathbb{P}^1$ into itself that is characterized by the property that*

$$\mu\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero.

Proof Let u and v be complex numbers that are not both zero. Then

$$d(au + bv) - b(cu + dv) = (ad - bc)u$$

and

$$a(cu + dv) - c(au + bv) = (ad - bc)v$$

Now $ad - bc \neq 0$ and also u and v are not both zero. It must therefore be the case that $au+bv$ and $cu+dv$ are not both zero. It therefore follows that u and v determine a well-defined element of the Riemann sphere represented by the fraction $\dfrac{au + bv}{cu + dv}$. Moreover if u, v, u' and v' are complex numbers, where u and v are not both zero, and where u' and v' are not both zero, and if $u/v = u'/v'$, then there exists some non-zero complex number w for which $u' = wu$ and $v' = wv$. But it then follows that

$$\frac{au + bv}{cu + dv} = \frac{au' + bv'}{cu' + dv'}.$$

It follows from what has been shown that a quadruple of complex numbers a, b, c and d satisfying the condition $ad - bc \neq 0$ does indeed determine a well-defined function μ mapping the Riemann sphere into itself that is characterized by the property that

$$\mu\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero, as claimed. ∎

A Möbius transformation of the Riemann sphere is determined by its coefficients. It is convenient to specify these coefficients in the form of a non-singular 2×2 matrix.

Accordingly let A be a non-singular 2×2 matrix. Then there exist complex numbers a, b, c and d for which

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Moreover the requirement that A be non-singular (i.e., invertible) ensures that $ad-bc \neq 0$. We denote by μ_A the Möbius transformation of the Riemann sphere defined so that

$$\mu_A\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero.

It then follows that

$$\mu_A(z) = \frac{az+b}{cz+d}$$

for all complex numbers z for which $cz + d \neq 0$. If $c \neq 0$ then

$$\mu_A\left(-\frac{d}{c}\right) = \infty \quad \text{and}$$

$$\mu_A(\infty) = \frac{a}{c}.$$

If $c = 0$ then $d \neq 0$ and accordingly $\mu_A(\infty) = \infty$ and $\mu_A(z) = (az + b)/d$ for all complex numbers z.

Proposition 1.7 *The composition of any two Möbius transformations is a Möbius transformation. Specifically let A and B be non-singular 2×2 matrices with complex coefficients, and let μ_A and μ_B be the corresponding Möbius transformations of the Riemann sphere. Then the composition $\mu_A \circ \mu_B$ of these Möbius transformations is the Möbius transformation μ_{AB} of the Riemann sphere determined by the product AB of the matrices A and B.*

Proof Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} f & g \\ h & k \end{pmatrix},$$

and let

$$AB = \begin{pmatrix} m & n \\ p & q \end{pmatrix}.$$

Then

$$m = af + bh, \quad n = ag + bk,$$

$$p = cf + dh \quad \text{and} \quad q = cg + dk.$$

Now let u and v be complex numbers that are not both zero. Then $fu+gv$ and $hu + kv$ are not both zero, because the matrix B is non-singular. The definition of the Möbius transformations μ_A, μ_B and μ_{AB} associated with the non-singular 2×2 matrices A, B and AB respectively ensures that

$$\begin{aligned}
\mu_A\left(\mu_B\left(\frac{u}{v}\right)\right) &= \mu_A\left(\frac{fu+gv}{hu+kv}\right) \\
&= \frac{a(fu+gv)+b(hu+kv)}{c(fu+gv)+d(hu+kv)} \\
&= \frac{mu+nv}{pu+qv} = \mu_{AB}\left(\frac{u}{v}\right).
\end{aligned}$$

The result follows. ∎

Corollary 1.8 *Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$, let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \textit{and} \quad C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and let μ_A and μ_C be the corresponding Möbius transformations, defined so that

$$\mu_A\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv} \quad \textit{and} \quad \mu_C(z) = \frac{du - bv}{-cu + av}$$

for all complex numbers u and v that are not both zero. Then the mapping $\mu_A \colon \mathbb{P}^1 \to \mathbb{P}^1$ is invertible, and its inverse is the Möbius transformation $\mu_C \colon \mathbb{P}^1 \to \mathbb{P}^1$.

Proof Let

$$M = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}.$$

Then $AC = CA = M$. It follows from Proposition 1.7 that

$$\mu_A \circ \mu_C = \mu_C \circ \mu_A = \mu_M = \mathrm{Id}_{\mathbb{P}^1},$$

where $\mathrm{Id}_{\mathbb{P}^1}$ denotes the identity map of the Riemann sphere. The result follows. ∎

Proposition 1.9 *Let a, b, c, d, f, g, h and k be complex numbers satisfying $ad \neq bc$ and $fk \neq gh$, and let μ_1 and μ_2 be the Möbius transformations of the Riemann sphere defined so that*

$$\mu_1(z) = \frac{az + b}{cz + d}, \quad \mu_2(z) = \frac{fz + g}{hz + k}$$

for all complex numbers with $cz + d \neq 0$ and $hz_2 + k \neq 0$. Then the Möbius transformations μ_1 and μ_2 coincide if and only if there exists some non-zero complex number m such that $f = ma$, $g = mb$, $h = mc$ and $k = md$.

Proof Clearly if there exists a complex number m with the stated properties then the Möbius transformations μ_1 and μ_2 coincide.

Conversely suppose that there is some Möbius transformation μ of the Riemann sphere with the property that

$$\mu(z) = \frac{az + b}{cz + d} = \frac{fz + g}{hz + k}$$

whenever $cz + d \neq 0$ and $hz + k \neq 0$.

First consider the case when $c = 0$. Then no real number is mapped by μ to the point ∞ of the Riemann sphere "at infinity" and therefore $h = 0$. But then $d \neq 0$, $k \neq 0$, $b/d = g/k$ and $a/d = f/k$. Therefore if we take $m = k/d$ in this case we find that $m \neq 0$, $f = ma$, $g = mb$, $h = mc$ and $k = md$. The existence of the required non-zero complex number m has therefore been verified in the case when $c = 0$.

Suppose then that $c \neq 0$. Then $h \neq 0$ and $\mu(-k/h) = \infty = \mu(-d/c)$, and therefore $k/h = d/c$. Let $m = h/c$. Then $k = md$. It then follows that

$$fz + g = (hz + k)\mu(z) = m(cz + d)\mu(z) = m(az + b)$$

for all complex numbers z distinct from $-d/c$, and therefore $f = ma$ and $g = mb$. The result follows. ∎

1.4 Straight Lines and Circles in the Complex Plane

We consider the forms of the equations that are commonly used to represent straight lines and circles in the complex plane.

Straight lines in the plane are represented with respect to standard Cartesian coordinates x and y by equations of the form $px + qy + h = 0$ where p, q and h are real numbers for which p and q are not both zero. If we represent the point (x, y) by the complex number $x + iy$, where $i = \sqrt{-1}$, then the equation of the line $px + qy + h = 0$ can be expressed, in the algebra of complex numbers, by the equation

$$2\mathrm{Re}[\overline{b}z] + h = 0,$$

where $b = \frac{1}{2}(p + iq)$. Moreover equations of this form, in which b is a non-zero complex number and h is a real number, determine straight lines in the complex plane.

Next we consider the form taken by the equation of a circle in the complex plane. If the centre of the circle is represented by the complex number m, and if the real number r represents the radius of the circle, where $r > 0$, then the circle consists of those complex numbers z that satisfy the equation $|z - m|^2 = r^2$. Expanding out, this equation can be presented in the form

$$|z|^2 - 2\mathrm{Re}[\overline{m}z] + |m|^2 - r^2 = 0.$$

It follows from this that, given an equation of the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

in which g and h are real numbers, and b is a complex number, that equation represents a circle in the complex plane if and only if $g \neq 0$ and $|b|^2 > gh$. (In cases where $|b|^2 = gh$ the equation is satisfied only at a single point; and if $|b|^2 < gh$ then the equation is not satisfied anywhere in the complex plane.)

We conclude from this discussion that straight lines and circles in the complex plane are those loci (or subsets) of the complex plane that can be specified by equations of the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

in which g and h are real numbers, b is a complex number, and $|b|^2 > gh$. The equation represents a circle if $g \neq 0$, but represents a straight line if $g = 0$.

Proposition 1.10 *Any Möbius transformation maps straight lines and circles in the complex plane to straight lines and circles.*

Proof The equation of a line or circle in the complex plane can be expressed in the form

$$g|z|^2 + 2\mathrm{Re}[\overline{b}z] + h = 0,$$

where g and h are real numbers, b is a complex number, and $|b|^2 > gh$. Moreover a locus of points in the complex plane satisfying an equation of this form is a circle if $g \neq 0$ and is a line if $g = 0$.

Let g and h be real constants, let b be a complex constant, and let $z = 1/w$, where $w \neq 0$ and w satisfies the equation

$$g|w|^2 + 2\mathrm{Re}[\overline{b}w] + h = 0,$$

Then

$$g|w|^2 + \overline{b}w + b\overline{w} + h = 0,$$

and therefore

$$\begin{aligned} g + \mathrm{Re}[bz] + h|z|^2 &= g + \overline{b}\overline{z} + bz + h|z|^2 \\ &= \frac{1}{|w|^2}\left(g|w|^2 + \overline{b}w + b\overline{w} + h\right) = 0. \end{aligned}$$

We deduce from this that the Möbius transformation that sends z to $1/z$ for all non-zero complex numbers z maps lines and circles to lines and circles.

Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation of the Riemann sphere. Then there exist complex numbers a, b, c and d satisfying $ad - bc \neq 0$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. The result is immediate when $c = 0$. We therefore suppose that $c \neq 0$. Then

$$\mu(z) = \frac{az + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c} \times \frac{1}{cz + d}$$

when $cz + d \neq 0$. The Möbius transformation μ is thus the composition of three maps that each send circles and straight lines to circles and straight lines, namely the maps

$$z \mapsto cz + d, \quad z \mapsto \frac{1}{z} \quad \text{and} \quad z \mapsto \frac{a}{c} - \frac{(ad - bc)z}{c}.$$

Thus the Möbius transformation μ must itself map circles and straight lines to circles and straight lines, as required. ∎

1.5 Cross Ratios

Let p_1, p_2, p_3 and p_4 be elements of the Riemann sphere, and, for $j = 1, 2, 3, 4$, let u_j, v_j, u'_j and v'_j be complex numbers that are such as to ensure that u_j and v_j are not both zero, u'_j and v'_j are not both zero and

$$p_j = \frac{u_j}{v_j} = \frac{u'_j}{v'_j}$$

for $j = 1, 2, 3, 4$. Then there exist non-zero complex numbers w_1, w_2, w_3 and w_4 that are such as to ensure that $u'_j = w_j u_j$ and $v'_j = w_j v_j$ for $j = 1, 2, 3, 4$ (see Lemma 1.3). Let complex numbers ρ, ρ', σ and σ' be defined so that

$$\begin{aligned} \rho &= (u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2), \\ \sigma &= (u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1), \\ \rho' &= (u'_1 v'_3 - u'_3 v'_1)(u'_2 v'_4 - u'_4 v'_2), \\ \sigma' &= (u'_2 v'_3 - u'_3 v'_2)(u'_1 v'_4 - u'_4 v'_1). \end{aligned}$$

Then $\rho' = w_1 w_2 w_3 w_4 \rho$ and Then $\sigma' = w_1 w_2 w_3 w_4 \sigma$. It follows that $\rho' = 0$ if and only if $\rho = 0$, $\sigma' = 0$ if and only if $\sigma = 0$, and $\dfrac{\rho}{\sigma} = \dfrac{\rho'}{\sigma'}$ in all cases where ρ and σ are not both zero.

Now $\rho = 0$ if and only if either $p_1 = p_3$ or $p_2 = p_4$. (This follows on applying Lemma 1.3.) Moreover $p_1 = p_3$ and $\sigma = 0$ if and only if either $p_1 = p_2 = p_3$ or $p_1 = p_3 = p_4$. Also $p_2 = p_4$ and $\sigma = 0$ if and only if either $p_2 = p_3 = p_4$ or $p_1 = p_2 = p_4$. If follows that ρ and σ are both equal to zero if and only if three of the elements p_1, p_2, p_3, p_4 coincide with one another.

We conclude that, in all cases where no three of the elements p_1, p_2, p_3 and p_4 of the Riemann sphere coincide with one another, there exists a well-defined element $(p_1, p_2; p_3, p_4)$ of the Riemann sphere that is determined so as to ensure that if u_j and v_j are complex numbers determined for $j = 1, 2, 3, 4$ so as to ensure that u_j and v_j are not both zero and $p_j = \dfrac{u_j}{v_j}$, then

$$(p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

This element $(p_1, p_2; p_3, p_4)$ of the Riemann sphere is referred to as the *cross-ratio* of p_1, p_2, p_3 and p_4.

Proposition 1.11 *Let p_1, p_2, p_3 and p_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let $q = (p_1, p_2; p_3, p_4)$. Then*

- $(p_1, p_2; p_3, p_4)$, $(p_2, p_1; p_4, p_3)$, $(p_3, p_4; p_1, p_2)$, $(p_4, p_3; p_2, p_1)$ *are all equal to q;*
- $(p_1, p_2; p_4, p_3)$, $(p_2, p_1; p_3, p_4)$, $(p_4, p_3; p_1, p_2)$, $(p_3, p_4; p_2, p_1)$ *are all equal to $\dfrac{1}{q}$.*
- $(p_1, p_3; p_2, p_4)$, $(p_3, p_1; p_4, p_2)$, $(p_2, p_4; p_1, p_3)$, $(p_4, p_2; p_3, p_1)$ *are all equal to $1 - q$;*
- $(p_1, p_4; p_2, p_3)$, $(p_4, p_1; p_3, p_2)$, $(p_2, p_3; p_1, p_4)$, $(p_3, p_2; p_4, p_1)$ *are all equal to $\dfrac{q-1}{q}$;*
- $(p_1, p_3; p_4, p_2)$, $(p_3, p_1; p_2, p_4)$, $(p_4, p_2; p_1, p_3)$, $(p_2, p_4; p_3, p_1)$ *are all equal to $\dfrac{1}{1-q}$;*
- $(p_1, p_4; p_3, p_2)$, $(p_4, p_1; p_2, p_3)$, $(p_3, p_2; p_1, p_4)$, $(p_2, p_3; p_4, p_1)$ *are all equal to $\dfrac{q}{q-1}$;*

Proof Let u_1, v_1, u_2, v_2, u_3, v_3, u_4 and v_4 be complex numbers with the properties that u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$ (where $u_j/v_j = \infty$ in cases where $u_j \neq 0$ and $v_j = 0$). Then

$$q = (p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

It follows directly that

$$(p_1, p_2; p_3, p_4), \quad (p_2, p_1; p_4, p_3), \quad (p_3, p_4; p_1, p_2) \text{ and } (p_4, p_3; p_2, p_1)$$

are all equal to q. Also

$$(p_1, p_2; p_4, p_3) = \frac{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)} = \frac{1}{q}.$$

Next we note that

$$(p_4, p_2; p_3, p_1) = \frac{(u_4 v_3 - u_3 v_4)(u_2 v_1 - u_1 v_2)}{(u_2 v_3 - u_3 v_2)(u_4 v_1 - u_1 v_4)}.$$

It follows that

$$\begin{aligned}
&1 - (p_4, p_2; p_3, p_1) \\
&\quad = \frac{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1) + (u_4 v_3 - u_3 v_4)(u_2 v_1 - u_1 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&\quad = \frac{u_1 u_2 v_3 v_4 - v_1 u_2 v_3 u_4 - u_1 v_2 u_3 v_4 + v_1 v_2 u_3 u_4}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&\qquad + \frac{v_1 u_2 v_3 u_4 - v_1 u_2 u_3 v_4 - u_1 v_2 v_3 u_4 + u_1 v_2 u_3 v_4}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&\quad = \frac{u_1 u_2 v_3 v_4 + v_1 v_2 u_3 u_4 - v_1 u_2 u_3 v_4 - u_1 v_2 v_3 u_4}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&\quad = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\
&\quad = q.
\end{aligned}$$

Consequently

$$(p_4, p_2; p_3, p_1) = 1 - q.$$

It then follows that

$$(p_4, p_2; p_1, p_3) = \frac{1}{1 - q}.$$

Furthermore

$$(p_3, p_2; p_1, p_4) = 1 - (p_4, p_2; p_1, p_3) = 1 - \frac{1}{1 - q} = \frac{q}{q - 1},$$

and therefore

$$(p_3, p_2; p_4, p_1) = \frac{q-1}{q}.$$

The remaining identities follow directly. ∎

Lemma 1.12 *Let z_1, z_2 z_3 and z_4 be distinct complex numbers. Then*

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

Proof This follows directly from the definition of cross ratios of quadruples of elements of the Riemann sphere on representing the complex number z_j as the fraction u_j/v_j with $u_j = z_j$ and $v_j = 1$ for $j = 1, 2, 3, 4$. ∎

Lemma 1.13 *Let z_1, z_2 and, z_3 be distinct complex numbers. Then*

$$(z_1, z_2; z_3, \infty) = \frac{z_1 - z_3}{z_2 - z_3}$$

Proof Let $u_1 = z_1$, $u_2 = z_2$, $u_3 = z_3$, $u_4 = 1$, $v_1 = v_2 = v_3 = 1$ and $v_4 = 0$. Then $z_j = u_j/v_j$ for $j = 1, 2, 3$ and $\infty = u_4/v_4$. It follows from the definition of cross-ratios that

$$(z_1, z_2; z_3, \infty) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} = \frac{z_1 - z_3}{z_2 - z_3},$$

as required. ∎

Lemma 1.14 *Let p_1, p_2, p_3, p_4 be a quadruple of points of the Riemann sphere satisfying the condition that no three of the points all coincide with one another. Then the following identities hold when two of the points coincide with one another:*

$$(p_1, p_2; p_3, p_4) = \infty \text{ whenever } p_2 = p_3 \text{ or } p_1 = p_4;$$

$$(p_1, p_2; p_3, p_4) = 0 \text{ whenever } p_1 = p_3 \text{ or } p_2 = p_4;$$

$$(p_1, p_2; p_3, p_4) = 1 \text{ whenever } p_1 = p_2 \text{ or } p_3 = p_4.$$

Proof Let complex numbers u_j and v_j be chosen for $j = 1, 2, 3, 4$ such that u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$. The definition of cross-ratios ensures that

$$(p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

Now, for distinct integers j and k between 1 and 4, $p_j = p_k$ if and only if $u_j v_k = u_k v_j$. Also there exists a non-zero complex number w for which $u_2 = w u_1$ and $v_2 = w v_1$ if and only if $p_1 = p_2$, and there exists a non-zero complex number w for which $u_4 = w u_3$ and $v_4 = w v_3$ if and only if $p_3 = p_4$. The required identities therefore follow directly. ∎

1.6 The Action of Möbius Transformations on the Riemann Sphere

Lemma 1.15 *Let p_1, p_2 and p_3 be distinct elements of the Riemann sphere, and let $\mu^*_{p_1,p_2,p_3}: \mathbb{P}^1 \to \mathbb{P}^1$ be the function mapping the Riemann sphere into itself defined such that*

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p)$$

for all elements p of the Riemann sphere. Then $\mu^*_{p_1,p_2,p_3}$ is Möbius transformation, and moreover $\mu^*_{p_1,p_2,p_3}(p_1) = \infty$, $\mu^*_{p_1,p_2,p_3}(p_2) = 0$ and $\mu^*_{p_1,p_2,p_3}(p_3) = 1$.

Proof Let $p_j = u_j/v_j$ for $j = 1, 2, 3$, where, for each of these values of j, the elements u_j and v_j are complex numbers that are not both zero. It then follows from the definition of cross-ratio that

$$\mu^*_{p_1,p_2,p_3}\left(\frac{u}{v}\right) = \frac{(u_1v_3 - u_3v_1)(u_2v - uv_2)}{(u_2v_3 - u_3v_2)(u_1v - uv_1)}.$$

Consequently $\mu^*_{p_1,p_2,p_3}$ is the Möbius transformation corresponding to the coefficient matrix

$$\begin{pmatrix} -(u_1v_3 - u_3v_1)v_2 & (u_1v_3 - u_3v_1)u_2 \\ -(u_2v_3 - u_3v_2)v_1 & (u_2v_3 - u_3v_2)u_1 \end{pmatrix}.$$

It then follows from Lemma 1.14 that $\mu^*_{p_1,p_2,p_3}(p_1) = \infty$, $\mu^*_{p_1,p_2,p_3}(p_2) = 0$ and $\mu^*_{p_1,p_2,p_3}(p_3) = 1$. as required. ∎

Proposition 1.16 *Let p_1, p_2, p_3 be distinct points of the Riemann sphere $\mathbb{P}^1$, and let q_1, q_2, q_3 also be distinct points of $\mathbb{P}^1$. Then there exists a unique Möbius transformation $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(p_j) = q_j$ for $j = 1, 2, 3$.*

Proof Let $\mu^*_{p_1,p_2,p_3}: \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu^*_{q_1,q_2,q_3}: \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformations of the Riemann sphere defined so that

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p) \quad \text{and} \quad \mu^*_{q_1,q_2,q_3}(p) = (q_1, q_2; q_3, p)$$

for all elements p of the Riemann sphere. Then

$$\begin{aligned} \mu^*_{p_1,p_2,p_3}(p_1) = \mu^*_{q_1,q_2,q_3}(q_1) &= \infty, \\ \mu^*_{p_1,p_2,p_3}(p_2) = \mu^*_{q_1,q_2,q_3}(q_2) &= 0, \\ \mu^*_{p_1,p_2,p_3}(p_3) = \mu^*_{q_1,q_2,q_3}(q_3) &= 1. \end{aligned}$$

It follows that $\mu(p_j) = q_j$ for $j = 1, 2, 3$, where $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ is the Möbius transformation of the Riemann sphere defined such that

$$\mu(p) = \mu^{*-1}_{q_1,q_2,q_3}(\mu^*_{p_1,p_2,p_3}(p))$$

for all elements p of the Riemann sphere.

Now let $\hat{\mu}: \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation of the Riemann sphere with the property that $\hat{\mu}(p_j) = q_j$ for $j = 1, 2, 3$, and let

$$\lambda(p) = \mu^*_{q_1,q_2,q_3}(\hat{\mu}(\mu^{*-1}_{p_1,p_2,p_3}(p)))$$

for all elements p of the Riemann sphere. Then

$$\begin{aligned} \lambda(\infty) &= \mu^*_{q_1,q_2,q_3}(\hat{\mu}(\mu^{*-1}_{p_1,p_2,p_3}(\infty))) = \mu^*_{q_1,q_2,q_3}(\hat{\mu}(p_1)) \\ &= \mu^*_{q_1,q_2,q_3}(q_1) = \infty, \end{aligned}$$

and similarly $\lambda(0) = 0$ and $\lambda(1) = 1$.

Now λ is a Möbius transformation. It follows that there exist complex coefficients a, b, c and d, where $ad - bc \neq 0$, such that

$$\lambda\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero. Then the identity $\lambda(\infty) = \infty$ implies that $c = 0$, the identity $\lambda(0) = 0$ implies that $b = 0$, and consequently the identity $\lambda(1) = 1$ implies that $a = d$. Consequently $\lambda(p) = p$ for all elements p of the Riemann sphere. It follows from this that

$$\hat{\mu}(p) = \mu^{*-1}_{q_1,q_2,q_3}(\mu^*_{p_1,p_2,p_3}(p)) = \mu(p),$$

for all elements p of the Riemann sphere. Thus the Möbius transformation μ is the unique Möbius transformation of the Riemann sphere that sends p_j to q_j for $j = 1, 2, 3$, as asserted. ∎

The following corollary follows immediately from Proposition 1.16).

Corollary 1.17 *Two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere*

Proposition 1.18 *Let p_1, p_2, p_3, p_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let q_1, q_2, q_3, q_4 also be distinct elements of $\mathbb{P}^1$. Then a necessary and sufficient condition for the existence of a Möbius transformation $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(p_j) = q_j$ for $j = 1, 2, 3, 4$ is that*

$$(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4).$$

Proof Let $\mu^*_{p_1,p_2,p_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu^*_{q_1,q_2,q_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformations of the Riemann sphere defined so that

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p) \quad \text{and} \quad \mu^*_{q_1,q_2,q_3}(p) = (q_1, q_2; q_3, p)$$

for all elements p of the Riemann sphere, and let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that

$$\mu(p) = \mu^{*-1}_{q_1,q_2,q_3}(\mu^*_{p_1,p_2,p_3}(p))$$

for all elements p of the Riemann sphere. Then (as shown in the proof of Proposition 1.16) the Möbius transformation μ is the unique Möbius transformation that satisfies $\mu(p_j) = q_j$ for $j = 1, 2, 3$. Now $\mu(p_4) = \mu(q_4)$ if and only if $\mu^*_{p_1,p_2,p_3}(p_4) = \mu^*_{q_1,q_2,q_3}(q_4)$, and this is the case if and only if

$$(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4).$$

The result follows. ∎

Proposition 1.19 *Four distinct complex numbers z_1, z_2, z_3 and z_4 lie on a single line or circle in the complex plane if and only if their cross-ratio $(z_1, z_2; z_3, z_4)$ is a real number.*

Proof Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that $\mu(p) = (z_1, z_2; z_3, p)$ for all $p \in \mathbb{P}^1$. Then $\mu(z_1) = \infty$, $\mu(z_2) = 0$ and $\mu(z_3) = 1$. Möbius transformations map lines and circles to lines and circles (Propostion 1.10). It follows that a complex number z distinct from z_1, z_2 and z_3 lies on the circle in the complex plane passing through the points z_1, z_2 and z_3 if and only if $\mu(z)$ lies on the unique line in the complex plane that passes through 0 and 1, in which case $\mu(z)$ is a real number. The result follows. ∎

1.7 Cross-Ratios and Angles

We recall some basic properties of the algebra of complex numbers. Any complex number z can be written in the form

$$z = |z| \, (\cos\theta + \sqrt{-1} \sin\theta)$$

where $|z|$ is the modulus of z and θ is the angle in radians, measured anticlockwise, between the positive real axis and the line segment whose endpoints are represented by the complex numbers 0 and z. Moreover

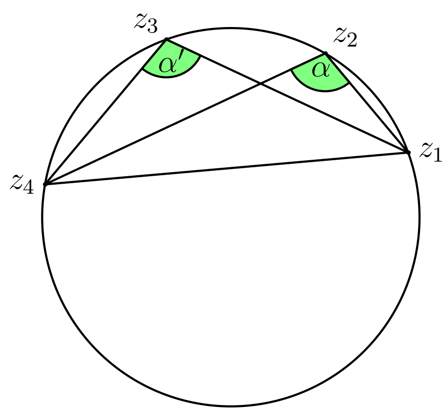
$$\frac{1}{\cos\alpha + \sqrt{-1}\,\sin\alpha} = \cos\alpha - \sqrt{-1}\,\sin\alpha$$

and

$$(\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\beta + \sqrt{-1}\,\sin\beta) = \cos(\alpha+\beta) + \sqrt{-1}\,\sin(\alpha+\beta)$$

for all real numbers α and β.

Proposition 1.20 *Let z_1, z_2, z_3 and z_4 be distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle. Then the angle between the lines joining z_2 to z_4 and z_1 is equal to the angle between the lines joining z_3 to z_4 and z_1.*



Proof Let α denote the angle between the lines joining z_2 to z_4 and z_1, and let α' be the angle between the lines joining z_3 to z_4 and z_1. We must show that $\alpha = \alpha'$. Now it follows from the standard properties of complex numbers that

$$\frac{z_1 - z_2}{z_4 - z_2} = \frac{|z_1 - z_2|}{|z_4 - z_2|}(\cos\alpha + \sqrt{-1}\sin\alpha),$$
$$\frac{z_1 - z_3}{z_4 - z_3} = \frac{|z_1 - z_3|}{|z_4 - z_3|}(\cos\alpha' + \sqrt{-1}\sin\alpha').$$

It now follows from the definition of cross-ratio that

$$(z_2, z_3; z_1, z_4) = \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)} = \frac{z_1 - z_2}{z_4 - z_2} \div \frac{z_1 - z_3}{z_4 - z_3}$$
$$= \frac{|z_1 - z_2|\,|z_4 - z_3|}{|z_1 - z_3|\,|z_4 - z_2|} \times \frac{\cos\alpha + \sqrt{-1}\,\sin\alpha}{\cos\alpha' + \sqrt{-1}\,\sin\alpha'}.$$

Now

$$\frac{1}{\cos\alpha' + \sqrt{-1}\sin\alpha'} = \cos\alpha' - \sqrt{-1}\sin\alpha',$$

and therefore

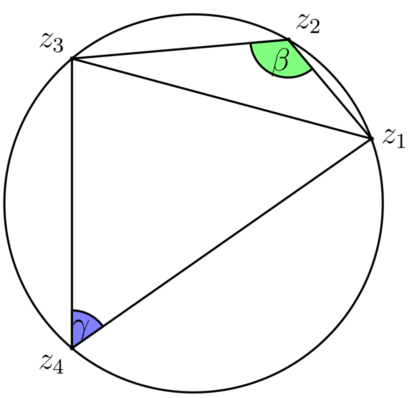
$$\begin{aligned}\frac{\cos\alpha + \sqrt{-1}\,\sin\alpha}{\cos\alpha' + \sqrt{-1}\,\sin\alpha'} &= (\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\alpha' - \sqrt{-1}\,\sin\alpha') \\ &= \cos(\alpha - \alpha') + \sqrt{-1}\,\sin(\alpha - \alpha').\end{aligned}$$

Consequently

$$(z_2, z_3; z_1, z_4) = |(z_2, z_3; z_1, z_4)|(\cos(\alpha - \alpha') + \sqrt{-1}\,\sin(\alpha - \alpha')).$$

But the cross ratio $(z_2, z_3; z_1, z_4)$ is a real number, because the complex numbers z_1, z_2, z_3 and z_4 lie on a circle (see Proposition 1.19), and consequently $\alpha - \alpha'$ must be an integer multiple of π. Also $0 < \alpha < \pi$ and $0 < \alpha' < \pi$, and therefore $-\pi < \alpha - \alpha' < \pi$. It follows that $\alpha - \alpha' = 0$, and thus $\alpha = \alpha'$, as required. ∎

Proposition 1.21 *Let z_1, z_2, z_3 and z_4 be distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle, let β be the angle between the lines joining z_2 to z_3 and z_1, and let γ be the angle between the lines joining z_4 to z_1 and z_3. Then $\beta + \gamma = \pi$.*



Proof It follows from the standard properties of complex numbers that

$$\begin{aligned}\frac{z_1 - z_2}{z_3 - z_2} &= \frac{|z_1 - z_2|}{|z_3 - z_2|}(\cos\beta + \sqrt{-1}\sin\beta), \\ \frac{z_3 - z_4}{z_1 - z_4} &= \frac{|z_3 - z_4|}{|z_1 - z_4|}(\cos\gamma + \sqrt{-1}\sin\gamma).\end{aligned}$$

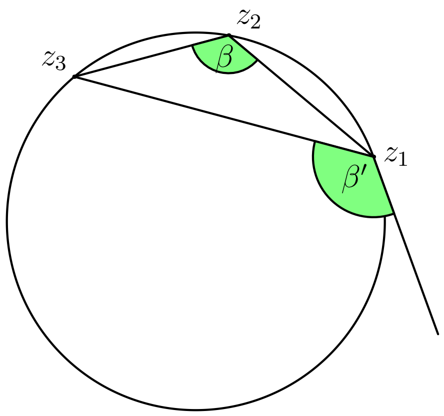
It now follows from the definition of cross-ratio that

$$(z_2, z_4; z_1, z_3)$$

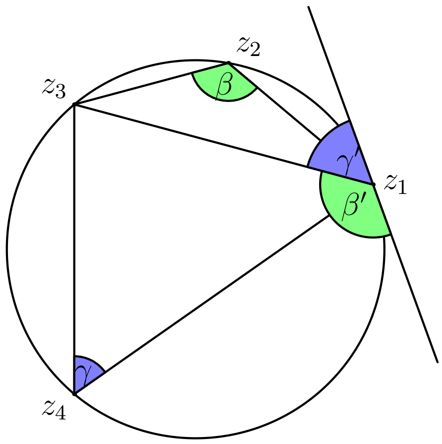
$$
\begin{aligned}
&= \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{z_1 - z_2}{z_3 - z_2} \times \frac{z_3 - z_4}{z_1 - z_4} \\
&= \frac{|z_1 - z_2|\,|z_3 - z_4|}{|z_1 - z_4|\,|z_3 - z_2|} (\cos\beta + \sqrt{-1}\,\sin\beta)(\cos\gamma + \sqrt{-1}\,\sin\gamma) \\
&= |(z_2, z_4; z_1, z_3)|\,(\cos(\beta + \gamma) + \sqrt{-1}\,\sin(\beta + \gamma)).
\end{aligned}
$$

But the cross ratio $(z_2, z_4; z_1, z_3)$ is a real number, because the complex numbers z_1, z_2, z_4 and z_3 lie on a circle (see Proposition 1.19), and consequently $\beta + \gamma$ must be an integer multiple of π. Also $0 < \beta < \pi$ and $0 < \gamma < \pi$, and therefore $0 < \beta + \gamma < 2\pi$. It follows that $\beta + \gamma = \pi$, as required. ∎

Proposition 1.22 *Let z_1, z_2 and z_3 distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle. Then the angle between the lines joining z_2 to z_3 and z_1 is equal to the angle between the line joining z_3 to z_1 and the ray tangent to the circle at z_1 that is directed so that the point z_2 and the tangent ray lie on opposite sides of the line that passes through the points z_1 and z_3.*

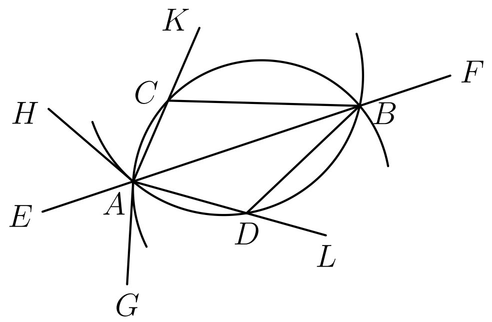


Proof Let β denote the angle between the lines joining z_2 to z_3 and z_1. Also let a point z_4 be taken on the circle so that z_1, z_2, z_3 and z_4 are distinct and moreover the points z_2 and z_4 lie on opposite sides of the line that passes through z_1 and z_3, and let γ denote the angle between the lines joining z_4 to z_1 and z_3. It follows from Proposition 1.21 that $\beta + \gamma = \pi$.



Now suppose that the point z_4 moves along the circle towards the point z_1. As the point z_4 approaches z_1 the direction of the chord of the circle from z_4 to z_1 approaches the direction of the ray tangent to the circle at z_1 that points into the side of the line through z_1 and z_3 in which z_2 lies. But the angle between the rays joining z_4 to z_1 and z_3 remains constant as z_4 approaches z_1. Consequently the angle γ' between the tangent ray at z_1 pointing into the side of the chord joining z_1 to z_3 and that chord itself is equal to the angle γ. The angle β' between the chord joining z_1 and z_3 and the tangent ray pointing into the side of that chord opposite to z_2 is then the supplement of the angle γ', where $\gamma' = \gamma$, and therefore $\beta' + \gamma = \pi = \beta + \gamma$. Consequently $\beta' = \beta$. The result follows. ∎

Proposition 1.23 *Let a geometrical configuration be as depicted in the accompanying figure. Thus let ACB and ADB be circular arcs that cut at the points A and B. Let the line joining points A and B be produced beyond A and B to E and F respectively. Let AG and AH be tangent to the circular arcs BCA and BDA respectively at A, where C and H lie on one side of AB and D and G lie on the other. Also let the lines AC and AD be produced to K and L respectively. Then the angle GAH is the sum of the angles KCB and LDB.*

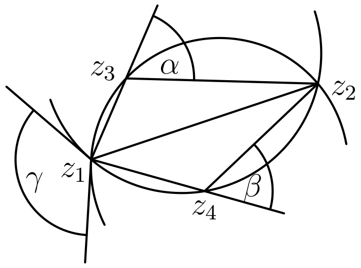


Proof Applying results of previous propositions, together with standard geometrical results, we find that

$$\begin{array}{rcll} \angle GAB &=& \angle ACB & \text{(Proposition 1.22)} \\ \Rightarrow \angle EAG &=& \angle KCB & \text{(supplementary angles)} \\ \angle HAB &=& \angle ADB & \text{(Proposition 1.22)} \\ \Rightarrow \angle EAH &=& \angle LDB & \text{(supplementary angles)} \\ \Rightarrow \angle GAH &=& \angle EAG + \angle EAH & \\ &=& \angle KCB + \angle LDB, & \end{array}$$

as required. ∎

Proposition 1.24 *Let two circles in the complex plane intersect at points represented by complex numbers z_1 and z_2, and let points represented by complex numbers z_3 and z_4 be taken on arcs of the respective circles joining z_1 and z_2 so that the point representing z_3 lies on the left hand side of the directed line from z_1 and z_2 and the point represented by the point z_4 lies on the right hand side of that line (as depicted in the accompanying figure).*



Then

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}\,(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent lines to the two circles at the intersection point represented by the complex number z_1.

Proof The configuration of the points z_1, z_2, z_3 and z_4 ensures that direction of the line from z_1 to z_3 is transformed into the direction of the line from z_3 to z_2 by rotation clockwise through an angle α less than two right angles. Similarly the direction of the line from z_1 to z_4 is transformed into the direction of the line from z_4 to z_2 by rotation anticlockwise through an angle β less than two right angles. Basic properties of complex numbers therefore ensure that

$$\begin{aligned} \frac{z_2 - z_3}{z_3 - z_1} &= \frac{|z_2 - z_3|}{|z_3 - z_1|}(\cos\alpha - \sqrt{-1}\,\sin\alpha). \\ \frac{z_2 - z_4}{z_4 - z_1} &= \frac{|z_2 - z_4|}{|z_4 - z_1|}(\cos\beta + \sqrt{-1}\,\sin\beta). \end{aligned}$$

Now

$$\begin{aligned} &\frac{\cos\beta + \sqrt{-1}\,\sin\beta}{\cos\alpha - \sqrt{-1}\,\sin\alpha} \\ &\quad = (\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\beta + \sqrt{-1}\,\sin\beta) \\ &\quad = \cos(\alpha + \beta) + \sqrt{-1}\,\sin(\alpha + \beta). \end{aligned}$$

Moreover the geometry of the configuration ensures that $\alpha + \beta = \gamma$ (Proposition 1.23). Thus

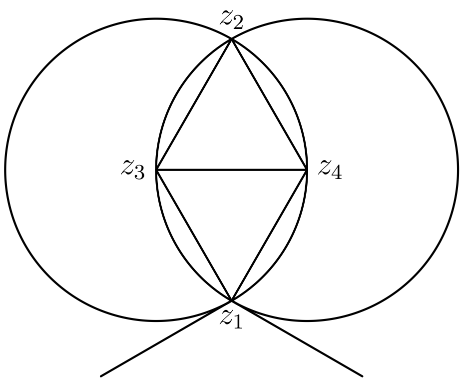
$$\begin{aligned} &\frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ &= \frac{|z_2 - z_4|\,|z_3 - z_1|}{|z_4 - z_1||z_2 - z_3|}(\cos\gamma + \sqrt{-1}\sin\gamma). \end{aligned}$$

But

$$\frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} = (z_1, z_2; z_3, z_4).$$

The result follows. ∎

Example The circles in the complex plane of radius 2 centred on -1 and 1 intersect at the points $\pm\sqrt{3}\,i$, where $i = \sqrt{-1}$. In this situation, take $z_1 = -\sqrt{3}\,i$, $z_2 = \sqrt{3}\,i$, $z_3 = -1$ and $z_4 = 1$. Then



$$\begin{aligned} (z_1, z_2; z_3, z_4) &= \frac{(-1 + \sqrt{3}\,i)(1 - \sqrt{3}\,i)}{(-1 - \sqrt{3}\,i)(1 + \sqrt{3}\,i)} = \frac{2 + 2\sqrt{3}\,i}{2 - 2\sqrt{3}i} \\ &= \frac{(2 + 2\sqrt{3}\,i)^2}{(2 - 2\sqrt{3}i)(2 + 2\sqrt{3}i)} \\ &= \frac{1}{2}(-1 + \sqrt{3}\,i) \end{aligned}$$

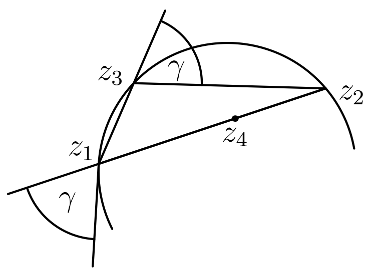
It follows that $(z_1, z_2; z_3, z_4) = \cos\gamma + \sqrt{-1}\,\sin\gamma$, where $\gamma = \frac{2}{3}\pi$. Thus the angle between the tangent lines to the circles at the intersection point z_1 is thus $\frac{4}{3}$ of a right angle. This is what one would expect from the basic geometry of the configuration, given that the triangle with vertices z_1, z_3 and z_4 is equilateral and the tangent lines to the circles are perpendicular to the lines joining the point of intersection to the centres of those circles.

Proposition 1.25 *Let z_1 and z_2 be complex numbers representing the end-points of a circular arc in the complex plane. Also, in the case where the circular arc lies on the left hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the circular arc and the straight line segment respectively, and, in the case where the circular arc lies on the right hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the straight line segment and the the circular arc respectively. Then*

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent line to the circle at the intersection point represented by the complex number z_1 and the line obtained by producing the chord joining z_2 and z_1 beyond z_1.

Proof We consider the configuration in which the circular arc lies on the left hand side of the directed line from z_1 to z_2. In that case the configuration is as depicted in the accompanying figure. In this configuration the angle made



at z_3 by the lines from z_1 and z_2 is equal to the angle between the chord from z_1 to z_2 and the depicted tangent line. The complements of those angles are then also equal to one another; these equal complements have been labelled γ in the figure.

Also the direction of the line from z_3 to z_2 is obtained from the direction of the line from z_1 to z_3 by rotation clockwise through an angle γ less than two right angles. It follows that

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|}(\cos\gamma - \sqrt{-1}\,\sin\gamma).$$

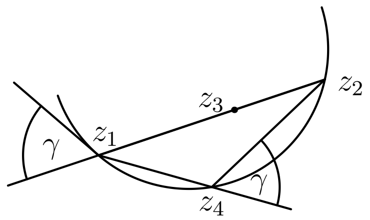
Also the direction of $z_2 - z_4$ is the same as that of $z_4 - z_1$, and therefore

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|}.$$

It follows that

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\ &= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ &= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma).\end{aligned}$$

We consider now the case in which the circular arc from z_1 to z_2 lies on the right hand side of the directed line from z_1 to z_2. In this case the complex numbers z_3 and z_4 represent points between z_1 and z_2 on the line and the circular arc respectively, as depicted in the following figure.



In this configuration, the angle sought is the angle γ, which in this case is equal both to the angle between the depicted tangent line to the circle at z_1 and the line that produces the chord joining z_2 to z_1 beyond z_1. Moreover, in this case

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma)$$

and

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|}.$$

It follows in this case also that

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\ &= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ &= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma).\end{aligned}$$

This completes the proof. ∎

Proposition 1.26 *Let two lines in the complex plane intersect at at point represented by the complex number* z_1*, and let points represented by* z_3 *and* z_4

be taken distinct from z_1, one on each of the two lines, where these points are labelled so that the direction of $z_3 - z_1$ is obtained from the direction of $z_4 - z_1$ by rotation anticlockwise through an angle γ less than two right angles. Then

$$(z_1, \infty; z_3, z_4) = \frac{|z_3 - z_1|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Proof The cross-ratio in this situation is defined so that

$$(z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1}.$$

Furthermore

$$\frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_3 - z_1|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

The result follows directly. ∎

Lines in the complex plane correspond to circles on the Riemann sphere that pass through the point at infinity. With that in mind, it can seen that Propositions 1.24, 1.25 and 1.26 conform to a common pattern, and show that, where two curves intersect at a point, each of those curves being either a circle or a straight line, the angle between the tangent lines to those curves at the point of intersection may be expressed in terms of the argument of an appropriate cross-ratio.

Indeed, to determine the angle the tangent lines to two circles on the Riemann sphere at a point p_1 where they intersect, one can determine the other point of intersection p_2, a point p_3 on one circular arc between p_1 to p_2, and a point p_4 on the other circular arc between p_1 and p_2. A positive real number R and a real number γ satisfying $-\pi < \gamma < \pi$ can then be determined so that

$$(p_1, p_2; p_3, p_4) = R(\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Then the angle between the tangent lines to those circles at the point p_1 of intersection, measured in radians, is then the absolute value $|\gamma|$ of γ.

Proposition 1.27 *Möbius transformations of the Riemann sphere $\mathbb{P}^1$ are angle-preserving. Thus if two circles on the Riemann sphere intersect at a point p of the Riemann sphere, and if a Möbius transformation μ maps p to a point q of the Riemann sphere, then the angle between the tangent lines to the original circles at the point p is equal to the angle between the tangent lines to the corresponding circles at the point q, the corresponding circles being the images of the original circles under the Möbius transformation.*

Proof The angle between the tangent lines to the original circles at p is determined by the value of a cross ratio of the form $(p_1, p_2; p_3, p_4)$, where p_1 and p_2 are the points of intersection of the original circles, and p_3 and p_4 lie on the circular arcs joining p_1 to p_2, with p_4 on the right hand side as the circle through p_3 is traversed in the direction from p_1 through p_3 to p_2. The angle between the tangent lines to the corresponding circles at q is determined in the analogous fashion by the value of the cross ratio $(q_1, q_2; q_3, q_4)$, where q_j is the image of p_j under the Möbius transformation sending the original circles to the corresponding circles. Proposition 1.18 ensures that $(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4)$. The result follows. ∎

1.8 The Orientation-Preserving Property of Möbius Transformations

A subset X of the complex plane $\mathbb{C}$ is said to be *open* if, given any complex number w belonging to X, there exists an open disk in the complex plane of sufficiently small radius centred on w that is wholly contained within the set X.

Definition An invertible function $\varphi: X \to Y$ between open subsets X and Y of the complex plane is said to be *orientation-preserving* if, given any point w of X, paths that traverse circles of sufficiently small radius centred on w once in the anticlockwise direction are mapped by φ to paths that wind around $\varphi(w)$ once in the anticlockwise direction.

Definition An invertible function $\varphi: X \to Y$ between open subsets X and Y of the complex plane is said to be *orientation-reversing* if, given any point w of X, paths that traverse circles of sufficiently small radius centred on w once in the anticlockwise direction are mapped by φ to paths that wind around $\varphi(w)$ once in the clockwise direction.

The transformation of the complex plane that maps each complex number to its complex conjugate is an example of an orientation-reversing transformation of the complex plane.

The composition of two orientation-preserving transformations between open subsets of the complex plane is orientation-preserving, as is the composition of two orientation-reversing transformations between such subsets. A transformation obtained on composing an orientation-preserving transformation with an orientation-reversing transformation is orientation-reversing, as is a transformation obtained on composing an orientation-reversing transformation with an orientation-preserving transformation.

Proposition 1.28 *A Möbius transformation of the Riemann sphere is orientation-preserving over the open subset of the complex plane consisting of those complex numbers that are not mapped to the element ∞ of the Riemann sphere.*

Proof Given complex numbers a and b, where $a \neq 0$, let $\tau_{a,b}$ denote the Möbius transformation of the Riemann sphere that maps ∞ to ∞ and maps each complex number z to $az + b$. Also let κ denote the Möbius transformation of the Riemann sphere that interchanges 0 and ∞ and maps z to $1/z$ for all non-zero complex numbers z. Then any Möbius transformation of the Riemann sphere can be expressed as a composition of Möbius transformations that are either of the form $\tau_{a,b}$ for appropriate coefficients a and b or else coincide with the Möbius transformation κ. (See the proof of Proposition 1.10.) It is not difficult to see that the transformations $\tau_{a,b}$ restrict to orientation-preserving transformations of the complex plane. The required result therefore follows from the observation that compositions of orientation-preserving transformations as orientation-preserving, once we establish that the Möbius transformation κ, when restricted to the non-zero complex numbers, is also an orientation-preserving transformation.

Consider a circle of radius s in the complex plane centred on 1, where $s < 1$. If that circle is traversed in the anticlockwise direction, starting at $1 + s$ and passing successively through $1 + s\sqrt{-1}$, $1 - s$ and $1 - s\sqrt{-1}$ before returning to $1 + s$, then then that path is mapped by the Möbius transformation κ to a path traversing a circle surrounding 1 and passing successively through the points

$$\frac{1}{1+s}, \quad \frac{1-s\sqrt{-1}}{1+s^2}, \quad \frac{1}{1-s}, \quad \frac{1+s\sqrt{-1}}{1+s^2}, \quad \frac{1}{1+s}.$$

This latter path is traversed in an anticlockwise direction. Thus if a circle centred on 1 of sufficiently small radius is traversed in an anticlockwise direction, then its image under the Möbius transformation κ will also be traversed in an anticlockwise direction. A path traversing a sufficiently small circles centred on any non-zero complex number w in the anticlockwise direction will then be mapped to a path traversing a circle centred on w^{-1} in an anticlockwise direction, because κ is equal to the composition of the successive orientation-preserving transformations $z \mapsto w^{-1}z$, $z \mapsto z^{-1}$ and $z \mapsto w^{-1}z$. Consequently κ restricts to an orientation-preserving transformation defined over the set of non-zero complex numbers. We can therefore conclude that any Möbius transformation of the Riemann sphere is indeed orientation-preserving when restricted to the open subset of the complex plane consisting of those complex numbers that are not mapped to the element ∞ of the Riemann sphere, as required. ∎

2 The Disk Model of the Hyperbolic Plane

2.1 Inversion of the Riemann Sphere in the Unit Circle

Let D denote the open unit disk in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let S denote the unit circle in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$S = \{z \in \mathbb{C} : |z| = 1\}$$

We define the *inversion* Ω of the Riemann sphere in the circle S bounding the open unit disk D to be the transformation of the Riemann sphere defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Then $\Omega(z) = z$ for all $z \in S$, and the composition $\Omega \circ \Omega$ of the inversion Ω with itself is the identity transformation of the Riemann sphere. Moreover Ω maps the open unit disk D into the region of the Riemann sphere that lies outside the unit circle S.

The transformation $\Omega\colon \mathbb{P}^1 \to \mathbb{P}^1$ is characterized by the property that

$$\Omega\left(\frac{u}{v}\right) = \frac{\overline{v}}{\overline{u}}$$

for all complex numbers v and w that are not both zero.

Lemma 2.1 *Let μ be a Möbius transformation of the Riemann sphere, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Also let a, b, c and d be complex coefficients determined so that*

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then $\Omega \circ \mu \circ \Omega$ is also a Möbius transformation, and moreover

$$\Omega(\mu(\Omega(z))) = \frac{\overline{d}z + \overline{c}}{\overline{b}z + \overline{a}}$$

for all complex numbers $z \in \mathbb{C}$ for which $\overline{b}z + \overline{a} \neq 0$.

Proof The definition of Möbius transformations and of the inversion Ω of the Riemann sphere in the unit circle ensure that

$$\mu\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv} \quad \text{and} \quad \Omega\left(\frac{u}{v}\right) = \frac{\overline{v}}{\overline{u}}$$

for all complex numbers u and v that are not both zero. Consequently

$$\Omega\left(\mu\left(\Omega\left(\frac{u}{v}\right)\right)\right) = \Omega\left(\mu\left(\frac{\overline{v}}{\overline{u}}\right)\right) = \Omega\left(\frac{a\,\overline{v} + b\,\overline{u}}{c\,\overline{v} + d\,\overline{u}}\right) = \frac{\overline{d}\,u + \overline{c}\,v}{\overline{b}\,u + \overline{a}\,v}$$

for all complex numbers u and v that are not both zero. The result follows. ∎

Proposition 2.2 *Let μ be a Möbius transformation of the Riemann sphere, let D be the open unit disk in the complex plane, where*

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let Ω be the inversion of the Riemann sphere in the unit circle that is defined so that

$$\Omega(0) = \infty, \quad \Omega(\infty) = 0 \quad \textit{and} \quad \Omega(z) = \frac{1}{\overline{z}} \textit{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Then the Möbius transformation μ maps the unit disk D onto itself if and only if both of the following two conditions are satisfied:

(i) $\Omega \circ \mu = \mu \circ \Omega$*;*

(ii) there exists at least one $z \in D$ for which $\mu(z) \in D$.

Proof First suppose that the Möbius transformation μ maps the unit disk D onto itself. Let z be a complex number satisfying $|z| = 1$. If it were the case that $|\mu(z)| < 1$ then there would exist some complex number w for which $|w| < 1$ and $\mu(w) = \mu(z)$, because μ maps the open unit disk onto itself. But this is not possible because all Möbius transformations are invertible. Next we note that if it were the case that $|\mu(z)| > 1$ then, for real numbers t that are less than 1 but sufficiently close to 1, it would follow that $|tz| < 1$ but $|\mu(tz)| > 1$, contradicting the requirement that the Möbius transformation μ map the open unit disk onto itself. Consequently $|\mu(z)| = 1$. We conclude therefore that the Möbius transformation μ maps the unit circle bounding the open unit disk into itself. The same is true of the inverse of μ. Consequently the Möbius transformation μ must map the unit circle onto itself.

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1). Now $\Omega(z) = z$ and $|\mu(z)| = 1$ for all complex numbers z

satisfying $|z| = 1$. It follows that $\hat{\mu}(z) = \mu(z)$ for all complex numbers z satisfying $|z| = 1$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Corollary 1.17). Consequently $\hat{\mu} = \mu$, and therefore $\Omega \circ \mu = \mu \circ \Omega$. It now follows directly that any Möbius transformation that maps the unit disk D onto itself must satisfy conditions (i) and (ii) in the statement of the proposition.

Conversely, suppose that Möbius transformation μ of the Riemann sphere satisfies conditions (i) and (ii) in the statement of the proposition. Then $\Omega \circ \mu = \mu \circ \Omega$. Let z be a complex number satisfying $|z| \neq 1$. Then $\Omega(z) \neq z$. It follows that $\mu(\Omega(z)) \neq \mu(z)$, because Möbius transformations are invertible transformations of the Riemann sphere, and therefore $\Omega(\mu(z)) \neq \mu(z)$, from which it follows that $|\mu(z)| \neq 1$. Consequently no complex number belonging to the open unit disk D is mapped by the Möbius transformation D to a point that lies on the unit circle. It follows that if one endpoint of a straight line segment or circular arc contained in the open disk D is mapped by μ into D, then the same must be true of the other endpoint of that straight line segment or circular arc.

Now the complex numbers belonging to the unit disk D can be joined to one another by straight line segments. Moreover condition (ii) in the statement of the proposition ensures that at least one complex number belonging to the unit disk D is mapped by the Möbius transformation μ into the unit disk D. Consequently the unit disk is mapped into itself by the Möbius transformation μ.

Moreover if the Möbius transformation μ has the property that $\Omega \circ \mu = \mu \circ \Omega$ then

$$\Omega \circ \mu^{-1} = \mu^{-1} \circ \mu \circ \Omega \circ \mu^{-1} = \mu^{-1} \circ \Omega \circ \mu \circ \mu^{-1} = \mu^{-1} \circ \Omega,$$

and consequently the inverse μ^{-1} of the Möbius transformation μ also satisfies (i) and (ii) in the statement of the proposition, and therefore maps the open unit disk D into itself. It follows that if the Möbius transformation μ satisfies conditions (i) and (ii) then it must map the open unit disk D onto itself, as required. ∎

Corollary 2.3 *Let μ be a Möbius transformation of the Riemann sphere, and let S be the unit circle consisting of all complex numbers z for which $|z| = 1$. Suppose that $\mu(S) \subset S$ and that $|\mu(0)| < 1$. Then the Möbius transformation μ maps the open unit disk onto itself. Moreover $\Omega \circ \mu = \mu \circ \Omega$, where Ω is the inversion of the Riemann sphere in the unit circle S, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.*

Proof Let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1), and moreover $\hat{\mu}(z) = \mu(z)$ for all $z \in S$, because $\mu(S) \subset S$ and $\Omega(z) = z$ for all $z \in S$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Corollary 1.17). It follows that $\hat{\mu} = \mu$, and therefore $\Omega \circ \mu = \mu \circ \Omega$. The required result now follows on applying Proposition 2.2. ∎

Lemma 2.4 *Given distinct complex numbers z_1 and z_2, where $|z_1| = |z_2| = 1$, there exists a Möbius transformation μ of the Riemann sphere mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$.*

Proof Choose a complex number z_3 distinct from z_1 and z_2 for which $|z_3| = 1$. Then there exists a unique Möbius transformation μ_1 with the properties that $\mu_1(z_1) = -1$, $\mu_1(z_2) = 1$ and $\mu_1(z_3) = i$. Möbius transformations map circles to circles, and, given any three distinct complex numbers that are not collinear, there exists exactly one circle in the complex plane passing through all three of these complex numbers. Consequently the Möbius transformation μ_1 must map the unit circle onto itself. If $|\mu_1(0)| < 0$ let the Möbius transformation μ be identical to μ_1; if $|\mu_1(0)| > 1$ or $\mu_1(0) = \infty$ let the Möbius transformation μ be defined so that $\mu(z) = 1/\mu_1(z)$ for all complex numbers z for which $\mu_1(z) \neq 0$. Then μ maps the unit circle onto itself, $\mu(z_1) = -1$, $\mu(z_2) = 1$ and $|\mu(0)| < 1$. Then $\mu(D)$ must map the open unit disk onto itself (see Corollary 2.3). The Möbius transformation μ then has the required properties. ∎

Proposition 2.5 *Let a and b be complex numbers satisfying $|b| < |a|$, and let μ be the Möbius transformation of the Riemann sphere defined so that*

$$\mu(z) = \frac{az + b}{\overline{b}\, z + \overline{a}} \quad \textit{whenever } \overline{b}\, z + \overline{a} \neq 0,$$

$\mu(-\overline{a}/\overline{b}) = \infty$ and $\mu(\infty) = a/\overline{b}$ in cases where $b \neq 0$ and $\mu(\infty) = \infty$ in cases where $b = 0$. Then $|\mu(z)| < 1$ whenever $|z| < 1$, $|\mu(z)| = 1$ whenever $|z| = 1$, and $|\mu(z)| > 1$ whenever $|z| > 1$ and $\overline{b}z + \overline{a} \neq 0$. Moreover the Möbius transformation μ maps the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself.

Proof Calculating, we find that

$$\begin{aligned} |\overline{b}z + \overline{a}|^2 - |az + b|^2 &= (\overline{b}z + \overline{a})(b\overline{z} + a) - (az + b)(\overline{a}\,\overline{z} + \overline{b}) \\ &= |b|^2|z|^2 + |a|^2 + a\,\overline{b}z + \overline{a}b\,\overline{z} \\ &\qquad - |a|^2|z|^2 - |b|^2 - a\,\overline{b}z - \overline{a}b\,\overline{z} \\ &= (|a|^2 - |b|^2)(1 - |z|^2). \end{aligned}$$

Consequently $|\mu(z)| < 1$ whenever $|z| < 1$, $|\mu(z)| = 1$ whenever $|z| = 1$ and $|\mu(z)| > 1$ whenever $|z| > 1$ and $\overline{b}z + \overline{a} \neq 0$.

Now the inverse μ^{-1} of the Möbius transformation μ is characterized by the property that

$$\mu^{-1}(z) = \frac{\overline{a}z - b}{-\overline{b}z + a}$$

for all complex numbers z for which $-\overline{b}z + a \neq 0$ (see Corollary 1.8). Because the coefficients of this Möbius transformation μ^{-1} have properties analogous to those of the Möbius transformation μ, we can conclude that μ^{-1} maps the open unit disk into itself, and therefore μ maps the open unit disk onto itself, as required. ∎

Corollary 2.6 *Let w be a complex number satisfying $|w| < 1$, and let μ_w be the Möbius transformation of the Riemann sphere that is defined so that $\mu_w(-1/\overline{w}) = \infty$, $\mu(\infty) = 1/\overline{w}$ and*

$$\mu_w(z) = \frac{z + w}{1 + \overline{w}\,z}$$

for all complex numbers z distinct from $-1/\overline{w}$. Then the Möbius transformation μ_w maps the open unit disk onto itself. Moreover

$$\mu_w(tw) = \frac{t + 1}{1 + |w|^2 t}\, w$$

for all real numbers t distinct from $-1/|w|^2$, and consequently the diameter of the unit circle passing through 0 and w is mapped onto itself by the Möbius transformation μ_w. In particular $\mu_w(0) = w$ and $\mu_w(-w) = 0$.

Proposition 2.7 *Let μ be a Möbius transformation of the Riemann sphere that maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ into itself and satisfies the condition $|\mu(0)| < 1$. Then there exist complex numbers a and b, where $|b| < |a|$, such that*

$$\mu(z) = \frac{az + b}{\overline{b}z + \overline{a}} \quad \textit{for all } z \in \mathbb{C} \textit{ for which } \overline{a}z + \overline{b} \neq 0.$$

Proof The Möbius transformation μ maps the unit circle into itself, and moreover $|\mu(0)| < 1$. It follows from Corollary 2.3 that $\Omega \circ \mu = \mu \circ \Omega$, where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Consequently $\mu = \Omega \circ \Omega \circ \mu = \Omega \circ \mu \circ \Omega$ because the composition of the inversion Ω with itself is the identity transformation of the Riemann sphere. Let a_0, b_0, c_0 and d_0 be complex coefficients determined so that

$$\mu(z) = \frac{a_0 z + b_0}{c_0 z + d_0} \quad \text{whenever } c_0 z + d_0 \neq 0.$$

Then the identity $\mu = \Omega \circ \mu \circ \Omega$ ensures that

$$\frac{a_0 z + b_0}{c_0 z + d_0} = \frac{\overline{d}_0 z + \overline{c}_0}{\overline{b}_0 z + \overline{a}_0}$$

for all complex numbers z for which $a_0 z + b_0 \neq 0$, $\overline{a}_0 + \overline{b}_0 z \neq 0$, $c_0 z + d_0 \neq 0$, and $\overline{c}_0 + \overline{d}_0 z \neq 0$ (see Lemma 2.1). Consequently there exists some non-zero complex number ω with the property that $\overline{a}_0 = \omega d_0$, $\overline{b}_0 = \omega c_0$, $\overline{c}_0 = \omega b_0$ and $\overline{d}_0 = \omega a_0$ (see Proposition 1.9). It then follows that

$$\overline{a}_0\, \overline{d}_0 = \omega^2 a_0 d_0.$$

But

$$|\overline{a}_0\, \overline{d}_0| = |a_0 d_0|.$$

It follows that $|\omega^2| = 1$, and therefore $|\omega| = 1$. Accordingly a real number θ can be found so that

$$\omega = \cos 2\theta + \sqrt{-1}\, \sin 2\theta.$$

Let

$$\eta = \cos\theta + \sqrt{-1}\, \sin\theta.$$

It then follows from De Moivre's Theorem that $\eta^2 = \omega$. Now $\overline{\eta}^2\, \eta^2 = |\eta|^4 = 1$. It follows that $\overline{\eta}^2 \omega = 1$. Let $a = \eta a_0$ and $b = \eta b_0$, $c = \eta c_0$ and $d = \eta d_0$. Then

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{whenever } cz + d \neq 0.$$

Also $a_0 = \overline{\eta} a$, $b_0 = \overline{\eta} b$, $c_0 = \overline{\eta} c$ and $d_0 = \overline{\eta} d$. Consequently

$$\overline{d} = \overline{\eta}\, \overline{d}_0 = \overline{\eta} \omega a_0 = \overline{\eta}^2 \omega a = a$$

and

$$\overline{c} = \overline{\eta}\, \overline{c}_0 = \overline{\eta} \omega b_0 = \overline{\eta}^2 \omega b = b.$$

Accordingly

$$\mu(z) = \frac{az + b}{\overline{b}\, z + \overline{a}} \quad \text{whenever } \overline{b}\, z + \overline{a} \neq 0.$$

Moreover $|\mu(0)| < 1$, and consequently $|b| < |a|$, as required. ∎

2.2 The Poincaré Distance Function on the Unit Disk

Definition Let D be the open unit disk in the complex plane $\mathbb{C}$, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The *Poincaré distance function* ρ on D is defined so that

$$\rho(z, w) = \log \left(\frac{|1 - \overline{w}\, z| + |z - w|}{|1 - \overline{w}\, z| - |z - w|} \right)$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$.

Note that

$$\frac{|z - w|}{|1 - \overline{w}\, z|} < 1$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$. (This follows directly from Corollary 2.6). Consequently the Poincaré distance $\rho(z, w)$ between any two points z and w of the unit disk is a well-defined positive real number.

Proposition 2.8 *Let s and t be real numbers satisfying $-1 < s < t < 1$. Then the Poincaré distance, in the unit disk, between s and t is given by the formula*

$$\rho(s, t) = \log \left(\frac{1 + t}{1 - t} \right) - \log \left(\frac{1 + s}{1 - s} \right).$$

Proof Evaluating, and noting that $1 - st > 0$ (because $|s| < 1$ and $|t| < 1$) and $|t - s| = t - s$ (since $s < t$ by assumption), we find that

$$\begin{aligned}
\rho(s, t) &= \log \left(\frac{|1 - st| + |t - s|}{|1 - st| - |t - s|} \right) \\
&= \log \left(\frac{1 - st + t - s}{1 - st + s - t} \right) \\
&= \log \left(\frac{(1 - s)(1 + t)}{(1 + s)(1 - t)} \right) \\
&= \log \left(\frac{1 + t}{1 - t} \right) - \log \left(\frac{1 + s}{1 - s} \right),
\end{aligned}$$

as required. ∎

Proposition 2.9 *Let ρ be the Poincaré distance function on the open unit disk D, and let δ be a positive real number. Then*

$$\{z \in D : \rho(z, 0) = \delta\} = \{z \in D : |z| = R\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof It follows from the definition of Poincaré distance function that all complex numbers z satisfying $\rho(z, 0) = \delta$ are equidistant from zero. They therefore constitute a circle centred on zero. It remains to determine the radius of that circle. Now it follows, on applying Proposition 2.8, that

$$\delta = \log\left(\frac{1 + R}{1 - R}\right).$$

Consequently

$$e^\delta - 1 = \frac{2R}{1 - R}, \quad e^\delta + 1 = \frac{2}{1 - R},$$

and therefore

$$R = \frac{e^\delta - 1}{e^\delta + 1},$$

as required. ∎

The Poincaré distance function ρ on the unit disk D has the property that $\rho(z, w) = \rho(w, z)$ for all $z, w \in D$. It therefore follows immediately from Proposition 2.8 that

$$\rho(s, t) = \left|\log\left(\frac{1 + t}{1 - t}\right) - \log\left(\frac{1 + s}{1 - s}\right)\right|$$

for all real numbers s and t satisfying $-1 < s < 1$ and $-1 < t < 1$.

Lemma 2.10 *Let z and w be complex numbers, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Then*

$$(z, \Omega(z); w, \Omega(w)) = \left|\frac{z - w}{1 - \overline{w}z}\right|^2$$

for all complex numbers z and w with the exception of those pairs z, w for which $|z| = 1$ and $z = w$.

Proof Let z and w be complex numbers. Suppose that it is not the case that $|z| = 1$ and $z = w$. Examination of possible cases shows that it is not then possible for three of the complex numbers z, $\Omega(z)$, w and $\Omega(w)$ to coincide with one another. Indeed if $|z| \neq 1$ and $|w| \neq 1$ then exactly two of the points $z, \Omega(z), w, \Omega(w)$ will lie in the unit disk consisting of those complex numbers whose modulus is less than one, and therefore it is not possible for any three of the four points to coincide with one another. If $|z| = 1$, it would only be possible for three of the points $z, \Omega(z), w, \Omega(w)$ to coincide with one another if it were also the case that $w = z$. Consequently the cross-ratio $(z, \Omega(z); w, \Omega(w))$ is defined in all cases with the exception of those where $|z| = 1$ and $w = z$.

Now let $u_1 = z$, $v_1 = 1$, $u_2 = 1$, $v_2 = \overline{z}$, $u_3 = w$, $v_3 = 1$, $u_4 = 1$, $v_4 = \overline{w}$. Then $u_1/v_1 = z$, $u_2/v_2 = \Omega(z)$, $u_3/v_3 = w$ and $u_4/v_4 = \Omega(w)$. The definition of cross-ratio then ensures that

$$\begin{aligned}(z, \Omega(z); w, \Omega(w)) &= \frac{(u_1v_3 - u_3v_1)(u_2v_4 - u_4v_2)}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} \\ &= \frac{(z - w)(\overline{w} - \overline{z})}{(1 - w\overline{z})(z\overline{w} - 1)} \\ &= \left| \frac{z - w}{1 - \overline{w}z} \right|^2,\end{aligned}$$

as required. ∎

Proposition 2.11 *Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z, w)$ denote the Poincaré distance between z and w. Then*

$$\rho(z, w) = \log \left(\frac{1 + \sqrt{(z, \Omega(z); w, \Omega(w))}}{1 - \sqrt{(z, \Omega(z); w, \Omega(w))}} \right),$$

where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Proof Evaluating, and applying the result of Lemma 2.10, we find that

$$\begin{aligned}\rho(z, w) &= \log \left(\frac{|1 - \overline{w}\, z| + |z - w|}{|1 - \overline{w}\, z| - |z - w|} \right) \\ &= \log \left(\frac{1 + \dfrac{|z - w|}{|1 - \overline{w}\, z|}}{1 - \dfrac{|z - w|}{|1 - \overline{w}\, z|}} \right) \\ &= \log \left(\frac{1 + \sqrt{(z, \Omega(z); w, \Omega(w))}}{1 - \sqrt{(z, \Omega(z); w, \Omega(w))}} \right),\end{aligned}$$

as required. ∎

Corollary 2.12 *Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z, w)$ denote the Poincaré distance between z and w. Then the cross-ratio $(z, \Omega(z); w, \Omega(w))$ is expressed in terms of the Poincaré distance according to the formula*

$$(z, \Omega(z); w, \Omega(w)) = \left(\frac{e^{\rho(z,w)} - 1}{e^{\rho(z,w)} + 1} \right)^2 .$$

Proof Let $q = (z, \Omega(z); w, \Omega(w))$ and $s = \rho(z, w)$. It follows from Proposition 2.11 that

$$s = \log \left(\frac{1 + \sqrt{q}}{1 - \sqrt{q}} \right) .$$

Consequently

$$e^s - 1 = \frac{2\sqrt{q}}{1 - \sqrt{q}}, \quad e^s + 1 = \frac{2}{1 - \sqrt{q}},$$

and thus

$$q = \left(\frac{e^s - 1}{e^s + 1} \right)^2 .$$

The result follows. ∎

Definition A transformation φ that maps the open unit disk D in the complex plane onto itself is said to be an *isometry* (with respect to Poincaré distance) if

$$\rho\Bigl(\varphi(z), \varphi(w)\Bigr) = \rho(z, w)$$

for all complex numbers z and w in the open unit disk D, where ρ denotes the Poincaré distance function on D.

Proposition 2.13 *Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$. Then every Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself is an isometry with respect to the Poincaré distance function on D.*

Proof The Möbius transformation μ has the property that $\mu \circ \Omega = \Omega \circ \mu$, because it maps the unit disk onto itself (see Proposition 2.2). Moreover the values of cross-ratios are preserved under the action of Möbius transformations (Proposition 1.18). Consequently

$$\begin{aligned} \Bigl(\mu(z), \Omega(\mu(z)); \mu(w), \Omega(\mu(w))\Bigr) &= \Bigl(\mu(z), \mu(\Omega(z)); \mu(w), \mu(\Omega(w))\Bigr) \\ &= \Bigl(z, \Omega(z); w, \Omega(w)\Bigr). \end{aligned}$$

The required result therefore follows immediately from an identity previously established (Proposition 2.11) expressing the Poincaré distance $\rho(z, w)$ in terms of the cross-ratio $(z, \Omega(z); w, \Omega(w))$. ∎

Proposition 2.14 *Let z_1, w_1, z_2 and w_2 be elements of the open unit disk D, where*

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, where ρ denotes the Poincaré distance function on D. Then there exists a Möbius transformation μ mapping the open unit disk D onto itself with the property that $\mu(z_1) = z_2$ and $\mu(w_1) = w_2$.

Proof The values of the cross-ratios

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) \quad \text{and} \quad (z_2, \Omega(z_2); w_2, \Omega(w_2))$$

are determined by the values of the Poincaré distances $\rho(z_1, w_1)$ and $\rho(z_2, w_2)$ respectively (see Corollary 2.12). Consequently

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) = (z_2, \Omega(z_2); w_2, \Omega(w_2)).$$

It follows from this that there exists a unique Möbius transformation μ with the properties that $\mu(z_1) = z_2$, $\mu(\Omega(z_1)) = \Omega(z_2)$, $\mu(w_1) = w_2$ and $\mu(\Omega(w_1)) = \Omega(w_2)$, (see Proposition 1.18).

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is itself a Möbius transformation (Lemma 2.1) Then

$$\begin{aligned}
\hat{\mu}(z_1) &= \Omega(\mu(\Omega(z_1))) = \Omega(\Omega(z_2)) = z_2, \\
\hat{\mu}(\Omega(z_1)) &= \Omega(\mu(\Omega(\Omega(z_1)))) = \Omega(\mu(z_1)) = \Omega(z_2), \\
\hat{\mu}(w_1) &= \Omega(\mu(\Omega(w_1))) = \Omega(\Omega(w_2)) = w_2, \\
\hat{\mu}(\Omega(w_1)) &= \Omega(\mu(\Omega(\Omega(w_1)))) = \Omega(\mu(w_1)) = \Omega(w_2).
\end{aligned}$$

Consequently the Möbius transformations μ and $\hat{\mu}$ both map z_1, $\Omega(z_1)$, w_1 and $\Omega(w_1)$ to z_2, $\Omega(z_2)$, w_2 and $\Omega(w_2)$ respectively. But two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Corollary 1.17). Consequently $\hat{\mu} = \mu$, and thus $\Omega \circ \mu = \mu \circ \Omega$. Moreover elements z_1 and z_2 of the open unit disk D are mapped into D. Applying Proposition 2.2, we conclude that the Möbius transformation μ maps the open unit disk D onto itself. This completes the proof. ∎

Proposition 2.15 *Let D be the open unit disk in the complex plane, let w_0 be a complex number lying in D, let δ be a positive real number, and let*

$$\Gamma = \{z \in D : \rho(z, w_0) = \delta\}.$$

Then Γ is a circle contained within the open unit disk D. Moreover if w_0 lies on the real line then the centre of the circle Γ also lies on the real line.

Proof Let
$$\Gamma_0 = \{z \in D : \rho(z,0) = \delta\}.$$
Then Γ_0 is a circle in the complex plane (see Proposition 2.9). Now there exists a Möbius transformation μ mapping the open unit disk D onto itself with the property that $\mu(0) = w_0$ (see Corollary 2.6). Now the image $\mu(\Gamma_0)$ of the circle Γ_0 must itself be a circle containined within the unit disk. Indeed Möbius transformations map circles and straight lines to circles and straight lines (Proposition 1.10), and obviously $\mu(\Gamma_0)$ cannot be a straight line. Moreover $\mu(\Gamma_0) = \Gamma$, because Möbius transformations mapping the open unit disk D onto itself are isometries with respect to the Poincaré distance function ρ on the open unit disk (Proposition 2.13). The result follows. ∎

Proposition 2.16 *Let ρ be the Poincaré distance function on the open unit disk D in the complex plane, let t be a real number satisfying $0 < t < 1$, and let w be a complex number distinct from 0 and t for which $|w| < 1$. Then*
$$\rho(0,w) \leq \rho(0,t) + \rho(t,w).$$
Moreover $\rho(0,w) = \rho(0,t) + \rho(t,w)$ if and only if the complex number w is a positive real number for which $t < w < 1$.

Proof We first note that
$$\rho(0,t) = \log\left(\frac{1+t}{1-t}\right)$$
(see Proposition 2.8).

Given a complex number w in the unit disk that is distinct from 0 and t, let real numbers s and u between -1 and 1 be determined so that
$$\log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right) = \rho(t,w)$$
and
$$\log\left(\frac{1+u}{1-u}\right) - \log\left(\frac{1+t}{1-t}\right) = \rho(t,w).$$
Then $-1 < s < t < u < 1$ and
$$\rho(s,t) = \rho(t,u) = \rho(t,w)$$
and consequently
$$\rho(s,u) = \rho(s,t) + \rho(t,u) = 2 \times \rho(t,u) < 2 \times \rho(0,u) = \rho(-u,u)$$

(again applying Proposition 2.8). It follows that $-u < s < t < u$.

Let

$$\Gamma_1 = \{z \in D : \rho(z,0) = \rho(u,0)\}$$

and

$$\Gamma_2 = \{z \in D : \rho(z,t) = \rho(u,t)\}.$$

It follows from Proposition 2.15 that Γ_1 and Γ_2 are circles in the complex plane, containined in the open unit disk D, whose centres lie on the real line. The circle Γ_1 passes through $-u$ and u, and the circle Γ_2 passes through s and u. Now $-u < s < u$. It follows from elementary geometry that all points of the circle Γ_2 with the exception of the point u lie within the circle Γ_1. Now the point w lies on the circle Γ_2. Therefore

$$\rho(0,w) \leq \rho(0,u) = \rho(0,t) + \rho(t,u) = \rho(0,t) + \rho(t,w).$$

Moreover $\rho(0,w) = \rho(0,t) + \rho(t,w)$ if and only if $w = u$, in which case w lies on the real line and $t < w < 1$. The result follows. ∎

Proposition 2.17 (Triangle Inequality for Poincaré Distance) *The Poincaré distance function ρ on the open unit disk D has the property that*

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3)$$

for all complex numbers z_1, z_2 and z_3 belonging to the disk D.

Proof This inequality follows directly in cases where any two of z_1, z_2 and z_3 coincide with one another. Accordingly it remains to prove that the inequality holds in cases where these three complex numbers are distinct.

Accordingly let z_1, z_2 and z_3 be any three distinct points of the unit disk D. There exists a real number t satisfying $0 < t < 1$ determined so that $\rho(0,t) = \rho(z_1, z_2)$. There then exists a Möbius transformation μ that maps the open unit disk onto itself and satisfies $\mu(0) = z_1$ and $\mu(t) = z_2$ (see Proposition 2.14). Let w be the unique point of the open unit disk for which $\mu(w) = z_3$. Then

$$\rho(0,w) \leq \rho(0,t) + \rho(t,w).$$

(see Proposition 2.16). But the Möbius transformation μ is an isometry of the Poincaré distance function (Proposition 2.13). Consequently

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3).$$

as required. ∎

Lemma 2.18 *Let u be a real number satisfying $0 < u < 1$ and let z be a point of the open unit disk that does not lie on the real line between 0 and u. Then*

$$\rho(0, u) < \rho(0, z) + \rho(z, u),$$

where ρ denotes the Poincaré distance function on the open unit disk.

Proof A positive real number θ can be chosen for which t is a positive real number, where

$$t = (\cos\theta + \sqrt{-1}\,\sin\theta)z.$$

Let

$$w = (\cos\theta + \sqrt{-1}\,\sin\theta)u.$$

The condition in the statement of the lemma regarding the location of z ensures that the complex number w is not a real number lying between t and 1. It follows from Proposition 2.16 that

$$\rho(0, w) < \rho(0, t) + \rho(t, w).$$

Now rotations of the open unit disk about zero are isometries of the Poincaré distance function defined on the unit disk. Consequently

$$\rho(0, u) < \rho(0, z) + \rho(z, u),$$

as required. ∎

2.3 Hyperbolic Length

Definition Let Γ be a straight line segment or circular arc contained in the open unit disk, and let p and q be points lying on Γ. We define the *hyperbolic length* of Γ between the points p and q to be the smallest non-negative real number L with the property that

$$\rho(z_0, z_1) + \rho(z_1, z_2) + \cdots + \rho(z_{m-1}, z_m) \leq L$$

for all choices of distinct points $z_0, z_1, z_2, \ldots, z_{m-1}, z_m$ lying in order along the line or curve Γ with $z_0 = p$ and $z_m = q$.

Remark Those familiar with the concept of *least upper bounds* will note that the hyperbolic length of Γ is, according to this definition, the least upper bound of the values of the sums of the prescribed form.

Now a basic principle of real analysis asserts that if a non-empty set of real numbers is bounded above, then that set has a least upper bound.

Accordingly, in order to prove that any straight line segment or circular arc contained within the open unit disk in the complex plane has a well-defined hyperbolic length, provided that the endpoints of that segment or arc lie within the open disk, it would be necessary to show that there exists some positive real number M that is large enough to ensure that, whenever points $z_0, z_1, \ldots, z_m$ are taken in order along that segment or arc, then

$$\sum_{j=1}^{m} \rho(z_j, z_{j-1}) \leq M.$$

Now suppose that the straight line segment or circular arc is contained within a disk of radius R centred on zero in the complex plane, where $0 < R < 1$. One can then establish the existence of a real constant K, determined by R, such that $\rho(z, z') \leq K|z - z'|$ for all complex numbers z and z' satisfying $|z| \leq R$ and $|z'| \leq R$. One can then show that

$$\sum_{j=1}^{m} \rho(z_j, z_{j-1}) \leq KN,$$

where N is the Euclidean length of the straight line segment or arc in question. Consequently the basic principle of real analysis described above guarantees that the segment or circular arc has a well-defined hyperbolic length.

Remark The definition given is applicable also to certain other curves besides straight line segments and circular arcs, provided that those curves are sufficiently well-behaved.

In particular, if the curve is parametrized by a real variable t so that the the points of the curve are of the form $x(t) + \sqrt{-1}\, y(t)$, where $x(t)$ and $y(t)$ are continuously differentiable real-valued functions of t as t increases from t_0 to t_1, then the hyperbolic length of the curve may be defined in the manner described. Its value can be shown to be equal to the value of the integral

$$\int_{t_0}^{t_1} \frac{2}{1 - x^2 - y^2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Given points p and q that lie on some straight line segment or circular arc Γ in the open unit disk, let us denote by

$$L_{\mathrm{hyp}}(\Gamma; p, q)$$

the hyperbolic length of Γ between the points p and q.

Lemma 2.19 *Let p, q be points lying on straight line segment or circular arc Γ in the open unit disk. Then*

$$L_{\rm hyp}(\Gamma; p, q) \geq \rho(p, q),$$

where $L_{\rm hyp}(\Gamma; p, q)$ denotes respectively the hyperbolic length of Γ between the points p and q and $\rho(p, q)$ denotes the Poincaré distance between p and q.

Proof This result follows directly from the definition of hyperbolic length. (The criterion in that definition applies in particular to the case where the collection of points along Γ between p and q just consists of the two points p and q, with $m = 1$, $z_0 = p$ and $z_1 = q$, employing the notation employed in the definition of hyperbolic length given above.) ∎

Proposition 2.20 *Let p, q and r be points lying in order along a straight line segment or circular arc Γ in the open unit disk. Then*

$$L_{\rm hyp}(\Gamma; p, r) = L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r).$$

Proof Let $z_0, z_1, z_2, \ldots, z_n$ be points in order along Γ with $z_0 = p$ and $z_n = r$. Then either $q = z_k$ for some integer k between 1 and $n-1$ or else q lies between z_{k-1} and z_k for some integer k between 1 and n. In the case where $q = z_k$ for some integer k between 1 and $n - 1$, we find that

$$\begin{aligned}\sum_{j=0}^{n} \rho(z_{j-1}, z_j) &= \sum_{j=0}^{k} \rho(z_{j-1}, z_j) + \sum_{j=k+1}^{n} \rho(z_{j-1}, z_j) \\ &\leq L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r).\end{aligned}$$

In the case where q lies between z_{k-1} and z_k for some integer k between 1 and n, the Triangle Inequality satisfied by the Poincaré distance function (Proposition 2.17) ensures that

$$\begin{aligned}\sum_{j=0}^{n} \rho(z_{j-1}, z_j) &= \sum_{j=0}^{k-1} \rho(z_{j-1}, z_j) + \rho(z_{k-1}, z_k) \\ &\qquad + \sum_{j=k+1}^{n} \rho(z_{j-1}, z_j) \\ &\leq \sum_{j=0}^{k-1} \rho(z_{j-1}, z_j) + \rho(z_{k-1}, q) \\ &\qquad + \rho(q, z_k) + \sum_{j=k+1}^{n} \rho(z_{j-1}, z_j) \\ &\leq L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r).\end{aligned}$$

It follows from these observations that

$$L_{\rm hyp}(\Gamma; p, r) \leq L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r).$$

Now let some positive real number ε be given. Then there exist points $z_0, z_1, \ldots, z_m$ in order along Γ with $z_0 = p$ and $z_m = q$ such that

$$\sum_{j=1}^{m} \rho(z_{j-1}, z_j) > L_{\rm hyp}(\Gamma; p, q) - \varepsilon.$$

There also exist points $z_m, z_{m+1}, \ldots, z_n$ in order along Γ with $z_m = q$ and $z_n = r$ such that

$$\sum_{j=m+1}^{n} \rho(z_{j-1}, z_j) > L_{\rm hyp}(\Gamma; q, r) - \varepsilon.$$

Consequently

$$\sum_{j=1}^{n} \rho(z_{j-1}, z_j) > L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r) - 2\varepsilon.$$

It follows that

$$L_{\rm hyp}(\Gamma; p, r) > L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r) - 2\varepsilon$$

for all positive real numbers ε, and therefore

$$L_{\rm hyp}(\Gamma; p, r) \geq L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r).$$

The inequalities established within the proof now enable us to conclude that

$$L_{\rm hyp}(\Gamma; p, r) = L_{\rm hyp}(\Gamma; p, q) + L_{\rm hyp}(\Gamma; q, r),$$

as required. ∎

Proposition 2.21 *Let Γ be the straight line segment in the open unit disk with endpoints p and q, where p and q are real numbers satisfying $-1 < p < q < 1$. Then the hyperbolic length of Γ is equal to the Poincaré distance $\rho(p, q)$ between p and q.*

Proof Let $t_0, t_1, \ldots, t_m$ be real numbers for which

$$p = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = q.$$

Applying Proposition 2.8 we find that

$$\begin{aligned}\sum_{j=1}^{m}\rho(t_{j-1},t_j) &= \sum_{j=1}^{m}\left(\log\left(\frac{1+t_j}{1-t_j}\right)-\log\left(\frac{1+t_{j-1}}{1-t_{j-1}}\right)\right)\\ &= \log\left(\frac{1+q}{1-q}\right)-\log\left(\frac{1+p}{1-p}\right)=\rho(p,q).\end{aligned}$$

The result follows. ∎

Proposition 2.22 *Let μ be a Möbius transformation mapping the open unit disk in the complex plane onto itself, and let Γ be a straight line segment or circular arc contained within the open unit disk. Then the hyperbolic length of the image $\mu(\Gamma)$ of Γ under the Möbius transformation μ is equal to the hyperbolic length of Γ itself.*

Proof This result follows from the definition of hyperbolic length, in view of the fact that Möbius transformations that map the open unit disk onto itself are isometries with respect to the Poincaré distance function (Proposition 2.13). ∎

2.4 Geodesics in the Open Unit Disk

Definition We say that a straight line segment or circular arc contained within the open unit disk in the complex plane is a *geodesic* if the hyperbolic length of the segment or arc between any two points lying on it is equal to the Poincaré distance between those two points.

Proposition 2.23 *Möbius transformations mapping the open unit disk onto itself map geodesics onto geodesics.*

Proof Möbius transformations mapping the open unit disk onto itself are isometries with respect to the Poincaré distance function (Proposition 2.13) and they preserve hyperbolic distance (Proposition 2.22) The result therefore follows immediately from these observations and the definition of geodesics in the open unit disk. ∎

Theorem 2.24 *Let Γ be a straight line segment or circular arc contained within the open unit disk in the complex plane. Then Γ is a geodesic if and only if the straight line or circle of which it forms part intersects the unit circle orthogonally.*

Proof First suppose that the straight line or circle of which Γ forms part intersects the unit circle orthogonlly at points z_1 and z_2. It follows from Lemma 2.4 that there exists a Möbius transformation μ of the Riemann sphere mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$. Now Möbius transformations map circles and straight lines to circles and straight lines (Proposition 1.10). Moreover they preserve the angles between circles and straight line segments at their points of intersection (see Proposition 1.27). Therefore the straight line or circle of which the image $\mu(\Gamma)$ under the the Möbius transformation μ forms part must intersect the unit circle orthogonally at -1 and 1, and consequently it must coincide with the real line. We conclude therefore that $\mu(\Gamma)$ must be contained within the real line.

It then follows from Proposition 2.21 $\mu(\Gamma)$ must be a geodesic. Now Möbius transformations that map the open unit disk onto itself map geodesics to geodesics (Proposition 2.23). Consequently Γ, being the image of geodesic under the inverse of the Möbius transformation μ, must itself be a geodesic.

Now suppose that Γ is a geodesic. Let p and q be points lying on Γ, and let u be the positive real number for which $\rho(0, u) = \rho(p, q)$, where ρ denotes the Poincaré distance function on the open unit disk. Then there exists a Möbius transformation μ, mapping the open unit disk onto itself, which is such as to ensure that $\mu(p) = 0$ and $\mu(q) = u$. Now Möbius transformations map circles and straight lines to circles and straight lines (Proposition 1.10). Consequently $\mu(\Gamma)$ is a straight line or circular arc on which lie the real numbers 0 and u.

Suppose that $\mu(\Gamma)$ were to pass through some point z of the unit disk that did not lie on the real line between 0 and u. Then, applying Lemma 2.18 and Proposition 2.20 it would follow that

$$\begin{aligned} L_{\mathrm{hyp}}(\mu(\Gamma); 0, u) &= L_{\mathrm{hyp}}(\mu(\Gamma); 0, z) + L_{\mathrm{hyp}}(\mu(\Gamma); z, u) \\ &\geq \rho(0, z) + \rho(z, u) > \rho(0, u). \end{aligned}$$

Consequently $\mu(\Gamma)$ would not be a geodesic. It follows that Γ would not be a geodesic, because Möbius transformations that map the open unit disk onto itself map geodesics to geodesics (Proposition 2.23).

We conclude therefore that if Γ is a geodesic, and if μ is a Möbius transformation mapping the points p and q of Γ to 0 and u respectively, where $0 < u < 1$ and $\rho(0, u) = \rho(p, q)$, then all points of $\mu(\Gamma)$ must lie on the real line.

Now the real line cuts the unit circle orthogonally at the points of intersection. Also Möbius transformations preserve the angles between circles and straight line segments at their points of intersection (see Proposition 1.27).

Therefore the straight line or circle of which Γ forms part must also intersect the unit circle orthogonally, as required. ∎

2.5 Complete Geodesics

Definition A geodesic contained within the open unit disk is said to be *complete* if it is the intersection of the open unit disk with a straight line or circle in the complex plane.

Proposition 2.25 *Given two complete geodesics in the open unit disk D, there exists a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and maps one complete geodesic onto the other.*

Proof Let Γ_1 and Γ_2 be complete geodesics in the open unit disk D, and let I be the geodesic joining -1 and 1 that is the intersection of the disk D with the real axis of the complex plane. Then, given distinct points p_1 and q_1 lying on Γ_1, there exists a Möbius transformation μ_1 that maps the segment of Γ_1 with endpoints p_1 and q_1 into the real line. Then μ_1 maps the complete geodesic Γ_1 onto the complete geodesic I. Similarly there exists a Möbius transformation that maps the complete geodesic Γ_2 onto the complete geodesic I. Then $\mu_2^{-1} \circ \mu_1$ is a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and also maps the complete geodesic Γ_1 onto the complete geodesic Γ_2, as required. ∎

2.6 Geodesic Rays and Segments

Definition A *geodesic segment* is a geodesic that is a straight line segment or circular arc whose endpoints both lie within the open unit disk.

Definition A *geodesic ray* is a geodesic that has an endpoint within the open unit disk and which includes that endpoint together with all points of a complete geodesic that lie between the endpoint and some point at which the straight line or circle of which the geodesic ray forms part crossses the unit circle that bounds the open unit disk.

2.7 The Group of Hyperbolic Motions of the Disk

Definition Let X be a subset of the complex plane. A collection of invertible transformations of the set X is said to be a *transformation group* acting on the set X if the following conditions are satisfied:

(i) the identity transformation belongs to the collection;

(ii) any composition of transformations belonging to the collection must itself belong to the collection;

(iii) the inverse of any transformation belonging to the collection must itself belong to the collection.

The collection of all Möbius transformations of the Riemann sphere that map the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself is a transformation group acting on the open unit disk. Indeed the identity transformation is a Möbius transformation mapping the open unit disk onto itself, the composition of any two Möbius transformations that each map the open unit disk onto itself must also map the open unit disk onto itself, and the inverse of any Möbius transformation that maps the open unit disk onto itself must also map the open unit disk onto itself.

Definition Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $\kappa \colon D \to D$ be the transformation of the open unit disk defined so that $\kappa(z) = \overline{z}$ for all $z \in D$, where $\overline{z}$ denotes the complex conjugate of the complex number z. A transformation of the open unit disk is said to be a *hyperbolic motion* of the unit disk if either it is a Möbius transformation mapping the unit disk D onto itself or else it expressible as a composition of transformations of the form $\mu \circ \kappa$, where μ is a Möbius transformation mapping the open unit disk onto itself.

Möbius transformations give rise to orientation-preserving transformations of the complex plane (see Proposition 1.28 and the discussion of orientation-preserving and orientation-reversing transformations of the complex plane that follows the proof of that proposition). Also the transformation $\kappa \colon D \to D$ that maps each complex number z in D to its complex conjugate $\overline{z}$ is orientation-reversing. Consequently a composition of two transformations in which some Möbius transformation follows the complex conjugation transformation κ is orientation-reversing.

Orientation-preserving hyperbolic motions are the analogues, in hyperbolic geometry, of transformations of the flat Euclidean plane that can be represented as the composition of a rotation followed by a translation.

Orientation-reversing hyperbolic motions are the analogues, in hyperbolic geometry, of reflections and glide reflections of the flat Euclidean plane.

Proposition 2.26 *Let D be the open unit disk in the complex plane, consisting of those complex numbers z that satisfy $|z| < 1$. Then, given any* *orientation-preserving hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$, such that*

$$\varphi(z) = \frac{az + b}{\overline{b}\, z + \overline{a}} \quad \text{for all } z \in D.$$

Similarly, given any orientation-reversing hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$ such that

$$\varphi(z) = \frac{a\, \overline{z} + b}{\overline{b}\, \overline{z} + \overline{a}} \quad \text{for all } z \in D.$$

Proof This result follows directly on applying Proposition 2.7. ∎

Proposition 2.27 *The collection of all hyperbolic motions of the open unit disk is a transformation group acting on the open unit disk.*

Proof The identity transformation is a Möbius transformation that maps the open unit disk onto itself and is thus a hyperbolic motion. Next let μ_1 and μ_2 be Möbius transformations that map the open unit disk onto itself, Then $\kappa \circ \mu_2 \circ \kappa$ is also a Möbius transformation that maps the open unit disk onto itself. Indeed there exist complex numbers a_2 and b_2, where $|b_2| < |a_2|$, such that

$$\mu_2(z) = \frac{a_2 z + b_2}{\overline{b}_2\, z + \overline{a}_2}$$

for all complex numbers z for which $\overline{b}_2\, z + \overline{a}_2 \neq 0$ (see Proposition 2.7). Then

$$\kappa(\mu_2(\kappa(z))) = \frac{\overline{a}_2 z + \overline{b}_2}{b_2\, z + a_2},$$

and therefore $\kappa \circ \mu \circ \kappa$ is also a Möbius transformation that maps the open unit disk D onto itself. Now

$$\mu_1 \circ (\mu_2 \circ \kappa) = (\mu_1 \circ \mu_2) \circ \kappa, \quad (\mu_1 \circ \kappa) \circ \mu_2 = (\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)) \circ \kappa$$

and

$$(\mu_1 \circ \kappa) \circ (\mu_2 \circ \kappa) = \mu_1 \circ (\kappa \circ \mu_2 \circ \kappa).$$

Moreover $\mu_1 \circ \mu_2$ and $\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)$, being compositions of Möbius transformations that map the open unit disk onto itself, are themselves Möbius transformations that map the open unit disk onto itself. It follows from this observation that any composition of hyperbolic motions of the open unit disk is itself a hyperbolic motion of the open unit disk. Also

$$(\mu_2 \circ \kappa)^{-1} = \kappa \circ \mu_2^{-1} = (\kappa \circ \mu_2^{-1} \circ \kappa) \circ \kappa,$$

and the inverse of any Möbius transformation that maps the open unit disk onto itself must itself be a Möbius transformation that maps the open unit disk onto itself. Consequently the inverse of any hyperbolic motion is itself a hyperbolic motion. It follows that the collection of all hyperbolic motions of the open unit disk is indeed a transformation group acting on the open unit disk. ∎

Proposition 2.28 *Let Γ be a complete geodesic in the open unit disk D. Then there exists an orientation-reversing hyperbolic motion φ with the property that $\varphi(z) = z$ for all complex numbers z that lie on the geodesic Γ and also those points of the open unit disk D that lie on one side of the geodesic Γ are mapped by φ to points that lie on the other side of Γ.*

Proof Let I be the set of real numbers t that satisfy the inequalities $-1 < t < 1$. Then I is a complete geodesic in the open unit disk D. There then exists a Möbius transformation μ that maps the geodesic I onto the geodesic Γ. (see Proposition 2.25). Let $\varphi = \mu \circ \kappa \circ \mu^{-1}$, where $\kappa(z) = \overline{z}$ for all $z \in D$. Then the orientation-reversing hyperbolic motion Γ has the required properties. ∎

Proposition 2.29 *Let z_1, w_1, z_2 and w_2 be complex numbers belonging to the open unit disk D. Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, and suppose also that one of the sides of the geodesic Γ_1 in D passing through z_1 and w_1 has been chosen, and that one of the sides of the geodesic Γ_2 in D passing through z_2 and w_2 has also been chosen. Then there exists a hyperbolic motion φ with the following properties: $\varphi(z_1) = z_2$; $\varphi(w_1) = w_2$; φ maps complex numbers on the chosen side of the geodesic Γ_1 to complex numbers on the chosen side of the geodesic Γ_2.*

Proof It follows from Proposition 2.14 that there exists a Möbius transformation that maps the open unit disk onto itself and also maps z_1 and w_1 to z_2 and w_2 respectively. If this Möbius transformation does not itself map the chosen side of Γ_1 to the chosen side of Γ_2, then it may be composed with an orientation-reversing hyperbolic motion that fixes all complex numbers of the geodesic Γ_2 whilst mapping complex numbers on one side of Γ_2 to complex numbers on the other side. The result follows. ∎

2.8 The Hyperbolic Centre of a Circle in the Disk

Proposition 2.30 *Let w be a complex number belonging to the open unit disk D in the complex plane, and let ρ denote the Poincaré distance function* *on D. Let δ be a positive real number. Then*

$$\{z \in D : \rho(z, w) < \delta\} = \left\{ z \in D : \left| \frac{z - w}{1 - \overline{w}\, z} \right| < R \right\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof Let

$$\mu_w(z) = \frac{z + w}{1 + \overline{w} z}$$

for all complex numbers z. Then μ_w is a Möbius transformation mapping the open unit disk onto itself for which $\mu_w(0) = w$ (see Corollary 2.6). Now Möbius transformations mapping the open unit disk onto itself are isometries with regard to the Poincaré distance function (see Proposition 2.13). Consequently

$$\{z \in D : \rho(z, w) < \delta\} = \{z \in D : \rho(\mu_w^{-1}(z), 0) < \delta\}.$$

The required result now follows on applying Proposition 2.9. ∎

Definition Let D be the open unit disk in the complex plane that consists of those complex numbers z satisfying $|z| < 1$, and let C be a circle in the complex plane that is contained within D. A complex number w is said to be the *hyperbolic centre* of the circle C if the Poincaré distance between z and w is the same for all points z that lie on the circle C.

Proposition 2.31 *Let C be a circle in the complex plane that is contained within the open unit disk D. Suppose that the circle C intersects the real axis at real numbers u and v, where $-1 < u < v < 1$. Suppose also that the hyperbolic centre of the circle C lies on the real axis, and is located at t, where $u < t < v$. Then*

$$\left(\frac{1 + t}{1 - t} \right)^2 = \frac{(1 + u)(1 + v)}{(1 - u)(1 - v)}.$$

Proof Applying Proposition 2.8, we find that t, u and v must satisfy the identity

$$\log \left(\frac{1 + v}{1 - v} \right) - \log \left(\frac{1 + t}{1 - t} \right) = \log \left(\frac{1 + t}{1 - t} \right) - \log \left(\frac{1 + u}{1 - u} \right).$$

Consequently

$$2 \log \left(\frac{1 + t}{1 - t} \right) = \log \left(\frac{1 + u}{1 - u} \right) + \log \left(\frac{1 + v}{1 - v} \right).$$

The required result then follows on taking the exponential of both sides of this identity. ∎