Contents

Geometrical Algebra and Associated Symbolic Notation

The term *geometrical algebra* is sometimes employed to describe the characteristic nature of the content of Book II of Euclid's *Elements of Geometry*. This label derives from a perception that the results stated in Euclid's propositions in this book correspond in some meaningful sense to simple identities in basic algebra that can be justified in algebraic terms by means of straightforward algebraic computations. However, in more recent decades, some historians of mathematics have deprecated the use of the term *geometrical algebra*, and have argued strenuously against interpretations of the content of Book II that suggests that the content is essentially algebraic, though presented in geometric language.

Some might however consider it convenient to adopt some form of symbolic notation to represent the essential ideas of the proofs in a form that bears some resemblance to basic algebra. This has been common practice for centuries.

For purposes of the commentaries which follow, let us establish here the following notation and conventions.

Suppose that a polygon $ABCD \ldots HK$ in some plane is bounded by straight line segments

$$AB,\ BC,\ CD, \ldots\ HK,\ KA.$$

This polygon may be identified and represented, with respect to its area, using the notation

$$\mathrm{Poly}(ABCD \ldots HK).$$

Further notation may then be introduced to represent, with respect to area, squares and rectangles with sides equal to given straight line segments.

Thus suppose that a straight line segment is given, and let its endpoints be A and B. Then all squares with sides equal to AB are equal to one another in area. The notation $\mathrm{Quad}(AB)$ may therefore be used to represent, with respect to area, a square whose sides are all equal to the straight line segment AB. Thus

$$\mathrm{Quad}(AB) = \mathrm{Poly}(EFGH)$$

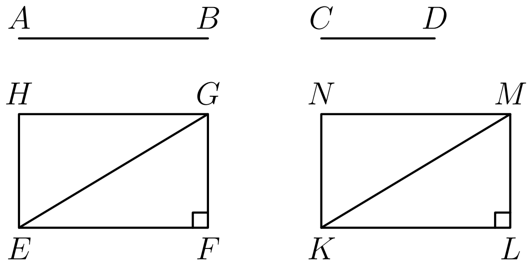
for all squares $EFGH$ for which

$$AB = EF = FG = GH = HE.$$

(The Latin word meaning "square" is *quadratum*.)

Next suppose that two straight line segments AB and CD are given. Let $EFGH$ and $KLMN$ be rectangles, where $EF = KL = AB$ and $FG =$

$LM = CD$. Then, with respect to area, the rectangle $EFGH$ is double the triangle EFG, and the rectangle $KLMN$ is double the triangle KLM. Now the two sides EF, FG are equal to the two sides KL, LM, and the



included angle EFG is equal to the included angle KLM (as both angles are right angles). Applying the SAS Congruence Rule (as stated in Euclid's *Elements of Geometry*, Book I, Proposition 4), we deduce that the triangles EFG and KLM are equal to one another with respect to area, and therefore the rectangles $EFGH$ and $KLMN$, being the doubles of the corresponding triangles, are equal in area to one another. Thus

$$\mathrm{Poly}(EFGH) = \mathrm{Poly}(KLMN).$$

Accordingly the notation $\mathrm{Rect}(AB, CD)$ may be employed to signify, with respect to area, a rectangle contained by sides equal to AB and CD. These containing sides accordingly share a common endpoint at some corner of the rectangle.

In modern geometry, but not in ancient Greek geometry, it is possible, and customary, to choose some straight line segment to represent a unit of length, and to express the ratio of the length that an arbitrary straight line segment bears to the chosen unit segment in the form of a real number. If, for example, the real number c represents the ratio that some straight line segment bears to the unit segment then we say that the length of straight line segment is c units in length. Also the ratio of the area of a rectilineal figure to that of a square constructed on the unit straight line segment may be expressed as a real number that is considered to specify the area of the figure in the appropriate units.

Nevetheless, in instances where the ratio that a straight line segment bears to the unit segment is expressed by a positive integer n, then it would be commonplace for the ancient Greeks to specify the length of the straight line segment as the appropriate number of units. Thus taking a 'foot' as the unit of length, one might describe a line segment as being, for example, four feet in length (see, for example, Plato, *Meno*, 82 C–85 B).

The ancient Greek mathematicians would say that, given two straight line segments of unequal length, the shorter *measures* the longer if the longer line segment can be subdivided into parts that are all equal in length to the shorter segment. The positive integer that is the number of segments arising from this subdivision would then express the ratio of the longer segment to the shorter segment (see the definitions at the beginning of Books VII and X of Euclid's *Elements of Geometry*). Similarly a smaller rectilineal plane figure *measures* a larger rectilinear plane figure if the larger plane figure can be subdivided into a number of parts each equal in area to the smaller plane figure. Again the number of parts expresses the ratio, in area, of the larger plane figure to the smaller plane figure.

Two straight line segments are said to be *commensurable* if there exists some straight line segment that measures both of the given line segments. Otherwise those two line segments are said to be *incommensurable*. Similarly two rectilineal plane figures are said to be *commensurable* if there exists some rectilineal plane figure that measures both of the given line segments. Otherwise those two figures are said to be *incommensurable* (see the definitions the beginning of Book X of Euclid's *Elements of Geometry*).

By the time of Plato, several generations before that of Euclid, the ancient Greek mathematicians knew that, if a smaller square measures a larger square, and if the positive integer that specifies the ratio of the two squares is not a square number, then the sides of the larger square are incommensurable with those of the smaller square (see Plato, *Theaetetus*, 147 C–148 B).

A particular case of the more general result just stated is the well-known result that the diagonal of an isosceles right-angled triangle is incommensurable with the short sides of that triangle.

Consequently any version of "elementary geometry" associating real numbers to all straight line segments and rectilineal plane figures so as to represent "numerically" the lengths of the line segments and the areas of the plane figures would constitute an anachronistic departure from an understanding of ancient Greek geometry that aimed to adhere, so far as is practicable, to the theoretical concepts and frameworks of the ancient Greek mathematicians.

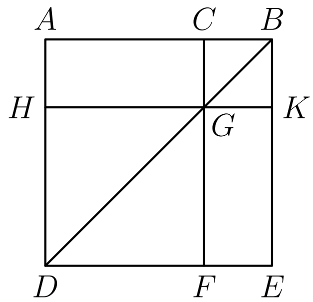
Commentary: Book II, Proposition 4

For this proposition we have to prove that, given a straight line segment AB, a square constructed on this straight line segment is equal in area to the combined area of the following geometric figures: a square constructed on the line segment AC; a square constructed on the line segment CB; two rectangles for which the two sides meeting at any corner have lengths equal to AC and CB respectively.

Accordingly the proposition can be stated in somewhat symbolic notation as follows:

$$\begin{aligned} \text{Quad}(AB) &= \text{Quad}(AC + CB) \\ &= \text{Quad}(AC) + \text{Quad}(CB) + 2 \times \text{Rect}(AC, CB). \end{aligned}$$

Here Quad(AB) may be considered to represent, with regard to area, a square (or *quadratum*, in Latin) on the line AB, and Rect(AC, CB) may be considered to represent, with regard to area, a rectangle (or *rectangulum*) with containing sides equal in length to the straight lines AC and CB.



Of course if, in accordance with 'modern' practice, we were to choose a unit of length, thereby determining a unit straight line and a unit square, and represent the ratios of the lengths of the line segments AC and CB to that of the unit straight line by real numbers represented by the algebraic variables x and y, as has been common practice from at least the early 17th century, then the result to be proved here would correspond to the familiar algebraic identity

$$(x + y)^2 = x^2 + y^2 + 2xy.$$

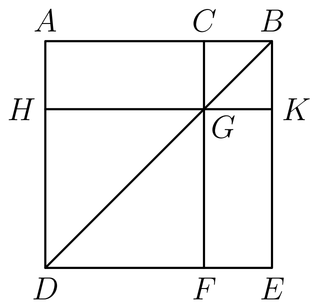
The symbolic language of algebra is not employed in this fashion in Euclid's *Elements of Geometry*.

The geometric equality to be proved seems obvious on considering the associated diagram, provided that the commonplace geometric properties taken for granted in drawing diagrams such as this one can indeed be justified on the basis of the "elements" set out in the propositions of the first book of Euclid's *Elements of Geometry*.

First we set out Euclid's construction for generating all the straight lines in the associated figure. Initially a straight line segment AB is given, together with a point C in the interior of that segment. A square $ADEB$ is constructed on AB (I. 46), then the diagonal BD is drawn, then the line CF starting at C is drawn parallel to AD and EB (I. 31). (Note that Proposition 30 of Book I of Euclid's *Elements of Geometry* ensures that a line parallel to at least one of the parallel lines AD and EB will necessarily be parallel to both.)

Now the straight line CF and the diagonal BD intersect at a point G lying between C and F. Euclid provides no explanation for this. Indeed the text does not even identify the point G as the intersection point of the straight lines CF and BD, and thus one needs to refer to the diagram to identify the location of the point in question. More modern axiomatic treatments of elementary synthetic geometry from the late nineteenth century onwards will typically include axioms whose consequences will include propositions and theorems establishing the circumstances in which lines intersect. In such a more modern axiomatic treatment one would be able to prove formally that the points C and F lie on opposite sides of the infinite straight line joining the points B and D, and therefore the straight line segment CF must intersect BD at some point G lying between B and D.

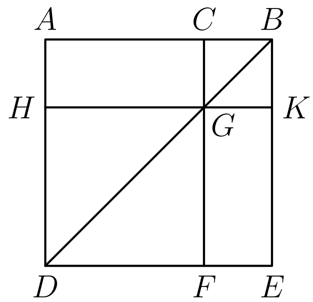
Having constructed the point G where CF and BD intersect, a line HK is drawn through that point parallel to the top and bottom edges AB and DE of the outermost square (I. 30 and I. 31).



Euclid next shows that $CGKB$ is a square. To verify this one must show that the figure $CGKB$ is both equilateral and right-angled. Now, because $ADEB$ is a square, ABD is an isosceles triangle, and therefore the angles ABD and ADB subtended by the equal sides are equal (I. 5). Also the corresponding angles ADB and CGB are equal because the straight lines AD and CF are parallel (I. 29). It follows that the angles CBG and CGB are equal, as both are equal to the angle ADB, and therefore CBG is an isosceles triangle with equal sides CB and CG. Basic properties of parallelograms

(I. 34) ensure that the straight line segments BK and GK are equal to CG and CB respectively. Therefore the figure $CGKB$ is equilateral.

Euclid also explains why the angles at the corners of the figure $CGKB$ are right angles. The angle KBC is right, because $ADEB$ is a square. The angles KBC and BCG add up to two right angles (I. 29), and the angle KBC is right, therefore the angle BCG is right. Also $CGKB$ is a parallelogram, and opposite angles of a parallelogram are equal (I. 34) Therefore angles CGK and GKB are equal to KBC and BCG respectively, and are therefore right angles. Thus all angles of the quadrilateral $CGKB$ are right angles. The sides of this quadrilateral have been shown to be equal. Therefore the quadrilateral $CGKB$ is a square, representing the square on the straight line segment CB. Similarly $HDFG$ is a square, equal to the square on AC.



Now the lines AC, HG, GF and KE are equal to one another because $HDFG$ is a square, $AHGC$ and $GFEK$ are rectangles, and are thus parallelograms, and opposite sides of any parallelogram are equal (I. 34). Also the lines CB, CG and GK are equal to one another for similar reasons. Therefore both the rectangles $AHGC$ and $GFEK$ are equal to one another, and both represent a rectangle contained by the straight line segments AC and CB. We have now shown that

$$\mathrm{Poly}(HDFG) = \mathrm{Quad}(AC), \quad \mathrm{Poly}(CGKB) = \mathrm{Quad}(CB)$$

and

$$\mathrm{Poly}(AHGC) = \mathrm{Poly}(GFEK) = \mathrm{Rect}(AC, CB).$$

It follows that

$$\begin{aligned}\mathrm{Quad}(AB) &= \mathrm{Poly}(HDFG) + \mathrm{Poly}(CGKB) \\ &\qquad + \mathrm{Poly}(AHGC) + \mathrm{Poly}(GFEK) \\ &= \mathrm{Quad}(AC) + \mathrm{Quad}(CB) + 2 \times \mathrm{Rect}(AC, CB).\end{aligned}$$

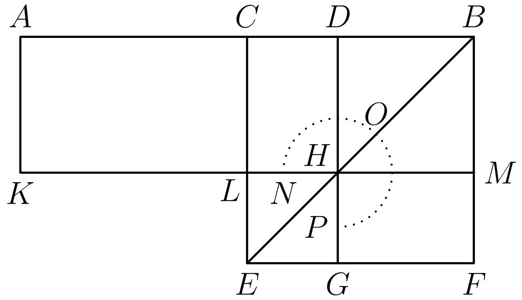
Commentary: Book II, Proposition 5

For this proposition we have to prove that, if a finite straight line AB is cut at C and D, where the point C bisects AB and the point D lies between C and B, and if a rectangle be constructed on AD, with its sides perpendicular to AD equal in length to DB, and if the area of this rectangle be added to that of a square with sides equal in length to CD, then the resultant area is equal to that of a square with sides equal in length to CB.

Accordingly the proposition can be stated in somewhat symbolic notation as follows:

$$\mathrm{Rect}(AD, DB) + \mathrm{Quad}(CD) = \mathrm{Quad}(CB).$$

Here $\mathrm{Quad}(CD)$ and $\mathrm{Quad}(CB)$ may be considered to represent, with regard to area, squares on the lines CD and CB respectively, and $\mathrm{Rect}(AD, DB)$ may be considered to represent, with regard to area, a rectangle with containing sides equal in length to the straight lines AD and DB.

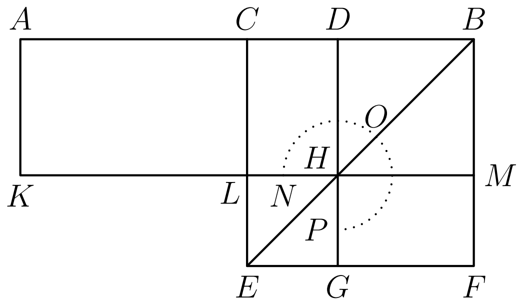


Of course if, in accordance with 'modern' practice, we were to choose a unit of length, thereby determining a unit straight line and a unit square, and represent the ratios of the lengths of the line segments CB and CD to that of the unit straight line by real numbers represented by the algebraic variables x and y, as has been common practice from at least the early 17th century, then the result to be proved here would correspond to the algebraic identity

$$(x + y)(x - y) + y^2 = x^2.$$

The symbolic language of algebra is not employed in this fashion in Euclid's *Elements of Geometry*.

In order to prove the proposition, let the square $CEFB$ be constructed on the finite line CB, let the vertices B and E of this square be joined by the diagonal BE, and let the point D on the edge CB of this square be joined to a point G on the opposite edge of the square by a line DG that is parallel to both CE and BF, and let that line DG intersect that diagonal BE of the square at the point H. Then let the rectangle $AKMB$ be completed so as to ensure that the side KM of this rectangle passes through the point H. Moreover let L be the point where CE intersects KM.



Now Proposition 43 in Book I of the *Elements of Geometry* ensures that the rectangles $CLHD$ and $HGFM$ are equal in area. Accordingly

$$\text{Poly}(CLHD) = \text{Poly}(HGFM).$$

Also $AK = DH = DB$. Consequently

$$\begin{aligned}
\text{Rect}(AD, DB) &= \text{Poly}(AKHD) = \text{Poly}(AKLC) + \text{Poly}(CLHD) \\
&= \text{Poly}(CLMB) + \text{Poly}(HGFM) \\
&= \text{Poly}(CLHGFB),
\end{aligned}$$

where $\text{Poly}(CLHGFB)$ represents, in area, the polygon $CLHGFB$. This polygon is the figure referred to by Euclid as the 'gnomon' NOP. Also

$$\text{Quad}(CD) = \text{Poly}(LEGH),$$

because the sides of the square $LEGH$ are equal in length to the finite line CD. Accordingly

$$\begin{aligned}
\text{Rect}(AD, DB) + \text{Quad}(CD) &= \text{Poly}(CLHGFB) + \text{Poly}(LEGH) \\
&= \text{Quad}(CB),
\end{aligned}$$

as required.

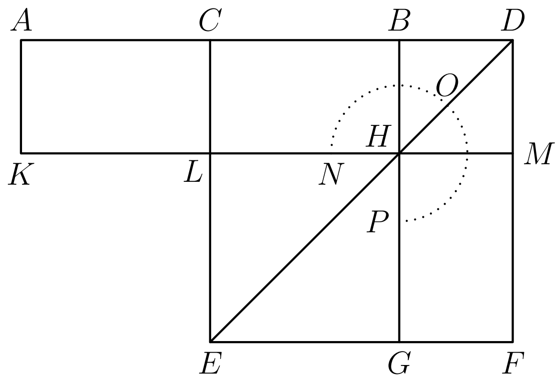
Commentary: Book II, Proposition 6

For this proposition we have to prove that, if a finite straight line AB is cut at bisected at C and is also produced in a straight line beyond the point C to end at a point D, and if a rectangle be constructed on AD, with its sides perpendicular to AD equal in length to BD, and if the area of this rectangle be added to that of a square with sides equal in length to CB, then the resultant area is equal to that of a square with sides equal in length to CD.

Accordingly the proposition can be stated in somewhat symbolic notation as follows:

$$\text{Rect}(AD, DB) + \text{Quad}(CB) = \text{Quad}(CD).$$

Here $\text{Quad}(CB)$ and $\text{Quad}(CD)$ may be considered to represent, with regard to area, squares on the lines CB and CD respectively, and $\text{Rect}(AD, DB)$ may be considered to represent, with regard to area, a rectangle with containing sides equal in length to the straight lines AD and DB.

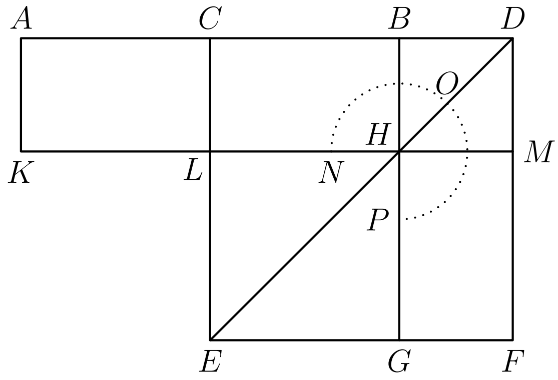


Of course if, in accordance with 'modern' practice, we were to choose a unit of length, thereby determining a unit straight line and a unit square, and represent the ratios of the lengths of the line segments CB and BD to that of the unit straight line by real numbers represented by the algebraic variables x and y, as has been common practice from at least the early 17th century, then the result to be proved here would correspond to the algebraic identity

$$(2x + y)y + x^2 = (x + y)^2.$$

The symbolic language of algebra is not employed in this fashion in Euclid's *Elements of Geometry.*

In order to prove the proposition, let the square $CEFD$ be constructed on the finite line CD, let the vertices D and E of this square be joined by the diagonal DE, and let the point B on the edge CD of this square be joined to a point G on the opposite edge of the square by a line BG that is parallel to both CE and DF, and let that line BG intersect that diagonal DE of the square at the point H. Then let the rectangle $AKMD$ be completed so as to ensure that the side KM of this rectangle passes through the point H. Moreover let L be the point where CE intersects KM.



Now Proposition 43 in Book I of the *Elements of Geometry* ensures that the rectangles $CLHB$ and $HGFM$ are equal in area. Also the rectangles $AKLC$ and $CLHB$ are equal in area, because AC and CB are equal in length. Accordingly

$$\text{Poly}(AKLC) = \text{Poly}(CLHB) = \text{Poly}(HGFM).$$

Also $AK = BH = BD$. Consequently

$$\begin{aligned} \text{Rect}(AD, DB) &= \text{Poly}(AKMD) = \text{Poly}(AKLC) + \text{Poly}(CLMD) \\ &= \text{Poly}(HGFM) + \text{Poly}(CLMD) \\ &= \text{Poly}(CLHGFD), \end{aligned}$$

where $\text{Poly}(CLHGFD)$ represents, in area, the polygon $CLHGFD$. This polygon is the figure referred to by Euclid as the 'gnomon' NOP. Also

$$\text{Quad}(CB) = \text{Poly}(LEGH),$$

because the sides of the square $LEGH$ are equal in length to the finite line CB. Accordingly

$$\begin{aligned} \text{Rect}(AD, DB) + \text{Quad}(CB) &= \text{Poly}(CLHGFD) + \text{Poly}(LEGH) \\ &= \text{Quad}(CD), \end{aligned}$$

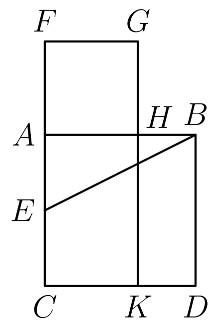
as required.

Commentary: Book II, Proposition 11

This proposition supplies a geometric construction using straightedge and compasses for cutting a line segment in a particular ratio named by the ancient Greeks as *extreme and mean ratio* (see Euclid, *Elements of Geometry*, Book VI, Definition 3). In more modern times, this ratio is often referred to as the *golden ratio*, having been named as such by the German mathematician Martin Ohm in 1835.

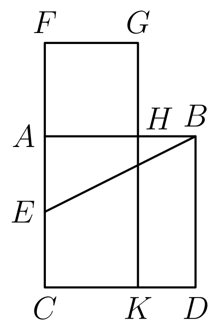
In the context of Euclid's *Elements of Geometry*, the term *ratio* does not appear before Book V, and consequently Euclid simply describes the properties that characterize the relationship between the line segments AH and AB without making explicit reference to the concept of 'ratios'.

The problem set out in this proposition requires us, given a line segment AB, to determine a point H in the interior of the segment so as to ensure that a square constructed on the side AH is equal in area to a rectangle with containing sides equal in length to AB and HB.



Of course if, in accordance with 'modern' practice, we were to choose a unit of length, thereby determining a unit straight line and a unit square, and represent the ratios of the lengths of the line segments AB and AH to that of the unit straight line by real numbers represented by the algebraic quantities b and x, as has been common practice from at least the early 17th century, then then our task would be to formulate a straightedge and compass construction, given a line segment of length b, to determine a line segment of length x, where $b(b - x) = x^2$. In other words, we would be seeking a positive real number x that is a root of the quadratic polynomial $x^2 + bx - b^2$. The symbolic language of algebra is not employed in this fashion in Euclid's *Elements of Geometry.*

In order to carry out the construction, a square $ACDB$ is constructed on the line segment AB, the side AC of that square is bisected at E, and that side is produced beyond A to a point F so as to ensure that EF and EB are equal in length. Of course this point F is the point at which the straight ray starting at E and passing through the point A intersects the circle centered on the point E that passes through the point A. A square $AHGF$ is then constructed on AF so that the side AH of the square is a part of the line segment AB. The point H then cuts the line segment AB in the required ratio.



The validity of the construction may be established by the following argument, which applies Pythagoras's Theorem (which is Proposition 47 in Book I) and Proposition 4 in Book II of Euclid's *Elements*. Note that

$$\begin{aligned}\mathrm{Quad}(EA)+\mathrm{Quad}(AB) &= \mathrm{Quad}(EB)=\mathrm{Quad}(EF)\\ &= \mathrm{Quad}(EA)+\mathrm{Quad}(AF)+2\times\mathrm{Rect}(EA,AF)\\ &= \mathrm{Quad}(EA)+\mathrm{Quad}(AH)+\mathrm{Rect}(AB,AH).\end{aligned}$$

Subtracting the area of the square on EA, we find that

$$\mathrm{Quad}(AB)=\mathrm{Quad}(AH)+\mathrm{Rect}(AB,AH).$$

But

$$\mathrm{Quad}(AB)=\mathrm{Rect}(AB,HB)+\mathrm{Rect}(AB,AH).$$

Consequently

$$\mathrm{Quad}(AH)=\mathrm{Rect}(AB,HB),$$

as required.

Euclid sets out the argument somewhat differently, applying Proposition 6 in Book II of the *Elements*. Applying that proposition in the context of the construction previously described, we find that

$$\mathrm{Rect}(CF,FA)+\mathrm{Quad}(EA)=\mathrm{Quad}(EF).$$

But the line segments EF and EB are equal in length. Consequently, applying Pythagoras's Theorem (Proposition 47 in Book I of the *Elements*), we find that

$$\mathrm{Quad}(EF)=\mathrm{Quad}(AB)+\mathrm{Quad}(EA).$$

Consequently

$$\mathrm{Rect}(CF, FA) = \mathrm{Quad}(AB),$$

and thus

$$\mathrm{Poly}(FGKC) = \mathrm{Poly}(ACDB).$$

But

$$\mathrm{Poly}(FGKC) = \mathrm{Quad}(AH) + \mathrm{Poly}(AHKC)$$

and

$$\mathrm{Poly}(ACDB) = \mathrm{Poly}(HKDB) + \mathrm{Poly}(AHKC).$$

Consequently

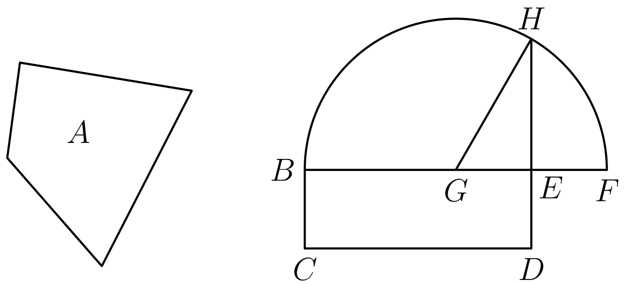
$$\mathrm{Quad}(AH) = \mathrm{Poly}(HKDB) = \mathrm{Rect}(AB, BH),$$

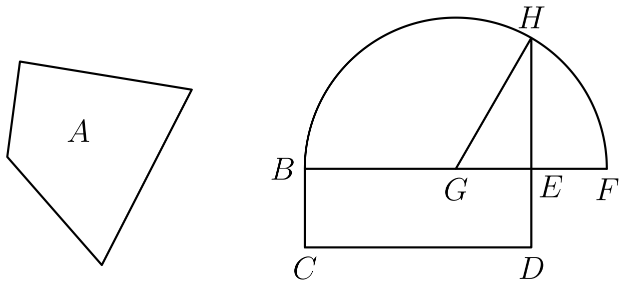
as required.

Commentary: Book II, Proposition 14

This proposition supplies a geometric construction using straightedge and compasses for drawing a square equal in area to a given rectangle. Now Proposition 45 in Book I of Euclid's *Elements of Geometry* describes how to construct a rectangle equal in area to any given plane rectilineal figure. Consequently the construction described in Proposition 45 in Book I can be followed by the construction set out by Euclid in this proposition in order to construct, using straightedge and compasses, a square equal in area to any given plane rectilineal figure.

Of course no further construction is needed when the given rectangle is itself a square. We therefore only have to describe and justify the construction when the given rectangle is oblong. Accordingly let $BCDE$ be a rectangle contained by sides BE and ED, where BE is longer than ED. In the context of Euclid's proposition, this rectangle is constructed, so as to be equal in area to a given plane rectilineal figure A: Proposition 45 in Book I of the *Elements* describes how the construction of the rectangle $BCDE$ might be performed. In Euclid's construction of the square equal in area to the rectangle $BCDE$, the side BE of that rectangle is produced in a straight line beyond E to a point F so as to ensure that ED and EF are equal in length. The straight line segment BF is bisected at the point G, and a semicircular arc from B to F is constructed, centred on the point G and lying on the opposite side of BF to the rectangle $BCDE$. The straight line segment DE is then produced in a straight line beyond E till it intersects the semicircular arc at the point H. Euclid then proves that a square constructed on the side EH would be equal in area to the rectangle $BCDE$.





Euclid's argument proceeds as follows. Proposition 5 in Book II of the *Elements of Geometry* ensures that

$$\mathrm{Rect}(BE, EF) + \mathrm{Quad}(EG) = \mathrm{Quad}(GF).$$

Now GF and GH are equal in length, because the points F and H lie on the semicircle centred on the point G. Also the angle GEH is a right angle. Pythagoras's Theorem (*Elements*, I.47) therefore ensures that

$$\mathrm{Quad}(GF) = \mathrm{Quad}(GH) = \mathrm{Quad}(HE) + \mathrm{Quad}(EG).$$

Consequently

$$\mathrm{Rect}(BE, EF) + \mathrm{Quad}(EG) = \mathrm{Quad}(HE) + \mathrm{Quad}(EG).$$

Also the straight line segments ED and EF are equal in length. Consequently

$$\mathrm{Poly}(BCDE) = \mathrm{Rect}(BE, EF) = \mathrm{Quad}(HE).$$

In other words, the rectangle $BCDE$ is equal in area to a square constructed on the straight line segment EH.

We now consider how Euclid's construction might be justified in the language and notation of modern geometry. Thus we suppose that the ratios of the the line segments BE, ED, GF, EG and HE to some chosen segment representing a unit of length are expressed by the real numbers a, b, r, x and y respectively. Then

$$a = r + x, \quad b = r - x \quad \text{and} \quad x^2 + y^2 = r^2.$$

The product ab then represents the ratio of the area of the rectangle $BCDE$ to that of the square on the unit line segment. Now

$$ab = (r + x)(r - x) = r^2 - x^2 = y^2.$$

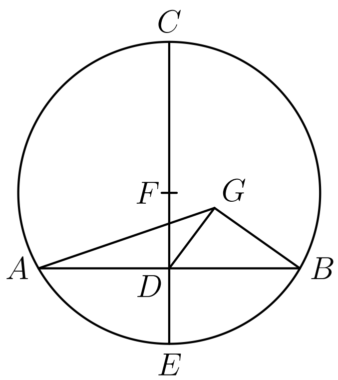
Thus the rectangle $BCDE$ is equal in area to a square constructed on the line segment HE.

Commentary: Book III, Proposition 1

Heath begins his commentary on this construction as follows:

> Todhunter observes that, when, in the construction, DC is said to be *produced* to E, it is assumed that D is within the circle, a fact which Euclid first demonstrates in III.2. This is no doubt true, although the word διήχθω, "let it be *drawn through*," is used instead of ἐκβεβλήσθω, "let it be *produced*." And, although it is not necessary to assume that D is within the circle, it is necessary for the success of the construction that the straight line drawn through D at right angles to AB shall meet the circle in two points (and no more): an assumption which we are not entitled to make on the basis of what has gone before only.
>
> Hence there is much to be said for the alternative procedure recommended by De Morgan as preferable to that of Euclid. De Morgan would first prove the fundamental theorem that "the line which bisects a chord perpendicularly must contain the centre," and then make III. 1, III. 25 and IV. 5 immediate corollaries of it.. . .

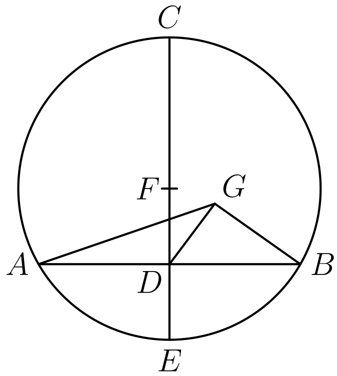
Notwithstanding the comments of De Morgan, Todhunter and Heath as described above, the Euclidean text could be considered to be structured so that the construction is first described as it would be performed by someone implementing the construction, thereby constructing a point D in the interior of the circle and a perpendicular bisector cutting AB at the point D and cutting the circle at two points C and E that would happen to lie on opposite sides of the line AB.



Having described the practical steps for performing the construction, the Euclidean text then supplies some theoretical justification for the procedure.

Given two points A and B lying on the circumference of the circle, the perpendicular bisector of the chord AB is constructed in the usual way. Euclid establishes that the centre of the circle cannot lie anywhere other than on this perpendicular bisector.

Indeed let G be a point in the plane of the circle that does not lie on the perpendicular bisector of the chord AB. Euclid shows that G cannot be the centre of the circle. Indeed suppose that G were the centre. Then $GA = GB$, and therefore the triangle GAB would be isosceles. But then $\angle GAB = \angle GBA$ (I. 5). Applying the SAS Congruence Rule (I. 4) to the triangles GAD and GBD at the vertices A and B respectively, it would then follow that $\angle GDA = \angle GDB$, and therefore both angles GDA and GDB would be right. Thus GD would be perpendicular to AB. But this is not possible, as G has been chosen to be a point not lying on the perpendicular bisector of AB. Thus G cannot be the centre of the circle.



In the Euclidean text it seems to be left as an exercise for the reader to show that the centre of the circle is indeed located at the point F, and thus that a correct construction has indeed been set out.

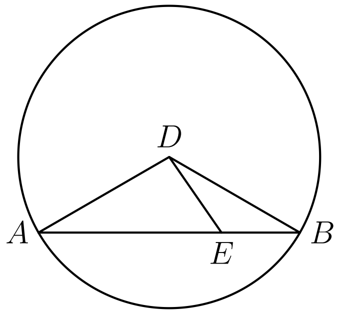
Now the circle must have a centre somewhere. Euclid has shown that this centre cannot lie away from the perpendicular bisector of the chord AB. Therefore the perpendicular bisector of AB must pass through the centre of the circle, and must therefore intersect that circle in exactly two points. Let F be the midpoint of the diameter of the circle terminated at these two points. Then F is the centre of the circle.

Furthermore the angle FAD is acute, because FDA is right and the sum of any two angles of a triangle must be less than two right angles (I. 17). The greater angle of a triangle is subtended by the greater side (I. 19). Therefore the radius FA of the circle is greater than FD, and thus the point D lies in the interior of the circle. This establishes the correctness of the construction set out above for finding the centre of a circle.

Commentary: Book III, Proposition 2

The proposition asserts that all points lying on a chord AB joining two points A and B located on the circumference of some circle must lie in the interior of that circle. Euclid employs a proof strategy that involves showing that any point not located in the interior of the circle cannot lie on the chord between the points A and B.

It would be more natural, in modern mathematics, to present a more direct proof. And indeed such a proof is to be found in nineteenth century textbooks that adapt and paraphrase Euclid's *Elements of Geometry.* Let A and B be points located on the circumference of a circle, and let E be a point lying on the chord AB between A and B. Let the point D be the centre of that circle, and let DA, DB and DE be joined.



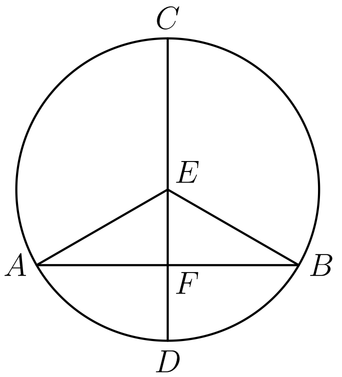
Then $DA = DB$, because the points A and B are located on a circle centred on the point D, and therefore DAB is an isosceles triangle. It follows that $\angle DAB = \angle DBA$ (I. 5). Now $\angle DEB$ is an exterior angle of the triangle DAE, whilst $\angle DAE$ is an interior angle of that triangle opposite to the exterior angle DEB. Therefore $\angle DEB > \angle DAE$ (I. 16). But

$$\angle DAE = \angle DAB = \angle DBA = \angle DBE.$$

It follows that $\angle DEB > \angle DBE$. Also, in any triangle, the greater angle is subtended by the greater side (I. 19). Consequently $DB > DE$, and therefore the point E lies closer to the centre D of the circle than the points A and B located on its circumference, and therefore lies in the interior of the circle. Thus the chord AB does indeed fall within the circle.

Commentary: Book III, Proposition 3

To prove this proposition one must establish, firstly, that if a diameter CD of a circle bisect a chord AD not through the centre, then CD cuts AB at right angles, and secondly, that if the diameter CD cut the chord AB at right angles, then it bisects the chord. Let the point E be the centre of the



circle, and let the diameter CD meet the chord AB at the point F. The centre E of the circle is the midpoint of the diameter CD.

Euclid uses the SSS Congruence Rule to prove that if the diameter CD bisects the chord, then it cuts the chord at right angles. Thus suppose that CD bisects AB at the point F. Then the sides AF, FE and EA of the triangle EAF are respectively equal to the sides the sides BF, FE and EB of the triangle EBF. The SSS Congruence Rule (*Elements*, I.8) ensures that the triangles EAF and EBF are congruent. Consequently the angles AFE and BFE are equal, and thus both angles are right angles.

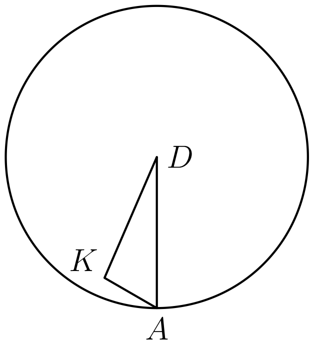
Note that applying the result that the angles subtending the equal sides of an isosceles triangle are equal (*Elements*, I.5), we can deduce that the angles EAB and EBA of the triangle EAB are equal to one another, whether or not the chord AB is bisected at F, and whether or not the line FE is at right angles to AB. Consequently, if the diameter CD bisects the chord AB at F then the SAS Congruence Rule can be used to prove the equality of the angles AFE and BFE, thus proving that the diameter CD cuts the chord AB at right angles.

In the situation where the diameter CD cuts the chord AB at right angles at F, and one is required to show that the chord AB is bisected at F, Euclid proves that the chord is indeed bisected by applying the SAA Congruence Rule established in Proposition 26 of Book I of the *Elements of Geometry*.

Commentary: Book III, Proposition 16

In proving that a straight line drawn at right angles to a diameter of a circle from its extremity will fall outside the circle, Euclid uses the fact that a straight line from a point A on the circumference of the circle and passing through a point in the interior of the circle will, when produced to the extent necessary, intersect the circumference again at some other point.

As an alternative to Euclid's argument, one may argue as follows. Let A be a point on the circumference of a circle, let D be the centre of that circle, and let K be some point in the interior of the circle that is not collinear with A and D. Join AK and KD.



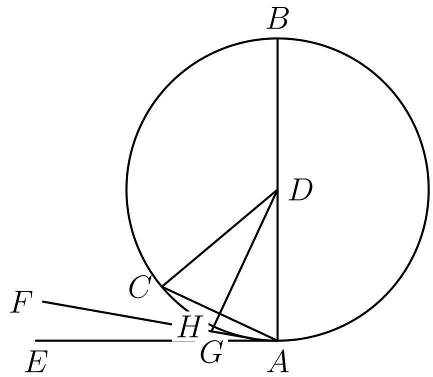
Now $DK < DA$, and the greater of two sides of any triangle subtends the greater angle (I. 18). It follows therefore that $\angle DKA > \angle DAK$, and therefore the sum of the angles of the triangle DAK at A and K exceeds twice the angle of that triangle at A. But the sum of any two angles of a triangle is less than two right angles (I. 17). It follows that the angle DAK is less than a right angle.

From this we conclude that a straight line drawn from a point A on the circumference of a circle at right angles to the line AD joining that point to the centre D of the circle must necessarily fall outside the circle.

This proposition also discusses properties of the *horn angle*. This is the "angle" between the circumference of the circle and its tangent line at the point A. It is not a rectilineal angle. Heath's commentary on this proposition includes extensive discussion of disputes and discussions of such horn angles from ancient times through to the seventeenth century.

Euclid asserts that "between the straight line" tangent to the circle at A "and the circumference another straight line cannot be interposed." Proving this is tantamount to showing that if a straight line AF from the point A on the circumference of the circle makes an an acute angle with the straight line

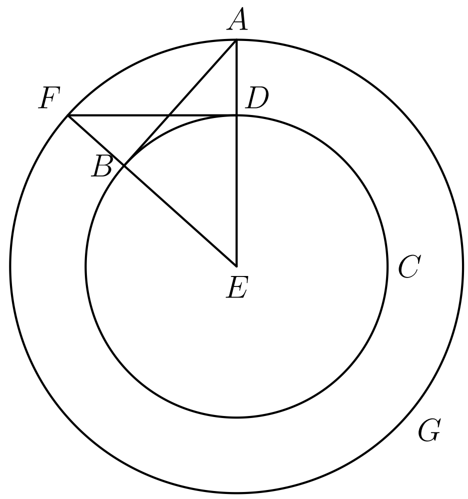
AD joining the point A to the centre D of the circle then the line AF must enter the interior of the circle. Euclid's proof can be paraphrased and slightly varied in the following manner. Drop a perpendicular from the centre D of the circle to the line AF, and let that perpendicular meet the line AF at the point G, as in Euclid's figure.



Then the triangle DGA has a right angle at G and an acute angle at A, and thus $\angle DGA > \angle DAG$. The greater of two angles of a triangle is subtended by the greater side (I. 19). It follows that $DA > DG$ and thus the point G lies in the interior of the circle.

Commentary: Book III, Proposition 17

The proposition describes and justifies a proposition for dropping, onto a given circle, a straight line from a given point outside the circle so as to ensure that the straight line so constructed touches the circle. In the configuration depicted in Euclid's diagram, the given circle is the circle BCD and the given point is the point A. The point A is first joined to the centre E of the circle. Then a line is drawn at right angles to AE at the point D at which AE intersects the circle. This line at right angles to AE at the point D is produced till it intersects, at the point F, the circle centred on E that passes through the given point A. The point F is joined to the centre E of the circle, and a line is drawn from the given point A to the point B at which the line FE intersects the circle BCD. Euclid shows that the line AB touches the circle BCD at the point B.



Now an application of the SAS Congruence Rule shows that the the triangles AEB and FED are congruent. Consequently the angles EDF and EBA are equal. But the angle EDF is a right angle. Consequently the angle EBA is also a right angle. It then follows from the preceding proposition, Proposition 16, applying the Porism associated with that proposition, that the line AB touches the circle BCD at the point B.

Commentary: Book III, Proposition 18

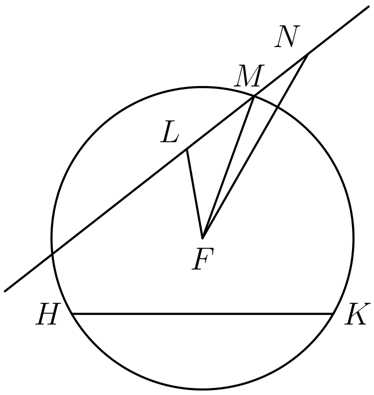
This proposition asserts that if a straight line touches a circle at a point C, and if a straight line segment FC is drawn that joins the centre F of the circle to the point C of contact, then the line FC is perpendicular to the straight line touching the circle at the point C of contact.

Before discussing how this result is proved, we first discuss what is meant by saying that a straight line *touches* a given circle, and then show that a straight line that touches a given circle cannot pass inside that circle.

Now, according to the definitions that commence Book III, a straight line touches a circle at a point C of contact if any only if it does not cut the the circle at that point. However the definitions prefixed to Book III of Euclid's *Elements of Geometry* provide no specification of which is meant by saying that a straight line cuts a circle at a point of intersection, or what is meant by saying that two circles cut one another at a point of intersection.

The straight line would cut the circle at that point if, following the line in one of the two directions along it, one passes from outside the circle to inside the circle on passing through the point C. If one wished to be more precise, one could say that the straight line cuts the circle at the point C of contact if and only if, for some sufficiently small circle centred on the point C, the point C separates the points on the straight line and within the small circle that lie inside the given circle from those points on the straight line and within the small circle that lie outside the given circle. A similar criterion could define, formally and precisely, what is meant by saying that two circles cut another.

In the argument which follows, we shall demonstrate that a straight line can touch a circle at at most one point of contact, and that a straight line which touches a circle at any point cannot pass through any point lying inside that circle. First let L, M and N be points lying on a given straight line, and let F be a point that does not line on that straight line. Suppose that FL is no longer than FM. We claim that FN is then longer than FM.



To prove this, note that the angle FML subtended by FL is no larger than the angle FLM subtended by FM (*Elements*, I.19). Moreover the exterior angle FMN of the triangle FLM is greater than the interior and opposite angle FLM of that triangle, and the exterior angle FML of the triangle FNM is greater than the interior and opposite angle FNM (*Elements*, I.16). Consequently

$$\angle FNM < \angle FML \leq \angle FLM < \angle FMN.$$

Consequently FN is longer than FM (*Elements*, I.19), as claimed.

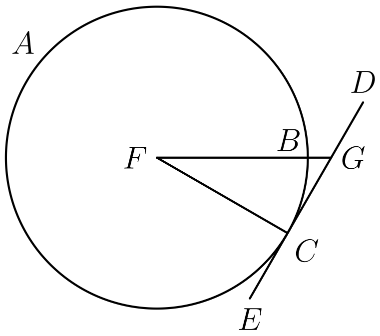
This result may be reformulated as follows. Suppose that points L, M and N lie on a straight line, with M lying between L and N, suppose that some circle is given and that the point L lies inside or on the circle, and the point M lies on the circle. Then the point N lies outside the circle.

It follows immediately from this result that if a straight line touches a circle, it cannot pass through any point that lies inside the circle.

Next we note that if a straight line that meets a circle at two distinct points cannot touch the circle at those two points, because the straight line joining those points lies within the circle (*Elements*, III.2), and consequently the straight line does not touch the circle. We conclude therefore that a straight line can touch a circle at at most one point.

Examination of Euclid's proof of Proposition 18 shows that he assumed that, when a straight line touches a circle at a some point of contact, then all points on the straight line other than the point of contact lie outside the circle.

Indeed in the proof he argues that, in the configuration depicted by the following figure, if the straight line DE touching the circle ABC at the point C were not perpendicular to FC, where F is the centre of the circle, then the perpendicular let fall from F onto the straight line would meet that straight line at a point G lying outside the circle. The sum of the two angles



FGC and FCG would be less than two right angles (*Elements*, I.17) and the angle FGC would be a right angle, therefore the angle FCG would be less than a right angle, and therefore would be less than the angle FGC. Consequently FG would be shorter than FC (*Elements*, I.19), contradicting the result that all points on the line DE other than the point C lie outside the circle.

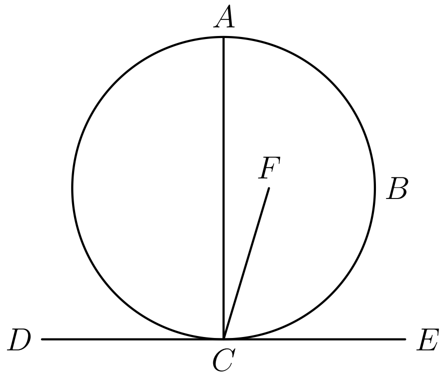
One may justify the assertion made in this proposition as follows. Suppose that a straight line touches a circle at a given point on the circle. Then, as noted above, all points on the straight line other than that point of contact lie outside the circle, and therefore the point of contact is the unique point on the line that closest to the centre of the circle.

Next we note that it follows immediately from Proposition 16 in Book III of the *Elements of Geometry* that if a perpendicular is dropped onto the straight line from the centre of the given circle, then the point at which that perpendicular intersects the straight line is the closest point on the straight line to the centre of the circle.

Combining these observations, we conclude that if a straight line touches a circle at some point, then the perpendicular let fall from the centre of the circle onto that straight line will intersect the straight line at the point of contact.

Commentary: Book III, Proposition 19

The result stated in this proposition is justified on the grounds that, from the point C of contact at which the line DE touches the circle ABC, only one infinite line can be drawn that is at right angles to to DE at C. The preceding proposition, Proposition 18, ensures that the line joining the point C to the centre of the circle ABC is at right angles to the line DE at the point C of contact. Therefore the infinite line drawn at right angles to the line DE at the point C must pass though the centre of the circle.



Commentary: Book III, Proposition 20

Let a circle be drawn with centre E, and let three distinct points P, Q and R be taken on the circumference of that circle. Then the configuration of the four points E, P, Q and R will be as described in exactly one of the following five cases:—

(Case A) the centre E of the circle lies within the triangle PQR;

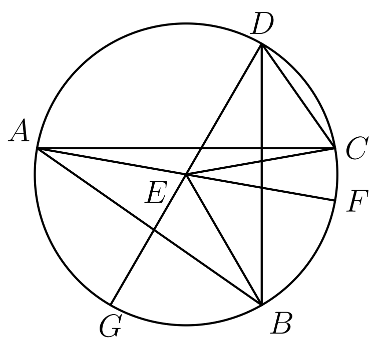
(Case B) the centre E of the circle lies on one or other of the sides the triangle PQR that join the point P to the points Q and R;

(Case C) the centre E of the circle does not line on the line QR, lies on the same side of that line as the point P, and lies outside the triangle PQR;

(Case D) the centre E of the circle lies on the line QR; circle;

(Case E) the point P and the centre E of the circle lie on opposite sides of the line QR.

The diagram associated to Proposition 20 in Book III of Euclid's *Elements of Geometry* is the following.

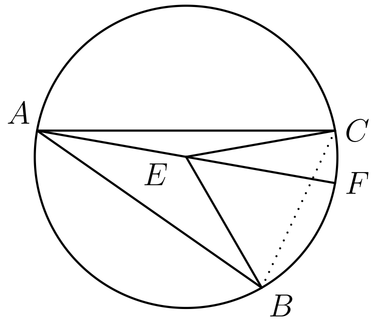


The statement of that proposition is applicable in cases A, B and C set out above, and is explicitly proved by Euclid in cases A and C. Moreover all these three cases A, B and C are represented within the configuration depicted by the above diagram.

We now consider the five individual cases.

Case A

In this case the points P, Q and R in the specification of the cases given above correspond to the points A, B and C of Euclid's diagram respectively. The centre E of the circle then lies within the triangle ABC. In this con-



figuration, the triangle EAB is isosceles, and therefore the angles EAB and EBA at the endpoints of the base AB of that triangle are equal to one another (*Elements*, I.5). But these two angles are the interior angles opposite the exterior angle FEB of that triangle, and that exterior angle is the sum of the two corresponding interior and opposite angles *Elements*, I.32). Consequently

$$\angle FEB = \angle EAB + \angle EBA = 2 \times \angle EAB.$$

Similarly

$$\angle FEC = \angle EAC + \angle ECA = 2 \times \angle EAC.$$

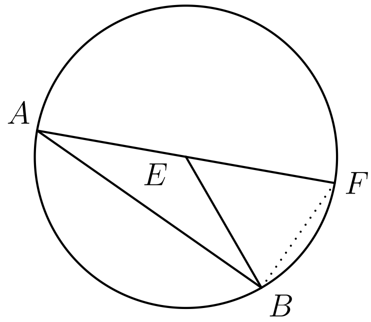
Therefore

$$\angle BEC = \angle FEB + \angle FEC = 2 \times \angle EAB + 2 \times \angle EAC = 2 \times \angle BAC.$$

The required result therefore follows in this case.

Case B

In this case the points P, Q and R in the specification of the cases given above correspond to the points A, B and F of Euclid's diagram respectively. The centre E of the circle then on the side AF of the triangle ABF. In this



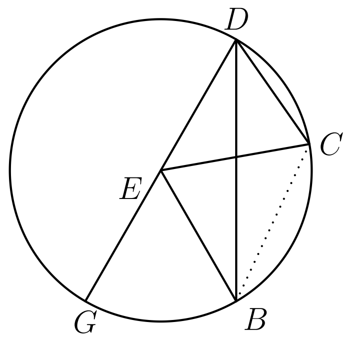
configuration, it follows, for the reasons set out in the discussion of Case A, that

$$\angle FEB = \angle EAB + \angle EBA = 2 \times \angle EAB.$$

The required result therefore follows in this case.

Case C

In this case the points P, Q and R in the specification of the cases given above correspond to the points D, B and C of Euclid's diagram respectively. The centre E of the circle then lies on the same side of the line BC as the point D, and lies outside the triangle DBC. In this configuration, the



triangles EDB and EDC are isosceles, and therefore

$$\angle EDB = \angle EBD \quad \text{and} \quad \angle EDC = ECD$$

(*Elements*, I.5). The exterior angles GEB and GEC of the triangles EDB and EDC respectively are the sums of the corresponding interior and opposite angles (*Elements*, I.32). Consequently

$$\angle GEB = \angle EDB + \angle EBD = 2 \times \angle EDB$$

and

$$\angle GEC = \angle EDC + \angle ECD = 2 \times \angle EDC.$$

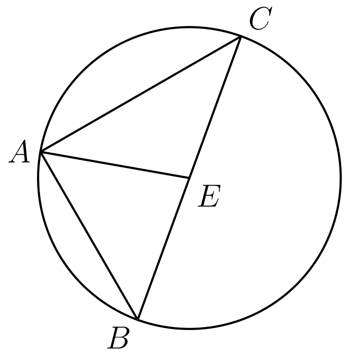
Therefore

$$\angle BEC = \angle GEC - \angle GEB = 2 \times \angle EDC - 2 \times \angle EDB = 2 \times \angle BDC.$$

The required result therefore follows in this case.

Case D

In this case the points P, Q and R in the specification of the cases given above correspond to the points A, B and C of the diagram accompanying Proposition 31 in Book III of Euclid's *Elements of Geometry*, and the centre E of the circle then lies on the side BC of the triangle ABC. That side BC is then a diameter of the circle. In Proposition 31 in Book III, Euclid



establishes that, in this configuration, the angle of triangle ABC at the vertex A is a right angle. Now the triangles EBA and ECA are isosceles, and therefore

$$\angle EAB = \angle EBA \quad \text{and} \quad \angle EAC = ECA$$

(*Elements*, I.5). The exterior angles CEA and BEA of the triangles EAB and EAC respectively are the sums of the corresponding interior and opposite angles (*Elements*, I.32). Consequently

$$\angle AEC = \angle EAB + \angle EBA = 2 \times \angle EAB$$

and

$$\angle AEB = \angle EAC + \angle ECA = 2 \times \angle EAC.$$

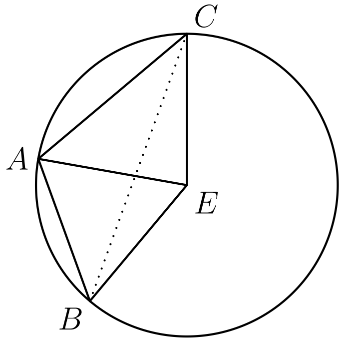
Therefore

$$\text{two right angles} = \angle AEB + \angle AEC = 2 \times \angle EAB + 2 \times \angle EAC = 2 \times \angle BAC.$$

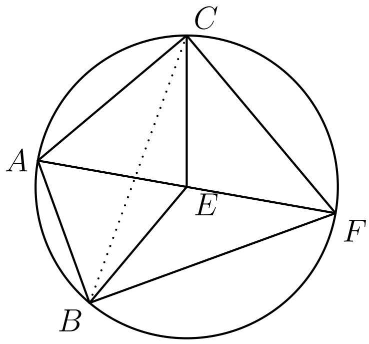
Consequently the angle BAC is a right-angle.

Case E

In this case the points P, Q and R in the specification of the cases given above correspond to the points A, B and C of the following diagram, and the centre E of the circle and the point A then lies on opposite sides of the line BC. We prove that, in this case, twice the angle BAC is equal to the remainder obtained on subtracting the angle BEC from four right angles.



To establish this result, let the line segment AE joining the point A to the centre E of the circle be produced in a straight line beyond E so as to intersect the circle again at F. Let the points B and C be joined to F as depicted in the diagram below. In this configuration, the line segment AF is a diameter of the circle $BACF$.



The discussion above regarding Case D establishes that the angles ABF and ACF are right angles. (Euclid establishes this result in Proposition 31 in Book III of the *Elements*.) Now the sum of the angles of any triangle is equal to two right angles (*Elements*, I.32). It follows therefore that

$$\angle BFA + \angle FAB = \text{one right angle}$$

and

$$\angle CFA + \angle FAC = \text{one right angle.}$$

Consequently

$$\begin{aligned}\angle BFC + \angle BAC &= \angle BFA + \angle FAB + \angle CFA + \angle FAC \\ &= \text{two right angles.}\end{aligned}$$

Now the results established in cases A, B and C above (and by Euclid in his proof of Proposition 20 in Book III for the configurations that he explicitly considers) ensure that

$$\angle BEC = 2 \times \angle BFC.$$

Consequently

$$2 \times \angle BAC + \angle BEC = \text{four right angles.}$$

Consequently twice the angle BAC is equal to the remainder obtained on subtracting the angle BEC from four right angles.

Moreover

$$\begin{aligned}\angle CEF + \angle BEF + \angle BEC &= \angle CEF + \angle BEF + \angle BEA + \angle CEA \\ &= \text{four right angles}\end{aligned}$$

(applying *Elements*, I.13). Consequently

$$2 \times \angle BAC + \angle BEC = \angle CEF + \angle BEF + \angle BEC,$$

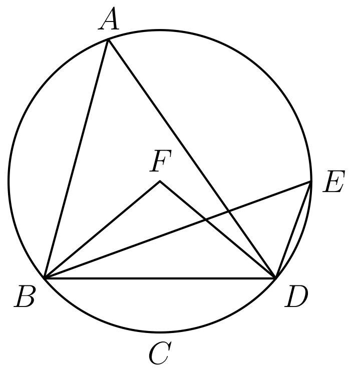
and therefore

$$2 \times \angle BAC = \angle CEF + \angle BEF.$$

Now, in modern geometry, the combination of the two angles CEF and BEF would together constitute a "reflex angle" exceeding two right angles, and twice the angle BAC would then be equal in magnitude to this reflex angle. However the only rectilineal angles recognized by the ancient Greeks are acute, right and obtuse angles. Thus, in ancient Greek geometry, all angles considered are less than two right angles.

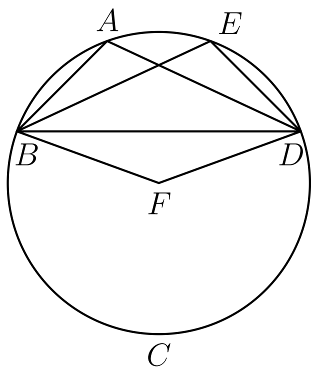
Commentary: Book III, Proposition 21

Euclid only proves Proposition 21 in Book III of the *Elements* for configurations in which centre of the circle lies inside the segment in question. Such a configuration is depicted in Euclid's diagram for this proposition, shown below.



In cases in which the segment $BAED$ is a semicircle, it can be shown that the triangles BAD and BED are right-angled, and consequently the result follows immediately.

It remains to consider configurations in which the centre F of the circle lies outside the segment $BAED$. Such configurations are as depicted in the following diagram



In such configurations, one can show that

$$\begin{aligned} 2 \times \angle BAD + \angle BFD &= \text{four right angles;} \\ 2 \times \angle BED + \angle BFD &= \text{four right angles.} \end{aligned}$$

Consequently

$$2 \times \angle BAD + \angle BFD = 2 \times \angle BED + \angle BFD,$$

and therefore the angles BAD and BED are equal, as required.

It is possible to give a proof of Proposition 21, covering all cases, which only requires, as prerequisite, the cases of Proposition 20 explicitly considered by Euclid, in which the segment considered in that proposition is greater than a semicircle. Such a proof was given by Robert Simson (1687–1768) in his translation of the first six books of Euclid's *Elements of Geometry*, in which he amended or extended those proofs that he considered inaccurate, insufficient or incomplete in the standard version of the Greek text available in the eighteenth century. Simson's statement and proof of Proposition 21 in Book III of the *Elements* are quoted below (copied from the 5th Edition, 1775).

PROP. XXI. THEOR.

> The angles in the same segment of a circle are equal to one another.
>
> Let ABCD be a circle, and BAD, BED angles in the same segment BAED: the angles BAD, BED are equal to one another.
> Take F the center of the circle ABCD: And, first, let the segment BAED be greater than a semicircle, and join BF, FD: And because the angle BFD is at the centre, and the angle BAD
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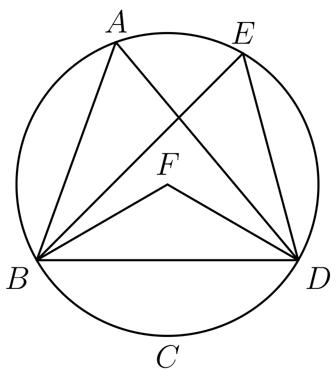
> 
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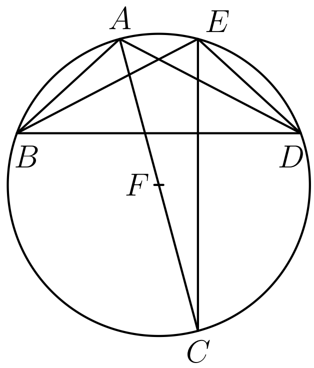
>
> at the circumference, and that they have the same part of the circumference, viz. BCD for their base; therefore the angle BFD is double of the angle BAD: For the same reason, the angle BFD is double of the angle BED: Therefore the angle BAD is equal to the angle BED.

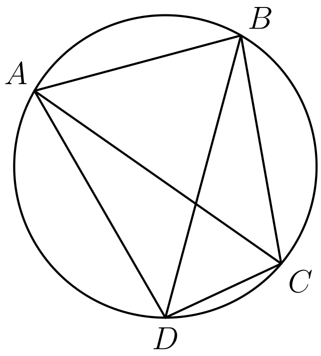
But, if the segment BAED be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to one another: Draw AF to the center, and produce it to C, and join CE: Therefore the segment BADC is greater than a semicircle;



and the angles in it BAC, BEC are equal, by the first case: For the same reason, because CBED is greater than a semicircle, the angles CAD, CED are equal: Therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the same segment, &c. Q. E. D.

Commentary: Book III, Proposition 22

The figure accompanying Euclid's proof of Proposition 22 of Book III of the *Elements of Geometry* depicts a quadrilateral $ABCD$ inscribed in a circle, where all four sides of the quadrilateral cut of a segment of the circle lying outside the quadrilateral that is less than a semicircle. In this configuration,



the segments $BADC$ and $ADCB$ are both greater than a semicircle, and therefore the Euclid's proof of Proposition 21, which explicitly considers only angles in segments greater than a semicircle, can be applied to show that

$$\angle CAB = \angle BDC \quad \text{and} \quad \angle ACB = \angle ADB.$$

Also the internal angles of the triangle ABC add up to two right angles (*Elements*, I.32). Consequently

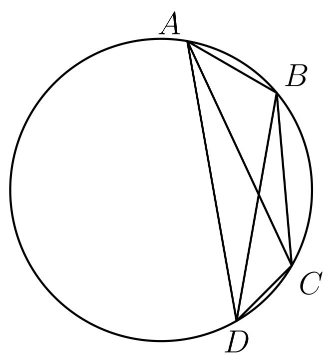
$$\begin{aligned} \angle ABC + \angle ADC &= \angle ABC + \angle ADB + \angle BDC \\ &= \angle ABC + \angle ACB + \angle CAB \\ &= \text{two right angles.} \end{aligned}$$

A similar argument would then show that

$$\angle BAD + \angle DCB = \text{two right angles.}$$

Alternatively one could deduce this equality by making use of the result that the sum of the four internal angles of the quadrilateral $ABCD$ is equal to four right angles.

Now let us consider how the proof of Proposition 22 of Book III of the *Elements of Geometry* can be applied when the four sides of the quadrilateral $ABCD$ cut off segments of the circle outside the quadrilateral are not all less than a semicircle. Now if any pair of these segments meet one another, they can only meet at vertices of the quadrilateral. Therefore at most one of the segments outside the quadrilateral cut off by the sides of the quadrilateral can be greater than a semicircle. Let us suppose that it is the segment outside the quadrilateral cut off by the side AD that is greater than a semicircle. Then the segments outside the quadrilateral cut off by the sides AB and BC are both less than a semicircle, and therefore the segments $BADC$ and $ADCB$ are both greater than a semicircle.



Consequently Euclid's proof of Proposition 21, valid for angles in segments greater than a semicircle, can be used to deduce that

$$\angle CAB = \angle BDC \quad \text{and} \quad \angle ACB = \angle ADB.$$

The argument presented above then shows that the angles ABC and ADC must sum to two right angles. Then, given that all four internal angles of the quadrilateral must sum to four right angles, we can deduce also that the angles BAD and DCB must also sum to two right angles.

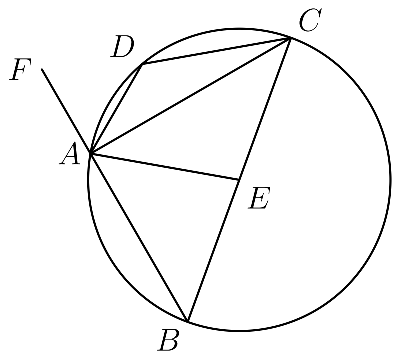
Given this result, we can prove that all angles in segments less than a semicircle must be equal to one another, using the result already established explicitly by Euclid in the case of angles in a segment greater than a semicircle. Indeed let B, A, E, D and C be points in cyclic order around a circle. Then Proposition 22, applied to the quadrilaterals $BADC$ and $BEDC$, ensures that

$$\angle BAD + \angle BCD = \text{two right angles} = \angle BED + \angle BCD.$$

Consequently the angles BAD and BED in the segment $BAED$ are equal to one another.

Commentary: Book III, Proposition 31

The various results stated in this proposition can be proved on the basis of the geometric configuration depicted in the diagram associated with the proposition.



Let a diameter AB be drawn across the given circle $ABCD$, and let E be the centre of that triangle. The line segments EA, EB and EC are then equal in length, and therefore

$$\angle BAE = \angle ABE = \angle ABC \quad \text{and} \quad \angle CAE = \angle ACE = \angle ACB$$

(*Elements*, I.5). Now

$$\angle BAC = \angle BAE + \angle CAE.$$

Consequently

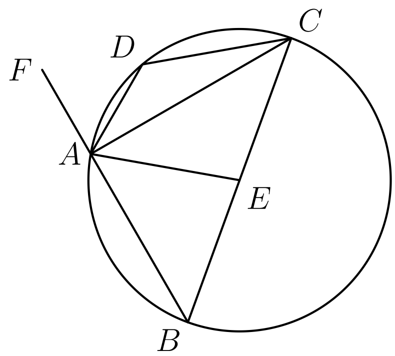
$$\angle BAC = \angle ABC + \angle ACB.$$

But the angle BAF is an exterior angle of the triangle ABC at the vertex A and is therefore equal to the sum of the interior angles of this triangle at vertices B and C (*Elements*, I.32). Consequently

$$\angle BAF = \angle ABC + \angle ACB.$$

It follows that the adjacent angles BAC and BAF are equal to one another, and are therefore by definition right angles. We conclude therefore that *the angle in any semicircle must be equal to a right angle*.

Let a chord be drawn across a given circle $ABCD$ that is not a diameter of the circle, and let that chord be the line segment AC. The chord partitions the circle into two segments: one segment is the segment ABC greater than a semicircle bounded by the chord AC and the circular arc ABC; the other segment is the segment ADC less than a semicircle bounded by the chord AC and the circular arc ADC. Let the diameter BD be drawn which has one endpoint located at the endpoint C of the chord AC, and let the configuration be completed as depicted in the diagram. The angle ABC is then the angle in the segment ABC greater than a semicircle.



Now the angle ABC must be less than a right angle, because BAC is a right angle and the sum of ABC and BAC is less than two right angles (*Elements*, I.17). We conclude therefore that *the angle in any segment greater than a semicircle must be less than a right angle.*

Also the sum of the angles ABC and ADC is equal to two right angles because the vertices of the quadrilateral $ABCD$ all lie on a circle (*Elements*, I.22). It follows that the angle ADC must be greater than a right angle. But this angle is the angle in the segment ADC. We conclude therefore that *the angle in any segment less than a semicircle must be greater than a right angle.*

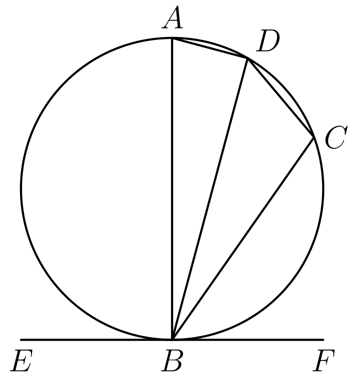
The *angle of the segment ABC* is represented by the angle between the chord AC and the arc ABC at the point A: this is not a rectilineal angle. This angle contains the right angle CAB. We conclude therefore that *the angle of any segment greater than a semicircle must be greater than a right angle.*

The *angle of the segment ADC* is represented by the angle between the chord AC and the arc AFC at the point A: this is not a rectilineal angle. This angle is contained within the right angle CAF. We conclude therefore that *the angle of any segment greater than a semicircle must be less than a right angle.*

Commentary: Book III, Proposition 32

Euclid's proof of this proposition covers the cases where the chord drawn across the circle is not a diameter of the circle.

Let the configuration be as depicted by Euclid in the figure below, in which the line EF is the tangent line to the circle touching the circle at the point B.



First let us consider the particular case in which the angle between the straight line touching the circle and the chord drawn across the circle that is under consideration is the angle FBD. In this case, the *alternate segment* is the segment BAD of the circle bounded by the chord BD and the circular arc BAD. The angle in this alternate segment is, by definition, the angle which, at any point on the circular arc BAD between B and D, is formed by the straight line segments joining that point to the endpoints B and D of the arc B. The magnitude of this angle is the same whichever point on the arc is chosen (*Elements*, III.21). We may therefore choose the point in question to be the point A for which AB is a diameter of the circle. The angle in the alternate segment BAD is then equal to the angle BAD. We must therefore prove in this case that the angles BAD and DBF are equal.

Now the angle ABD is a right angle (*Elements*, III.31), and the sum of the angles of any triangle is equal to two right angles (*Elements*, I.32). Consequently

$$\angle BAD + \angle ABD = \text{one right angle}.$$

But the angle ABF is also a right angle (*Elements*, III.16), and consequently

$$\angle DBF + \angle ABD = \angle ABF = \text{one right angle}.$$

Consequently

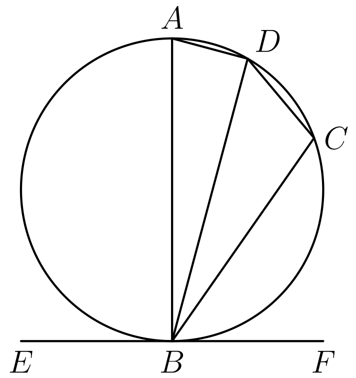
$$\angle BAD + \angle ABD = \angle DBF + \angle ABD,$$

and therefore, subtracting the angle ABD from both sides of this equality,

$$\angle BAD = \angle DBF,$$

as required in this case.

Next let us consider the particular case in which the angle between the straight line touching the circle and the chord drawn across the circle that is under consideration is the angle EBD. In this case, the *alternate segment* is the segment DCB of the circle bounded by the chord BD and the circular arc DCB. The angle in this alternate segment is, by definition, the angle which, at any point on the circular arc DCB between D and B, is formed by the straight line segments joining that point to the endpoints B and D of the arc B. The magnitude of this angle is the same whichever point on the arc is chosen (*Elements*, III.21). We may therefore choose the choose the point in question to be the point C indicated on Euclid's diagram. The angle in the alternate segment DCB is then equal to the angle DCB. We must therefore prove in this case that the angles DBE and DCB are equal.



Now, in the configuration depicted by Euclid, the quadrilateral $ABCD$ is inscribed in a circle. Consequently the sum of opposite angles of this quadrilateral is equal to two right angles (*Elements*, III.22). Consequently

$$\angle BAD + \angle DCB = \text{two right angles}.$$

But

$$\angle DBF + \angle DBE = \text{two right angles}$$

(*Elements*, I.13). Consequently

$$\angle DBF + \angle DBE = \angle BAD + \angle DCB.$$

But we have already shown that

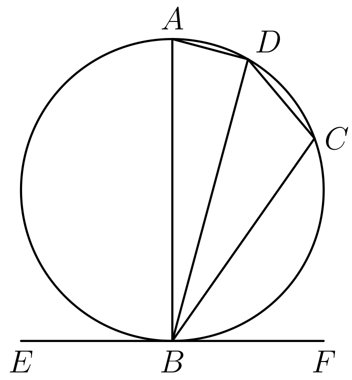
$$\angle DBF = \angle BAD.$$

Consequently

$$\angle DBE = \angle DCB,$$

as required in this case.

Euclid does not explicitly consider the case in which the straight line drawn across the circle is a diameter of the circle. The result in this case follows directly from the preceding proposition (*Elements*, III.31). Moreover the result in this case can be established employing the configuration depicted in Euclid's diagram.



Thus consider the case where the straight line drawn across the circle is the diameter AB and the angle under consideration is that between the tangent line is the angle ABE. Then the alternate segment is the semicircle bounded by the diameter AB and the circular arc $ADCB$. The angle in the alternate segment is thus equal to the angle ADB. Now the angle between the tangent line BE and the diameter BA is a right angle (*Elements*, III.18). Morever the preceding proposition (*Elements*, III.31) ensures that the angle ADB in the semicircle is also a right angle. Consequently

$$\angle ABE = \angle ACB,$$

as required in this case.

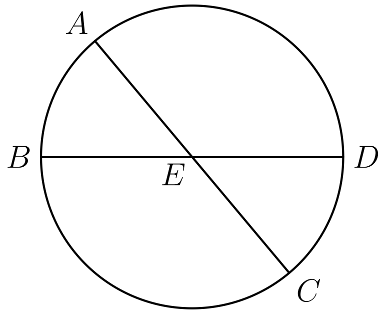
Commentary: Book III, Proposition 35

Euclid considers two cases, depending on whether or not the two given lines pass through the centre of the circle.

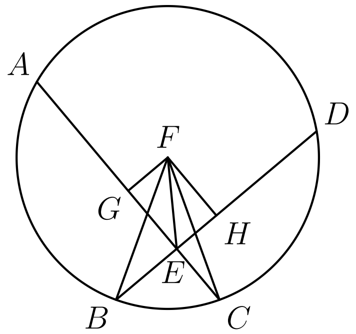
First suppose that the lines AC and BD pass through the centre E of the circle. Then

$$AE = EC = DE = EB.$$

Therefore both $\text{Rect}(AE, EC)$ and $\text{Rect}(BE, ED)$ are equal to the square on AE, and are therefore equal to one another.



In the second case the lines AC and BD intersect at a point E that is not the centre of the circle. Let F be the centre of the circle, join FE and drop perpendiculars from the centre F to the lines AC and BD (I. 12), meeting those lines at G and H respectively. Then AC and BD are bisected at G and H respectively (III. 3). We label the points A, B, C and D so that the point E of intersection lies between C and G, and between B and H.



Now

$$\text{Rect}(AE, EC) + \text{Quad}(EG) = \text{Quad}(GC)$$

(see II 5). Adding the square $\text{Quad}(GF)$ on GF to both sides, we find that

$$\text{Rect}(AE, EC) + \text{Quad}(EG) + \text{quad}(GF) = \text{Quad}(GC) + \text{Quad}(GF).$$

But FGE and FGC are right-angled triangles with the right angle at G. It follows from Pythagoras's Theorem (I. 47) that

$$\mathrm{Quad}(EG) + \mathrm{Quad}(GF) = \mathrm{Quad}(FE)$$

and

$$\mathrm{Quad}(CG) + \mathrm{Quad}(GF) = \mathrm{Quad}(FC).$$

It follows that

$$\mathrm{Rect}(AE, EC) + \mathrm{Quad}(FE) = \mathrm{Quad}(FC).$$

The same argument ensures that

$$\mathrm{Rect}(DE, EB) + \mathrm{Quad}(FE) = \mathrm{Quad}(FB).$$

But $FC = FB$, because the points B and C lie on a circle with centre F. It follows that

$$\begin{aligned} \mathrm{Rect}(AE, EC) + \mathrm{Quad}(FE) &= \mathrm{Quad}(FC) = \mathrm{Quad}(FB) \\ &= \mathrm{Rect}(DE, EB) + \mathrm{Quad}(FE). \end{aligned}$$

On subtracting the square $\mathrm{Quad}(FE)$ on FE, it follows that

$$\mathrm{Rect}(AE, EC) = \mathrm{Rect}(DE, EB).$$

The required geometric equality has now been verified in the case in which the lines AC and BD intersect at some point E that is not the centre of the circle, and has therefore been verified in all required cases.

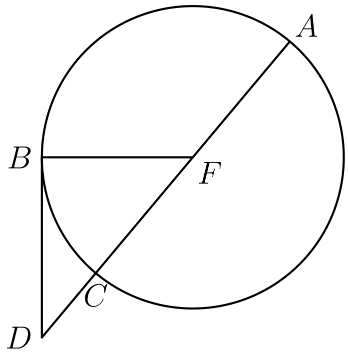
Commentary: Book III, Proposition 36

Euclid considers two cases, depending on whether or not the line from the point outside the circle passes through the centre of the circle.

First suppose that the line from the point D outside the circle passes through the centre F of the circle, and cuts the circumference of the circle at points A and C, where C lies between D and F. Also let DB be a line from the point D which touches the circle at the point B on its circumference. It is required to prove that

$$\text{Rect}(AD, DC) = \text{Quad}(DB).$$

(Here $\text{Rect}(AD, DC)$ represents, with respect to area, a rectangle whose sides meeting at a corner have lengths equal to the finite lines AD and DC, and $\text{Quad}(AB)$ represents, with respect to area, a square whose sides are equal to the finite line AB.)



Now the angle FBD is a right angle, because BD is tangent to the circle at the point B (III. 16). It follows from Pythagoras's Theorem (I. 47) that

$$\text{Quad}(FD) = \text{Quad}(FB) + \text{Quad}(BD).$$

Also

$$\text{Rect}(AD, DC) + \text{Quad}(FC) = \text{Quad}(FD)$$

(II. 6). (N.B., This geometric equality corresponds to the algebraic identity

$$(2a + b)b + a^2 = (a + b)^2,$$

on taking a and b to represent, in basic algebra, the lengths of the finite lines FC and CD respectively.) It follows that

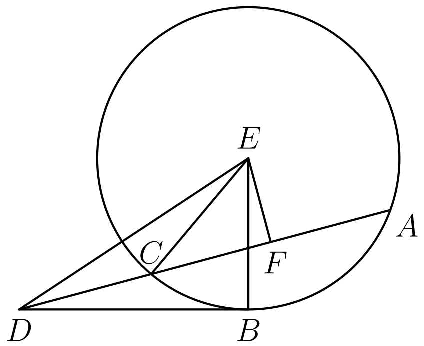
$$\text{Rect}(AD, DC) + \text{Quad}(FC) = \text{Quad}(FB) + \text{Quad}(BD).$$

But $FC = FB$, because the points B and C both lie on the circumference of a circle with centre D. It follows that $\mathrm{Quad}(FC) = \mathrm{Quad}(FB)$, and therefore

$$\mathrm{Rect}(AD, DC) = \mathrm{Quad}(BD),$$

Thus the required geometric equality is satisfied in the case where the line ACD passes through the centre F of the circle.

We must also establish the stated geometric equality in the case where the line ACD does not pass through the centre of the circle. In this case let DB touch the circle at B, and let F be the point on the line ACD that is the foot of the perpendicular dropped to the line ACD from the centre E of the circle.



Now the point F bisects the chord AC because the line EF cuts that chord at right angles (III. 3), and thus $AF = FC$. It then follows that

$$\mathrm{Rect}(AD, DC) + \mathrm{Quad}(FC) = \mathrm{Quad}(FD)$$

(II. 6). Adding $\mathrm{Quad}(EF)$ to both sides, we find that

$$\mathrm{Rect}(AD, DC) + \mathrm{Quad}(EF) + \mathrm{Quad}(FC) = \mathrm{Quad}(EF) + \mathrm{Quad}(FD).$$

Now $\angle EFC$, $\angle EFD$ and $\angle EBD$ are right angles. It follows from Pythagoras's Theorem (I. 47) that

$$\begin{aligned}
\mathrm{Quad}(EF) + \mathrm{Quad}(FC) &= \mathrm{Quad}(EC),\\
\mathrm{Quad}(EF) + \mathrm{Quad}(FD) &= \mathrm{Quad}(ED),\\
\mathrm{Quad}(EB) + \mathrm{Quad}(BD) &= \mathrm{Quad}(ED).
\end{aligned}$$

It follows that

$$\mathrm{Rect}(AD, DC) + \mathrm{Quad}(EC) = \mathrm{Quad}(ED) = \mathrm{Quad}(EB) + \mathrm{Quad}(BD).$$

But $EB = EC$, because the points B and C lie on the circumference of a circle with centre E, and therefore $\mathrm{Quad}(EB) = \mathrm{Quad}(EC)$. It follows that

$$\mathrm{Rect}(AD, DC) = \mathrm{Quad}(BD).$$

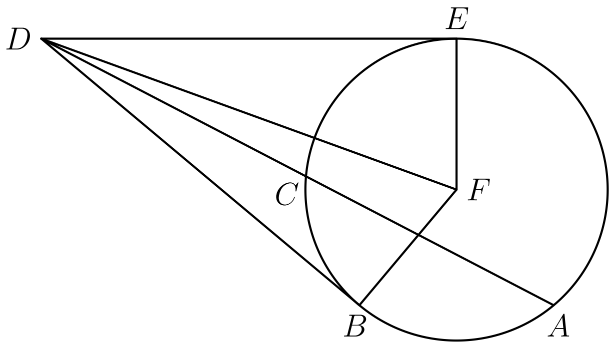
The required geometric equality has now been verified in the case in which the line ACD does not pass through the centre of the circle, and has therefore been verified in all required cases.

Commentary: Book III, Proposition 37

The proposition asserts that if a point D is taken outside a circle, if a line through D cut the circle in points A and C, where C lies between A and D, if a line joins D to a point B on the circle, and if

$$\mathrm{Quad}(DB) = \mathrm{Rect}(AD, DC),$$

then the line DB touches the circle at the point B.



As Sir Thomas L. Heath points out in his commentary on this proposition, it is not necessary, in proving this proposition, to construct the line DE touching the circle on the opposite side of the line DCA to the point B.

Indeed suppose that a line DHG through with endpoints D and G cuts the circle at points G and H. The preceding proposition, Proposition 36, ensures that

$$\mathrm{Rect}(GD, DH) = \mathrm{Rect}(AD, DC).$$

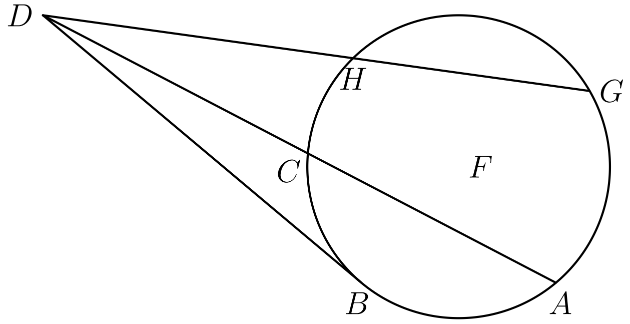
Consequently

$$\mathrm{Quad}(DH) < \mathrm{Rect}(GD, DH) = \mathrm{Rect}(AD, DC)$$

and

$$\mathrm{Quad}(DG) > \mathrm{Rect}(GD, DH) = \mathrm{Rect}(AD, DC)$$

It follows that if the point B is located such as to ensure that

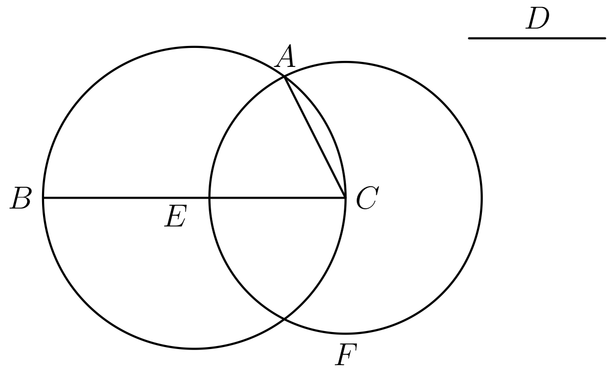


$$\mathrm{Quad}(DB) = \mathrm{Rect}(AD, DC)$$

then the point B cannot be located on a line through the point D that cuts the circle at two distinct points, and therefore the line DB joining D to B must touch the circle at the point B, as required.

Commentary: Book IV, Proposition 1

The problem discussed in this proposition is to construct a chord joining two points of a given circle ABC, where the chord is to be equal in length to a given straight line segment D that does not exceed in length the diameter of the given circle.



The construction is straightforward. Note that to perform the construction with straightedge and compasses, one would need to perform the construction set out in Proposition 2 of Book I of the *Elements of Geometry* in order to locate some point that can be joined to the point C by a line segment equal in length to the given line segment. Once such a point has been found, the circle centred on C can be drawn so as to pass through the point E on the diameter BC for which CE and D are equal in length.

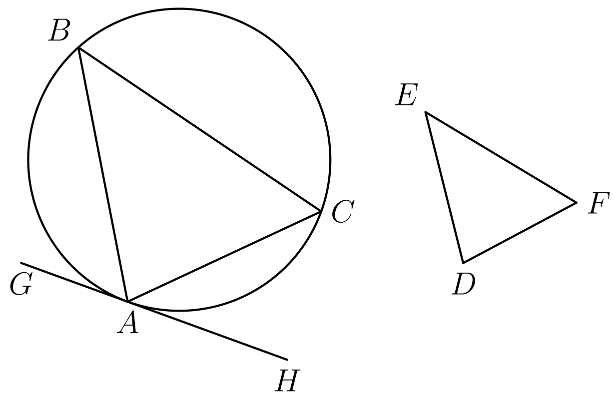
Commentary: Book IV, Proposition 2

The problem discussed in this proposition is that of inscribing, in a given circle, a triangle with the same angles as some given triangle. Thus, given a circle, and given a triangle DEF, we seek to construct a triangle ABC whose vertices lie on the given circle, where the angles of the triangle ABC at A, B and C are respectively equal to the angles of the triangle DEF at D, E and F respectively.

To achieve the construction, a line GH is drawn touching the given circle at some point A. This can be done by constructing a line through the point A that is at right angles to the line joining that point A to the centre of the circle (*Elements*, III.16, Porism). Then chords AB and AC of the given circle are drawn across the circle so as to ensure that

$$\angle HAC = \angle DEF \quad \text{and} \quad \angle GAB = \angle DFE.$$

The points B and C are then joined so as to construct a triangle ABC inscribed in the given circle.



Now the angle between a tangent line and a chord of a circle is equal to the angle in the alternate segment cut off by the chord (*Elements*, III.32). Accordingly

$$\angle HAC = \angle ABC \quad \text{and} \quad \angle GAB = \angle ACB.$$

Consequently

$$\angle ABC = \angle DEF \quad \text{and} \quad \angle ACB = \angle DFE.$$

Finally we note that

$$\angle BAC = \angle EDF.$$

because the interior angles of each of the triangles ABC and DEF add up to two right angles (*Elements*, I.32). The required construction has therefore been achieved.

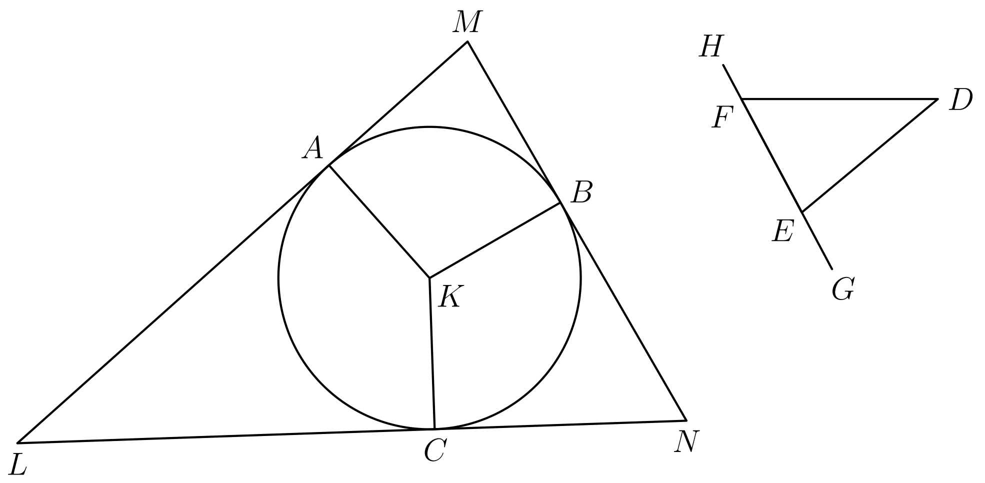
Commentary: Book IV, Proposition 3

This proposition discusses the problem of circumscribing, around a given circle, a triangle with the same angles as some given triangle. Thus, given a circle, and given a triangle DEF, we seek to construct a triangle ABC whose sides touch the given circle, where the angles of the triangle ABC at A, B and C are respectively equal to the angles of the triangle DEF at D, E and F respectively.

To achieve the construction, the side EF of the triangle DEF is produced in a straight line beyond E and F to points G and H respectively. A point B is chosen at random on the circle and is joined by a line segment to the centre K of the circle. Points B and C are then determined on the given circle, on opposite sides of the line BK, so as to ensure that

$$\angle BKA = \angle DEG \quad \text{and} \quad \angle BKC = \angle DFH$$

(*Elements*, I.23). Then lines LM, MN and NL are drawn, touching the given circle at points A, B and C respectively. These lines are determined so that LM and KA are at right angles, MN and KB are at right angles and NL and KC are at right angles (*Elements*, III.16, Porism). The points L, M and N are then determined so that the lines LM and NL intersect at the point L, the lines LM and MN intersect at the point M, and the lines MN and NL intersect at the point N.



The angles of any quadrilateral add up to four right angles. Moreover the angles KAM and KBM are right angles. Consequently

$$\angle AKB + \angle AMB = \text{two right angles.}$$

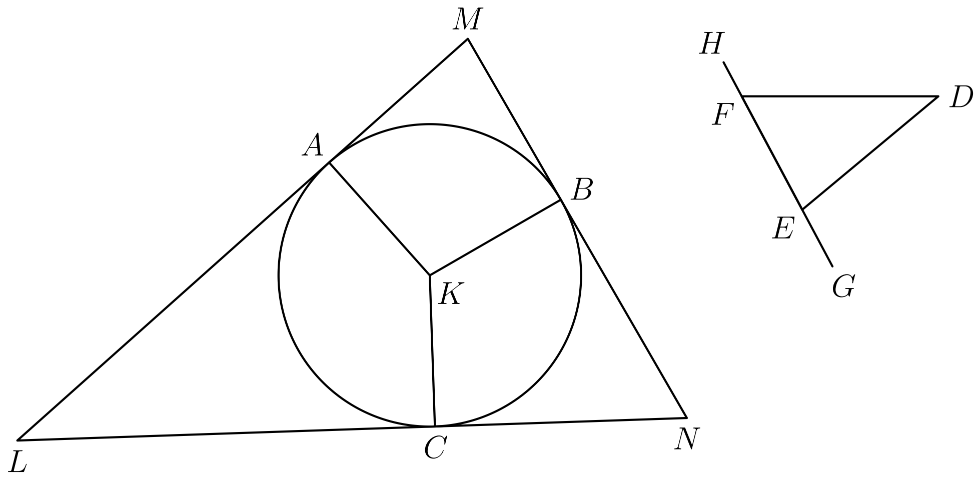
But $\angle AKB = \angle DEG$ and

$$\angle DEG + \angle DEF = \text{two right angles}$$

(*Elements*, I.13). Consequently

$$\angle LMN = \angle AMB = \angle DEF.$$

Thus the angles of the triangles DEF and LMN at the vertices M and E

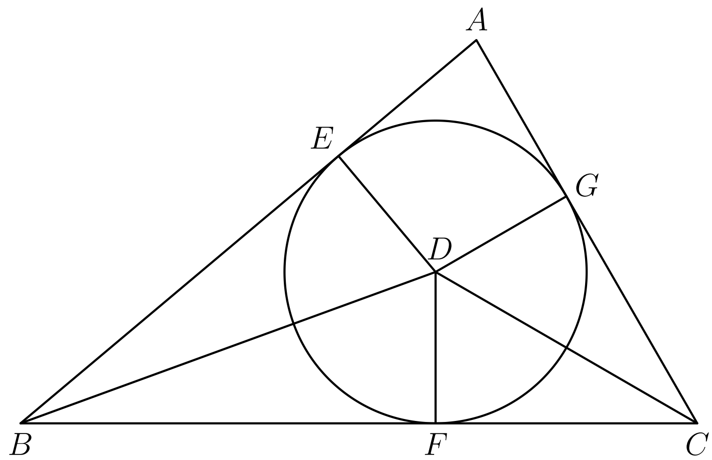


are equal. Similarly the angles of those triangles at the vertices N and F are equal, and the angles of those triangles at the vertices L and D are equal, as required.

Commentary: Book IV, Proposition 4

This proposition discusses the problem of inscribing a circle in a triangle so that the circle touches all three sides of the triangle.

Let a triangle ABC be given. To achieve the construction, the angles of the triangle at B and C are bisected by straight lines BD and CD that are produced so as to intersect at a point D inside the triangle. Straight lines DE, DF and DG are then drawn from the point D so as to intersect the sides AB, BC and CA of the triangle at the points E, F and G respectively. It can be shown that the straight line segments DE, DF and DG are equal in length. Accordingly a circle can be drawn passing through the points E, F and G. Moreover this circle will touch the sides of the triangle ABC at those points.



It remains to prove that the straight line segments DE, DF and DG are indeed equal in length. This can be established by applying that SAA Congruence Rule. The side DB is common to the two triangles EBD and FBD and the angles of the triangle EBD at E and B are respectively equal to the angles of the triangle FBD at F and B. Consequently the triangles EBD and FBD are congruent (*Elements*, I.26), and therefore $DE = DF$. Similarly the triangles FCD and GCD are congruent, and consequently $DF = DG$. Thus the three straight line segments DE, DF and DG are equal in length. Now, by construction, the sides of the triangle ABC intersect DE, DF and DG at right angles at the points E, F, G. Consequently the sides of the triangle ABC touch the circle EFG at those points (*Elements*, III.16, Porism).

Commentary: Book IV, Proposition 5

The problem discussed in this proposition is that of circumscribing a circle around a triangle so that the circle passes through all three vertices of the triangle.

The construction is standard and well-known, and is readily justified on applying the results proved in Propositions 1 and 3 in Book III of the *Elements of Geometry*.

Euclid divides the demonstration into three cases: but the essence of the construction is the same in all three cases, and thus the proof as presented by Euclid is, for this reason, repetitive.

In all the configurations identified by Euclid, the procedure, given the triangle ABC, is to construct the perpendicular bisectors DF and EF of the sides AB and AC of the triangle. The point at which these perpendicular bisectors intersect is then the centre of a circle that passes through all three vertices of the triangle ABC.

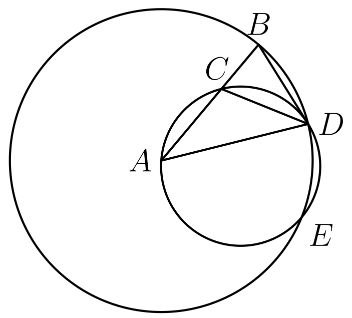
If it is regarded as appropriate to consider three cases separately the appropriate cases are the following: the case in which the point F at which the perpendicular bisectors intersect lies inside the triangle ABC; the case in which the point F at which the perpendicular bisectors intersect lies on one of the three sides of the triangle ABC; the case in which the point F at which the perpendicular bisectors intersect lies outside the triangle ABC. In the second of these cases, the vertices of the triangle may be relabelled, if necessary, so as to ensure that the point F lies on the side BC. In the third of these cases, the vertices of the triangle may be relabelled, if necessary, so as to ensure that the points A and F lie on opposite sides of the straight line BC.

When the vertices of the triangle are relabelled in the manner just described, if necessary, an immediate application of the results stated in Proposition 31 of Book III of the *Elements of Geometry* establishes that the angle BAC is acute in the first case, right in the second case, and obtuse in the third case.

Commentary: Book IV, Proposition 10

The objective of this proposition is to show that it is possible construct, using straightedge and compasses, an isosceles triangle in which the two equal angles are double the remaining angle. For such an isosceles triangle, five times the smallest angle will be equal to two right angles. Therefore, given ten such triangles, all congruent to one another, the triangles could be arranged with their smallest angles positioned at a single vertex, so as to form a regular decagon with centre located at the common vertex of the smallest angles of those triangles.

Euclid's construction of the isosceles triangle is as depicted in the following diagram in which ABD is an isosceles triangle, the sides AB and AD being equal in length, the point C is located on AC so as to ensure that the square on AC is equal in area to a rectangle contained by sides of lengths AD and DC, and in which BD is equal in length to AC.



Let a straight line segment AB be taken to serve as one of the two equal sides of the isosceles triangle. Proposition 11 of Book II of the *Elements of Geometry* sets out a straightedge and compasses construction for finding a point C on the straight line segment AB with the property that a square constructed with side AC is equal in area to a rectangle with containing sides equal to AB and BD. Then, in the symbolic notation employed in these commentaries,

$$\mathrm{Quad}(AC) = \mathrm{Rect}(AB, BC).$$

A point can then be located, by means of an appropriate straightedge and compasses construction, so that the straight line segment joining that point to the point B is equal in length to the straight line segment AC (see *Elements*, I.2). A circle passing through this point and centred on the point B will intersect the circle centred on A and passing through the point B at two points. Let D be one of those points of intersection. Then

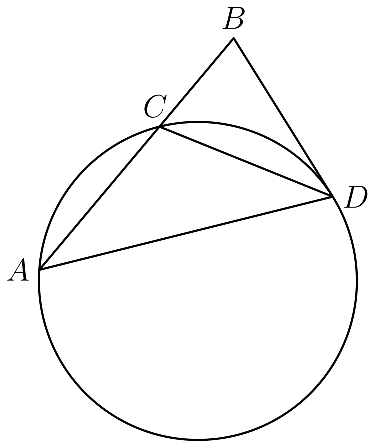
$$AD = AB \quad \text{and} \quad BD = AC.$$

The triangle ABD is then an isosceles triangle on the side BD of which is located a point C for which

$$\text{Quad}(BD) = \text{Quad}(AC) = \text{Rect}(AB, BC).$$

Euclid proves that the isosceles triangle ABD so constructed has the required properties.

Now Euclid has established that a straightedge and compasses construction can be used to circumscribe a circle around any triangle (*Elements*, IV.5). Accordingly let such a circle be circumscribed about the triangle ACD, as in the following diagram. Now the triangle ABD was constructed so as to



ensure that the square on BD is equal in area to a rectangle with containing sides equal to AD and DC. Proposition 37 in Book III of the *Elements of Geometry* then ensures that the line BD is a tangent line to the circle ACD at the point D. Then Proposition 32 in that book ensures that the angle BDC between the tangent line BD and the chord DC of the circle is equal to the angle DAC in the alternate segment. The latter angle is the same as the angle DAB. We have thus shown that

$$\angle BDC = \angle DAC = \angle ADB.$$

(Note that this equality has been established through the application of Propositions 32 and 37 of Book III of the *Elements of Geometry*. The results established in the first four books of the *Elements of Geometry* do not supply any alternative method for proving the equality of these two angles in a convenient fashion.)

Following Euclid, we now add the angle CDA to the equal angles BDC and DAC. We find that

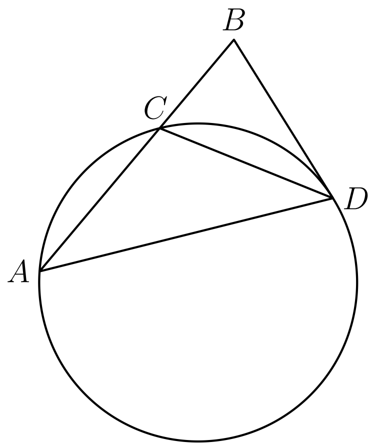
$$\angle BDA = \angle BDC + \angle CDA = \angle DAC + \angle CDA = \angle BCD,$$

because the exterior angle BCD of the triangle ACD is equal to sum of the interior and opposite angles of that triangle at the vertices A and D (*Elements*, I. 32). But

$$\angle BDA = \angle DBA,$$

because these two angles are the angles subtended by the equal sides AB and AD of the isosceles triangle ABD (*Elements*, I.5). We have now established that

$$\angle BDA = \angle DBA = \angle BCD.$$



The equality of the angles of the triangle BCD at C and D now ensures that that triangle is an isosceles triangle with equal sides DC and DB (*Elements*, I.6). But the construction of the triangle ensured that the line segments AC and BD are equal in length. Consequently

$$CA = BD = CD.$$

Thus the triangle CAD is isoceles, and therefore

$$\angle CDA = \angle DAC$$

(*Elements*, I.32). But, as previously noted

$$\angle DAC + \angle CDA = \angle BCD.$$

Consequently

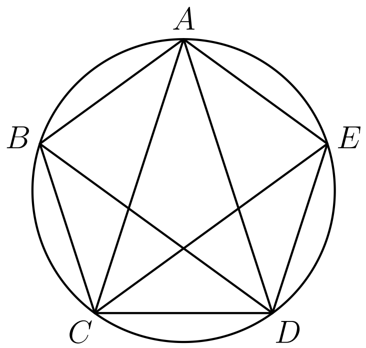
$$\angle BDA = \angle DBA = \angle BCD = 2 \times \angle DAB.$$

Thus an isoceles triangle ABD has indeed been constructed with the required properties.

Commentary: Book IV, Proposition 11

The problem discussed in this proposition is that of inscribing a regular pentagon inside a given circle. Specifically the problem is that of inscribing a pentagon inside a given circle that is both equilateral and equiangular. One therefore needs to prove both that the sides of the pentagon are equal in length and also that the interior angles of the pentagon are equal to one another.

The preceding proposition, Proposition 10, establishes that one can construct, with straightedge and compasses, an isosceles triangle in which the two equal angles are double the third angle. Applying Proposition 2 of this book, one can then inscribe an isosceles triangle ACD with these properties inside the given circle. The equal angles of this isosceles triangle at C and D are then bisected, and the bisecting straight lines are produced till they meet the circle at the points E and B.



We now analyse the geometry of the depicted configuration without following closely Euclid's argument. The isosceles triangle ACD has been constructed so as to ensure that

$$\angle ACD = \angle ADC = 2 \times \angle CAD.$$

Also

$$\angle ADB = \angle CDB = \angle ACE = \angle DCE = \angle CAD$$

because the straight lines CE and DB bisect the angles BCD and EDC. It follows that the angles in the larger segments cut off by the chords AB, BC, CD, DE and EA are equal to one another. Consequently the angles subtended by those chords at the centre of the circle are equal to one another (*Elements*, III.20). Applying the SAS Congruence Rule (*Elements*, I.4), it follows that the chords AB, BC, CD, DE and EA are equal in length. Thus the pentagon is equilateral. Furthermore the interior angle at each of the vertices A, C and D of the pentagon is composed of three angles of equal magnitude. Those angles are therefore equal to three times the angle CAD. Also

$$\angle ABD = \angle ACD = 2 \times \angle CAD \quad \text{and} \quad \angle AEC = \angle ADC = 2 \times \angle CAD,$$

and consequently the interior angles of the pentagon at each of the vertices B and E is equal to three times the angle CAD. The pentagon is therefore equiangular. Thus an equilateral and equiangular pentagon has indeed been inscribed in the given circle.