Module MAU23302: Euclidean and Non-Euclidean Geometry
Hilary Term 2022
Part II, (Sections 1 and 2)
Introduction to Non-Euclidean Geometry

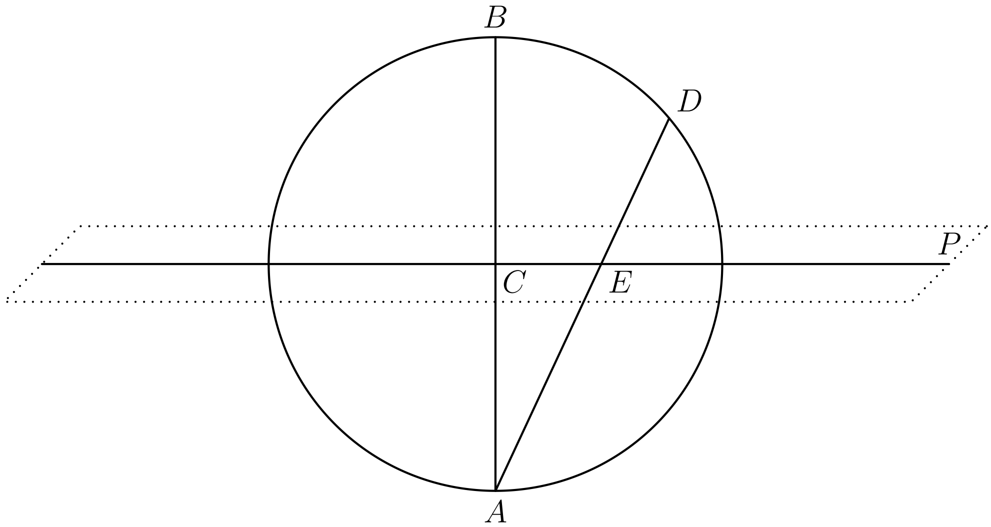
D. R. Wilkins

Contents

1 Möbius Transformations and Cross-Ratios

1.1 Stereographic Projection

Let a sphere in three-dimensional spaces be given, let C be the centre of that sphere, let AB be a diameter of that sphere with endpoints A and B, and let P be the plane through the centre of the sphere that is perpendicular to the diameter AB. Given a point D of the sphere distinct from the point A, the image of D under *stereographic projection* from the point A is defined to be the point E at which the line passing through the points A and D intersects the plane P.



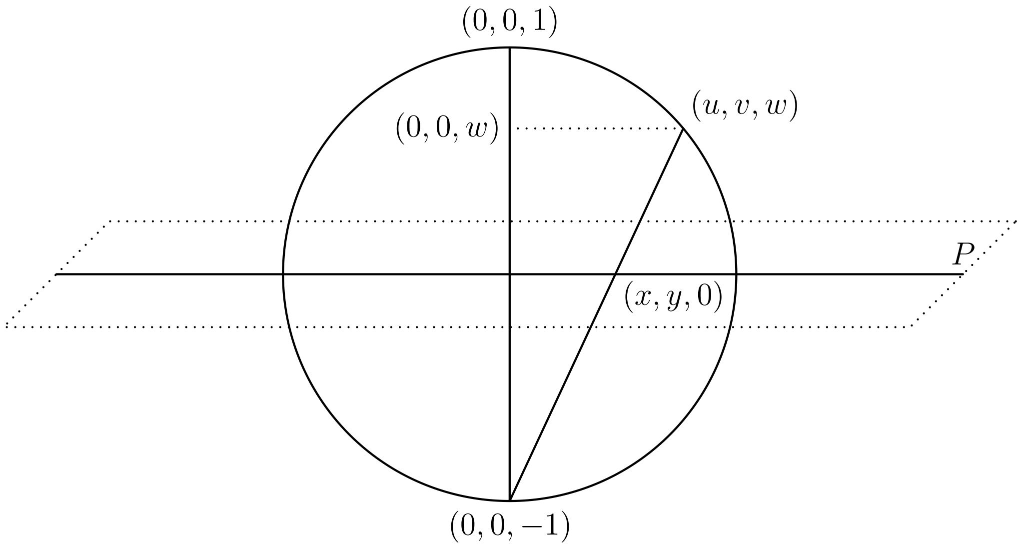
Proposition 1.1 *Let S^2 be the unit sphere in $\mathbb{R}^3$, consisting of those points (u, v, w) of $\mathbb{R}^3$ that satisfy the equation $u^2 + v^2 + w^2 = 1$, and let P be the plane consisting of those points (u, v, w) of $\mathbb{R}^3$ for which $w = 0$. Then, for each point (u, v, w) of S^2 distinct from the point $(0, 0, -1)$, the straight line passing through the points (u, v, w) and $(0, 0, -1)$ intersects the plane P at the point $(x, y, 0)$ at which*

$$x = \frac{u}{w+1} \quad \textit{and} \quad y = \frac{v}{w+1}.$$

Proof Let $A = (0, 0, -1)$, $D = (u, v, w)$ and $E = (x, y, 0)$. Then the displacements of the points D and E from the point A are represented by the vectors $(u, v, w + 1)$ and $(x, y, 1)$ respectively. These vectors are parallel because the points A, D and E are collinear. Consequently

$$\frac{x}{u} = \frac{y}{v} = \frac{1}{w+1}.$$

The result follows. ∎



Definition Let (u, v, w) be a point on the unit sphere distinct from the point $(0, 0, -1)$, where $u^2 + v^2 + w^2 = 1$, and let (x, y) be a point of the plane $\mathbb{R}^2$. We say that the point (x, y) is the *image* of the point (u, v, w) under *stereographic projection* from the point $(0, 0, -1)$ if

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Proposition 1.2 *Each point (x, y) of $\mathbb{R}^2$ is the image, under stereographic projection from the point $(0, 0, -1)$, of the point (u, v, w) of the unit sphere for which*

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \textit{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

This point (u, v, w) is distinct from the point $(0, 0, -1)$.

Proof Given a point (x, y) of $\mathbb{R}^2$, the straight line passing through the points $(0, 0, -1)$ and $(x, y, 0)$ is not tangent to the unit sphere, and therefore intersects the unit sphere at some point distinct from $(0, 0, -1)$. It follows that every point of $\mathbb{R}^2$ is the image, under stereographic projection from $(0, 0, -1)$, of some point of the unit sphere distinct from the point $(0, 0, -1)$.

Let (x, y) be the image, under stereographical projection from the point $(0, 0, -1)$, of a point (u, v, w), where $u^2 + v^2 + w^2 = 1$ and $w \neq -1$. Then

$$x = \frac{u}{w+1}, \quad y = \frac{v}{w+1}.$$

It follows that

$$x^2 + y^2 = \frac{u^2 + v^2}{(w+1)^2} = \frac{1 - w^2}{(w+1)^2} = \frac{1-w}{w+1}.$$

It follows that

$$w(x^2 + y^2) + x^2 + y^2 = 1 - w,$$

and therefore

$$w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

But then

$$1 + w = 1 + \frac{1 - x^2 - y^2}{1 + x^2 + y^2} = \frac{2}{1 + x^2 + y^2},$$

and therefore

$$\begin{aligned} u &= (1+w)x = \frac{2x}{1 + x^2 + y^2}, \\ v &= (1+w)y = \frac{2y}{1 + x^2 + y^2}. \end{aligned}$$

Conversely if

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

then

$$u^2 + v^2 + w^2 = \frac{4(x^2 + y^2) + (1 - x^2 - y^2)^2}{(1 + x^2 + y^2)^2} = 1,$$

because

$$\begin{aligned} &4(x^2 + y^2) + (1 - x^2 - y^2)^2 \\ &\qquad = 4(x^2 + y^2) + 1 - 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &\qquad = 1 + 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &\qquad = (1 + x^2 + y^2)^2. \end{aligned}$$

Also $w > -1$ and

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

The result follows. ∎

1.2 The Riemann Sphere

The *Riemann sphere* $\mathbb{P}^1$ may be defined as the set $\mathbb{C} \cup \{\infty\}$ obtained by augmenting the system $\mathbb{C}$ of complex numbers with an additional element, denoted by ∞, where ∞ is not itself a complex number, but is an additional element added to the set, with the additional conventions that

$$z + \infty = \infty, \quad \infty \times \infty = \infty, \quad \frac{z}{\infty} = 0 \quad \text{and} \quad \frac{\infty}{z} = \infty$$

for all complex numbers z, and

$$z \times \infty = \infty, \qquad \text{and} \quad \frac{z}{0} = \infty$$

for all non-zero complex numbers z. The symbol ∞ cannot be added to, or subtracted from, itself. Also 0 and ∞ cannot be divided by themselves.

Note that, because the sum of two elements of $\mathbb{P}^1$ is not defined for every single pair of elements of $\mathbb{P}^1$, this set cannot be regarded as constituting a group under the operation of addition. Similarly its non-zero elements cannot be regarded as constituting a group under multiplication. In particular, the Riemann sphere cannot be regarded as constituting a field.

The following proposition follows directly from Proposition 1.2.

Proposition 1.3 *Let $\varphi\colon \mathbb{P}^1 \to \mathbb{R}^3$ be the mapping from the Riemann sphere $\mathbb{P}^1$ to $\mathbb{R}^3$ defined such that $\varphi(\infty) = (0, 0, -1)$ and*

$$\varphi(x + y\sqrt{-1}) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right)$$

for all real numbers x and y. Then the map φ sets up a one-to-one correspondence between points of the Riemann sphere $\mathbb{P}^1$ and points of the unit sphere S^2 in $\mathbb{R}^3$. To each point of the Riemann sphere $\mathbb{P}^1$ there corresponds exactly one point of the unit sphere S^2 in three-dimensional Euclidean space, and vice versa. Moreover if (u, v, w) is a point of the unit sphere S^2 distinct from $(0, 0, -1)$ then $(u, v, w) = \varphi(x + y\sqrt{-1})$, where

$$x = \frac{u}{w + 1} \quad \textit{and} \quad y = \frac{v}{w + 1}.$$

1.3 Möbius Transformations

Definition Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$. The *Möbius transformation* $\mu_{a,b,c,d}\colon \mathbb{P}^1 \to \mathbb{P}^1$ with coefficients a, b, c and d is defined to be the function from the Riemann sphere $\mathbb{P}^1$ to itself determined by the following properties:

$$\mu_{a,b,c,d}(z) = \frac{az+b}{cz+d}$$

for all complex numbers z for which $cz + d \neq 0$; $\mu_{a,b,c,d}(-d/c) = \infty$ and $\mu_{a,b,c,d}(\infty) = a/c$ if $c \neq 0$; $\mu_{a,b,c,d}(\infty) = \infty$ if $c = 0$.

Note that the requirement in the above definition of a Möbius transformation that its coefficients a, b, c and d satisfy the condition $ad - bc \neq 0$ ensures that there is no complex number for which $az+b$ and $cz+d$ are both zero.

Let A be a non-singular 2×2 matrix whose coefficients are complex numbers, and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We denote by μ_A the Möbius transformation $\mu_{a,b,c,d}$ with coefficients a, b, c, d, defined so that

$$\begin{aligned} \mu_A(z) &= \begin{cases} \dfrac{az+b}{cz+d} & \text{if } cz+d \neq 0; \\ \infty & \text{if } c \neq 0 \text{ and } z = -d/c; \end{cases} \\ \mu_A(\infty) &= \begin{cases} \dfrac{a}{c} & \text{if } c \neq 0; \\ \infty & \text{if } c = 0. \end{cases} \end{aligned}$$

Lemma 1.4 *Let A be a non-singular 2×2 matrix with complex coefficients, and let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The corresponding Möbius transformation μ_A can then be characterized as the unique function mapping the Riemann sphere $\mathbb{P}^1$ to itself with the property that

$$\mu_A\left(\frac{u}{v}\right) = \frac{au+bv}{cu+dv}$$

for all complex numbers u and v that are not both zero (where $u/v = \infty$ in all cases, and in only those cases, where $u \neq 0$ and $v = 0$).

Proof Every point of the Riemann sphere may be expressed as a quotient of the form u/v, where u and v are complex numbers that are not both zero, and where $u/v = \infty$ in all cases, and in only those cases, where $u \neq 0$ and

$v = 0$. Let u, v, u' and v' are complex numbers, where u and v are not both zero, where u' and v' are not both zero, and where $u/v = u'/v'$. Then either v and v' are both non-zero or else $u/v = \infty$, in which case $v = v' = 0$. If v and v' are both non-zero then there exists a unique non-zero complex number w for which $v' = wv$, and then $u' = v'u/v = wu$. If $v = v' = 0$ then $u \neq 0$ and $u' \neq 0$, and then $u' = wu$ and $v' = wv$, where $w = u'/u$.

We conclude that, in all cases with u and v not both zero, u' and v' not both zero and $u/v = u'/v'$, there exists some non-zero complex number w such that $u' = wu$ and $v' = wv$. But then $au + bv$ and $cu + dv$ are not both zero, because the matrix A is non-singular, $au' + bv'$ and $cu' + dv'$ are not both zero, for the same reason, and

$$\frac{au' + bv'}{cu' + dv'} = \frac{w(au + bv)}{w(cu + dv)} = \frac{au + bv}{cu + dv}.$$

Consequently there exists a well-defined function $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$, mapping the Riemann sphere to itself, characterized by the property that

$$\mu\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v with the property that u and v are both zero.

Now if $v \neq 0$ and $z = u/v$ then

$$\mu(z) = \mu\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv} = \frac{azv + bv}{czv + dv} = \frac{az + b}{cz + d} = \mu_A(z).$$

On the other hand, if $v = 0$ then $u \neq 0$ and $u/v = \infty$, and therefore

$$\mu(\infty) = \mu\left(\frac{u}{v}\right) = \frac{au}{cu} = \frac{a}{c} = \mu_A(\infty).$$

We conclude therefore that $\mu = \mu_A$. The result follows. ∎

Proposition 1.5 *The composition of any two Möbius transformations is a Möbius transformation. Specifically let A and B be non-singular 2×2 matrices with complex coefficients, and let μ_A and μ_B be the corresponding Möbius transformations of the Riemann sphere. Then the composition $\mu_A \circ \mu_B$ of these Möbius transformations is the Möbius transformation μ_{AB} of the Riemann sphere determined by the product AB of the matrices A and B.*

Proof Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} f & g \\ h & k \end{pmatrix},$$

and let

$$AB = \begin{pmatrix} m & n \\ p & q \end{pmatrix}.$$

Then

$$m = af + bh, \quad n = ag + bk,$$

$$p = cf + dh \quad \text{and} \quad q = cg + dk.$$

Now let u and v be complex numbers that are not both zero. Then $fu+gv$ and $hu+kv$ are not both zero, because the matrix B is non-singular. Applying Lemma 1.4, we see that

$$\begin{aligned} \mu_A\left(\mu_B\left(\frac{u}{v}\right)\right) &= \mu_A\left(\frac{fu+gv}{hu+kv}\right) \\ &= \frac{a(fu+gv)+b(hu+kv)}{c(fu+gv)+d(hu+kv)} \\ &= \frac{mu+nv}{pu+qv} = \mu_{AB}\left(\frac{u}{v}\right). \end{aligned}$$

The result follows. ∎

Corollary 1.6 *Let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$, let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and let μ_A and μ_C be the corresponding Möbius transformations, defined so that

$$\mu_A\left(\frac{u}{v}\right) = \frac{au+bv}{cu+dv} \quad \text{and} \quad \mu_C(z) = \frac{du-bv}{-cu+av}$$

for all complex numbers u and v that are not both zero. Then the mapping $\mu_A\colon \mathbb{P}^1 \to \mathbb{P}^1$ is invertible, and its inverse is the Möbius transformation $\mu_C\colon \mathbb{P}^1 \to \mathbb{P}^1$.

Proof Let

$$M = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}.$$

Then $AC = CA = M$. It follows from Proposition 1.5 that

$$\mu_A \circ \mu_C = \mu_C \circ \mu_A = \mu_M = \mathrm{Id}_{\mathbb{P}^1},$$

where $\mathrm{Id}_{\mathbb{P}^1}$ denotes the identity map of the Riemann sphere. The result follows. ∎

1.4 Inversion of the Riemann Sphere in its Equatorial Circle

Let S^2 denote the unit sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\},$$

and let us refer to the points $(0, 0, 1)$ and $(0, 0, -1)$ as the *North Pole* and *South Pole* respectively. Let E denote the *Equatorial Plane* in $\mathbb{R}^3$, consisting of those points whose Cartesian coordinates are of the form $(x, y, 0)$, where x and y are real numbers.

Stereographic projection from the South Pole maps each point (u, v, w) of the unit sphere S^2 distinct from the South Pole to the point $(x, y, 0)$ of the equatorial plane E for which

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Moreover a point $(x, y, 0)$ of the Equatorial Plane E is the image under stereographic projection from the South Pole of the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2}, \quad w = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

We can also stereographically project from the North Pole. Note that, given a point in the Equatorial Plane, reflection in that Equatorial Plane will interchange the points of the sphere corresponding to it under stereographic projection from the North and South Poles. Thus a point (u, v, w) of the unit sphere S^2 distinct from the North Pole corresponds under stereographic projection to the point $(x, y, 0)$ of the Equatorial Plane E for which

$$x = \frac{u}{1-w} \quad \text{and} \quad y = \frac{v}{1-w}.$$

In the other direction, a point $(x, y, 0)$ of the Equatorial Plane E corresponds under stereographic projection from the North Pole to the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2}, \quad w = \frac{x^2+y^2-1}{1+x^2+y^2}.$$

Proposition 1.7 *Let O denote the origin $(0, 0, 0)$ of the Equatorial Plane E, where*

$$E = \{(x, y, z) \in \mathbb{R}^3 : z = 0\},$$

and let A be a point $(x, y, 0)$ of E distinct from the origin O. Let C be the point on the unit sphere S^2 that corresponds to A under stereographic projection from the North Pole $(0, 0, 1)$, and let B be the point of the Equatorial Plane E that corresponds to C under stereographic projection from the South Pole. Then $B = (p, q, 0)$, where

$$p = \frac{x}{x^2 + y^2} \quad \textit{and} \quad q = \frac{y}{x^2 + y^2}.$$

Thus the points O, A and B are collinear, and the points A and B lie on the same side of the origin O. Also the distances $|OA|$ and $|OB|$ of the points A and B from the origin satisfy $|OA| \times |OB| = 1$.

Proof Let $(x, y, 0)$ be a point of the Equatorial plane E distinct from the origin. This point is the image, under stereographic projection from the North Pole $(0, 0, 1)$ of the point (u, v, w) of the unit sphere S^2 for which

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2}, \quad w = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}.$$

This point then gets mapped under stereographic projection from the South Pole to the point $(p, q, 0)$ of the Equatorial Plane E for which

$$p = \frac{u}{w + 1} \quad \text{and} \quad q = \frac{v}{w + 1}.$$

Now

$$w + 1 = \frac{2(x^2 + y^2)}{1 + x^2 + y^2}.$$

It follows that

$$p = \frac{x}{x^2 + y^2} \quad \text{and} \quad q = \frac{y}{x^2 + y^2}.$$

Finally we note that O, A and B are collinear, where $0 = (0, 0, 0)$, $A = (x, y, 0)$ and $B = (p, q, 0)$, and the points A and B lie on the same side of the origin O. Also

$$|OA| = \sqrt{x^2 + y^2}, \quad \text{and} \quad |OB| = \frac{1}{\sqrt{x^2 + y^2}},$$

and therefore $|OA| \times |OB| = 1$, as required. ∎

1.5 The Action of Möbius Transformations on the Riemann Sphere

Proposition 1.8 *Let p_1, p_2, p_3 be distinct points of the Riemann sphere $\mathbb{P}^1$, and let q_1, q_2, q_3 also be distinct points of $\mathbb{P}^1$. Then there exists a unique Möbius transformation $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(p_j) = q_j$ for $j = 1, 2, 3$.*

Proof First we show that, given distinct points p_1, p_2 and p_3 of the Riemann sphere, there exists a Möbius transformation $\mu^*_{p_1,p_2,p_3}: \mathbb{P}^1 \to \mathbb{P}^1$ with the property that $\mu^*_{p_1,p_2,p_3}(p_1) = \infty$, $\mu^*_{p_1,p_2,p_3}(p_2) = 0$ and $\mu^*_{p_1,p_2,p_3}(p_3) = 1$. Now there exist complex numbers u_j and v_j for $j = 1, 2, 3$ such that u_j and v_j are not both zero and $u_j/v_j = p_j$ for $j = 1, 2, 3$. Then $u_1v_3 - u_3v_1$ and $u_2v_3 - u_3v_2$ are non-zero, because the points p_1, p_2 and p_3 of the Riemann sphere are specified to be distinct.

Also let u and v be complex numbers that are not both zero. Were it the case that

$$u_1v - uv_1 = u_2v - uv_2 = 0$$

then the point u/v of the Riemann sphere would coincide with both p_1 and p_2, which is impossible, given that p_1 and p_2 are specified to be distinct.

We conclude therefore that, for distinct points p_1, p_2, p_3 of the Riemann sphere, and for any complex numbers u and v that are not both zero, the complex numbers

$$(u_1v_3 - u_3v_1)(u_2v - uv_2) \quad \text{and} \quad (u_2v_3 - u_3v_2)(u_1v - uv_1)$$

are not both zero, and consequently there is a well-defined element $\mu^*_{p_1,p_2,p_3}(u/v)$ of the Riemann sphere characterized by the property that

$$\mu^*_{p_1,p_2,p_3}\left(\frac{u}{v}\right) = \frac{(u_1v_3 - u_3v_1)(u_2v - uv_2)}{(u_2v_3 - u_3v_2)(u_1v - uv_1)}$$

for all complex numbers u and v that are not both zero. Then the function sending u/v to $\mu^*_{p_1,p_2,p_3}(u/v)$ for all complex numbers u and v that are not both zero is a Möbius transformation of the Riemann sphere. Moreover

$$\mu^*_{p_1,p_2,p_3}(p_1) = \infty, \quad \mu^*_{p_1,p_2,p_3}(p_2) = 0 \quad \text{and} \quad \mu^*_{p_1,p_2,p_3}(p_3) = 1.$$

Now let p_1, p_2 and p_3 be distinct points of the Riemann sphere and also let q_1, q_2 and q_3 be distinct points of the Riemann sphere. Then there exist Möbius transformations $\mu^*_{p_1,p_2,p_3}: \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu^*_{q_1,q_2,q_3}: \mathbb{P}^1 \to \mathbb{P}^1$ characterized by the properties that

$$\mu^*_{p_1,p_2,p_3}(p_1) = \infty, \quad \mu^*_{p_1,p_2,p_3}(p_2) = 0, \quad \mu^*_{p_1,p_2,p_3}(p_3) = 1,$$

$$\mu^*_{q_1,q_2,q_3}(q_1) = \infty, \quad \mu^*_{q_1,q_2,q_3}(q_2) = 0 \quad \text{and} \quad \mu^*_{q_1,q_2,q_3}(q_3) = 1.$$

Let $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that

$$\mu = \mu^{*-1}_{q_1,q_2,q_3} \circ \mu^*_{p_1,p_2,p_3}.$$

Then

$$\mu(p_1) = q_1, \quad \mu(p_2) = q_2 \quad \text{and} \quad \mu(p_3) = q_3.$$

Now suppose let $\hat{\mu}: \mathbb{P}^1 \to \mathbb{P}^1$ be any Möbius transformation of the Riemann sphere with the properties that

$$\hat{\mu}(p_1) = q_1, \quad \hat{\mu}(p_2) = q_2 \quad \text{and} \quad \hat{\mu}(p_3) = q_3,$$

and let $\sigma: \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that

$$\sigma = \mu^*_{q_1,q_2,q_3} \circ \hat{\mu} \circ \mu^{*-1}_{p_1,p_2,p_3}.$$

Then $\sigma(\infty) = \infty$, $\sigma(0) = 0$ and $\sigma(1) = 1$. There then exist complex coefficients a, b, c and d, where $ad - bc \neq 0$, such that

$$\sigma\left(\frac{u}{v}\right) = \frac{au + bv}{cu + dv}$$

for all complex numbers u and v that are not both zero. Evaluating the Möbius transformation σ at the points ∞, 0 and 1 of the Riemann sphere, we find that

$$\frac{a}{c} = \infty, \quad \frac{b}{d} = 0 \quad \text{and} \quad \frac{a+b}{c+d} = 1.$$

Consequently $c = 0$, $a \neq 0$, $b = 0$, $d \neq 0$ and $a = d$. It follows that σ is the identity map of the Riemann sphere, and therefore

$$\hat{\mu} = \mu^{*-1}_{q_1,q_2,q_3} \circ \mu^*_{p_1,p_2,p_3} = \mu.$$

We conclude therefore that μ is the unique Möbius transformation of the Riemann sphere with the properties that $\mu(p_j) = q_j$ for $j = 1, 2, 3$, as required. ∎

Proposition 1.9 *Let p_1, p_2 and p_3 be three distinct points of the Riemann sphere, and let μ_1 and μ_2 be Möbius transformations of the Riemann sphere. Suppose that $\mu_1(p_j) = \mu_2(p_j)$ for $j = 1, 2, 3$. Then the Möbius transformations μ_1 and μ_2 coincide.*

Proof Let $q_j = \mu_1(p_j)$ for $j = 1, 2, 3$. Then both μ_1 and μ_2 must be identical to the unique Möbius transformation of the Riemann sphere that maps p_1, p_2 and p_3 to q_1, q_2 and q_3 respectively, and therefore μ_1 and μ_2 must be identical to one another, as required. ∎

Proposition 1.10 *Let a, b, c, d, f, g, h and k be complex numbers satisfying $ad \neq bc$ and $fk \neq gh$, and let μ_1 and μ_2 be the Möbius transformations of the Riemann sphere defined so that*

$$\mu_1(z) = \frac{az+b}{cz+d}, \quad \mu_2(z) = \frac{fz+g}{hz+k}$$

for all complex numbers with $cz + d \neq 0$ and $hz_2 + k \neq 0$. Then the Möbius transformations μ_1 and μ_2 coincide if and only if there exists some non-zero complex number m such that $f = ma$, $g = mb$, $h = mc$ and $k = md$.

Proof Clearly if there exists a complex number m with the stated properties then the Möbius transformations μ_1 and μ_2 coincide.

Conversely suppose that there is some Möbius transformation μ of the Riemann sphere with the property that

$$\mu(z) = \frac{az+b}{cz+d} = \frac{fz+g}{hz+k}$$

whenever $cz + d \neq 0$ and $hz + k \neq 0$.

First consider the case when $c = 0$. Then no real number is mapped by μ to the point ∞ of the Riemann sphere "at infinity" and therefore $h = 0$. But then $d \neq 0$, $k \neq 0$, $b/d = g/k$ and $a/d = f/k$. Therefore if we take $m = k/d$ in this case we find that $m \neq 0$, $f = ma$, $g = mb$, $h = mc$ and $k = md$. The existence of the required non-zero complex number m has therefore been verified in the case when $c = 0$.

Suppose then that $c \neq 0$. Then $h \neq 0$ and $\mu(-k/h) = \infty = \mu(-d/c)$, and therefore $k/h = d/c$. Let $m = h/c$. Then $k/d = m$. It then follows that

$$fz + g = (hz + k)\mu(z) = m(cz + d)\mu(z) = m(az + b)$$

for all complex numbers z distinct from $-d/c$, and therefore $f = ma$ and $g = mb$. The result follows. ∎

Proposition 1.11 *Any Möbius transformation of the Riemann sphere maps straight lines and circles to straight lines and circles.*

Proof The equation of a line or circle in the complex plane can be expressed in the form

$$g|z|^2 + 2\mathrm{Re}[\bar{b}z] + h = 0,$$

where g and h are real numbers, and b is a complex number. Moreover a locus of points in the complex plane satisfying an equation of this form is a circle if $g \neq 0$ and is a line if $g = 0$.

Let g and h be real constants, let b be a complex constant, and let $z = 1/w$, where $w \neq 0$ and w satisfies the equation

$$g|w|^2 + 2\mathrm{Re}[\overline{b}w] + h = 0,$$

Then

$$g|w|^2 + \overline{b}w + b\overline{w} + h = 0,$$

Then

$$\begin{aligned} g + \mathrm{Re}[bz] + h|z|^2 &= g + \overline{b}\overline{z} + bz + h|z|^2 \\ &= \frac{1}{|w|^2}\left(g|w|^2 + \overline{b}w + b\overline{w} + h\right) = 0. \end{aligned}$$

We deduce from this that the Möbius transformation that sends z to $1/z$ for all non-zero complex numbers z maps lines and circles to lines and circles.

Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation of the Riemann sphere. Then there exist complex numbers a, b, c and d satisfying $ad - bc \neq 0$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. The result is immediate when $c = 0$. We therefore suppose that $c \neq 0$. Then

$$\mu(z) = \frac{az + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c} \times \frac{1}{cz + d}$$

when $cz + d \neq 0$. The Möbius transformation μ is thus the composition of three maps that each send circles and straight lines to circles and straight lines and preserve angles between lines and circles, namely the maps

$$z \mapsto cz + d, \quad z \mapsto \frac{1}{z} \quad \text{and} \quad z \mapsto \frac{a}{c} - \frac{(ad - bc)z}{c}.$$

Thus the Möbius transformation μ must itself map circles and straight lines to circles and straight lines, as required. ∎

1.6 Cross-Ratios of Points of the Riemann Sphere

Definition The *cross-ratio* $(z_1, z_2; z_3, z_4)$ of four distinct complex numbers z_1, z_2, z_3 and z_4 is defined so that

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

We now extend the definition of cross-ratio so that, given any quadruple p_1, p_2, p_3, p_4 of points of the Riemann sphere satisfying the condition that no three of the points all coincide with one another, a corresponding point $(p_1, p_2; p_3, p_4)$ of the Riemann sphere is determined to represent the cross-ratio of the points p_1, p_2, p_3 and p_4.

Proposition 1.12 *There is a well-defined function, defined on quadruples p_1, p_2, p_3, p_4 of points of the Riemann sphere that satisfy the condition that no three of the members of the quadruple all coincide with one another, and sending such a quadruple p_1, p_2, p_3, p_4 to the point $(p_1, p_2; p_3, p_4)$ of the Riemann sphere characterized by the property that*

$$(p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

for all complex numbers u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 that are such as to ensure that u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$. The function defined in this fashion generalizes the definition of cross-ratio previously given for quadruples of distinct complex numbers.

Proof Let p_1, p_2, p_3, p_4 be a quadruple of points of the Riemann sphere. Then, for each integer j between 1 and 4, complex numbers u_j and v_j can be chosen, not both zero, such that $p_j = u_j/v_j$, where $u_j/v_j = \infty$ in cases where $u_j \neq 0$ and $v_j = 0$. Moreover, $p_j = p_k$, where j and k are integers between 1 and 4, if and only if $u_j v_k - u_k v_j = 0$.

Now if the points p_1, p_2, p_3, p_4 and ∞ are all distinct (so that p_1, p_2, p_3 and p_4 are distinct complex numbers), then v_1, v_2, v_3, v_4 are all non-zero, and also

$$(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1) \neq 0,$$

and, in this case, the definition of cross-ratios of distinct complex numbers requires that

$$\begin{aligned}(p_1, p_2; p_3, p_4) &= \frac{\left(\dfrac{u_1}{v_1} - \dfrac{u_3}{v_3}\right)\left(\dfrac{u_2}{v_2} - \dfrac{u_4}{v_4}\right)}{\left(\dfrac{u_2}{v_2} - \dfrac{u_3}{v_3}\right)\left(\dfrac{u_1}{v_1} - \dfrac{u_4}{v_4}\right)} \\ &= \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)} \\ &= \frac{u}{v}\end{aligned}$$

where

$$u = (u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)$$

and

$$v = (u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1),$$

and where $u/v = \infty$ in cases where $u \neq 0$ and $v = 0$.

Now suppose that p_1, p_2, p_3, p_4 are any points of the Riemann sphere that satisfy the requirement that no three of the listed points all coincide with one another. Suppose also that, for each integer j between 1 and 4, u_j, v_j, u'_j and v'_j are complex numbers, u_j and v_j and not both zero, u'_j and v'_j are not both zero, and

$$p_j = u_j/v_j = u'_j/v'_j.$$

Then there exist non-zero complex numbers w_1, w_2, w_3 and w_4 such that $u'_j = w_j u_j$ and $v'_j = w_j v_j$ for $j = 1, 2, 3, 4$. Let

$$u = (u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2),$$

$$v = (u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1),$$

$$u' = (u'_1 v'_3 - u'_3 v'_1)(u'_2 v'_4 - u'_4 v'_2)$$

and

$$v' = (u'_2 v'_3 - u'_3 v'_2)(u'_1 v'_4 - u'_4 v'_1).$$

Then $u' = w_1 w_2 w_3 w_4 u$ and $v' = w_1 w_2 w_3 w_4 v$, and therefore $u'/v' = u/v$. Moreover the requirement that no three of the points p_1, p_2, p_3, p_4 all coincide with one another ensures that the complex numbers u and v are not both zero. Indeed if it were the case that $u = v = 0$, then at least one of the following four conditions would need to hold:

- $u_1 v_3 - u_3 v_1 = 0$ and $u_2 v_3 - u_3 v_2 = 0$;
- $u_1 v_3 - u_3 v_1 = 0$ and $u_1 v_4 - u_4 v_1 = 0$;
- $u_2 v_4 - u_4 v_2 = 0$ and $u_2 v_3 - u_3 v_2 = 0$;
- $u_2 v_4 - u_4 v_2 = 0$ and $u_1 v_4 - u_4 v_1 = 0$.

in the first case we would have $p_1 = p_2 = p_3$; in the second $p_1 = p_3 = p_4$; in the third $p_2 = p_3 = p_4$; and in the fourth $p_1 = p_2 = p_4$.

Accordingly, given points p_1, p_2, p_3 and p_4 of the Riemann sphere $\mathbb{P}^1$, where no three of these points all coincide with one another, the quadruple of points p_1, p_2, p_3, p_4 determines a point $(p_1, p_2; p_3, p_4)$ of the Riemann sphere characterized by the property that, given any complex numbers u_j and v_j with the properties that u_j and v_j are not both zero and $p_j = u_j/v_j$ for

$j = 1, 2, 3, 4$, the point $(p_1, p_2; p_3, p_4)$ of the Riemann sphere is determined so that

$$(p_1, p_2; p_3, p_4) = u/v,$$

where

$$u = (u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)$$

and

$$v = (u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1),$$

and where $u/v = \infty$ in cases where $u \neq 0$ and $v = 0$. This completes the proof. ∎

Accordingly we can define the cross-ratio of appropriate quadruples of points of the Riemann sphere in the following manner.

Definition The cross-ratio of points of the Riemann sphere assigns points $(p_1, p_2; p_3, p_4)$ of the Riemann sphere to those those quadruples p_1, p_2, p_3, p_4 of points of the Riemann sphere for which no three points all coincide with one another, so as to ensure that, given complex numbers u_1, v_1, u_2, v_2, u_3, v_3, u_4 and v_4, where u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$, and where no three of the points p_1, p_2, p_3, p_4 all coincide with one another, the cross-ratio of those points is determined so that

$$(p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

We now show that, given four elements p_1, p_2, p_3, p_4 of the Riemann sphere satisfying the condition that no three of the points all coincide with one another, the value of the cross-ratio $(p_1, p_2; p_3, p_4)$ taken with respect to any one particular ordering of those four elements determines the value of the cross-ratio taken with respect to any other ordering of those elements.

Proposition 1.13 *Let p_1, p_2, p_3 and p_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let $q = (p_1, p_2; p_3, p_4)$. Then*

- $(p_1, p_2; p_3, p_4)$, $(p_2, p_1; p_4, p_3)$, $(p_3, p_4; p_1, p_2)$, $(p_4, p_3; p_2, p_1)$ *are all equal to q;*

- $(p_1, p_2; p_4, p_3)$, $(p_2, p_1; p_3, p_4)$, $(p_4, p_3; p_1, p_2)$, $(p_3, p_4; p_2, p_1)$ *are all equal to $\dfrac{1}{q}$.*

- $(p_1, p_3; p_2, p_4)$, $(p_3, p_1; p_4, p_2)$, $(p_2, p_4; p_1, p_3)$, $(p_4, p_2; p_3, p_1)$ *are all equal to* $1 - q$*;*

- $(p_1, p_4; p_2, p_3)$, $(p_4, p_1; p_3, p_2)$, $(p_2, p_3; p_1, p_4)$, $(p_3, p_2; p_4, p_1)$ *are all equal to* $\dfrac{q-1}{q}$*;*

- $(p_1, p_3; p_4, p_2)$, $(p_3, p_1; p_2, p_4)$, $(p_4, p_2; p_1, p_3)$, $(p_2, p_4; p_3, p_1)$ *are all equal to* $\dfrac{1}{1-q}$*;*

- $(p_1, p_4; p_3, p_2)$, $(p_4, p_1; p_2, p_3)$, $(p_3, p_2; p_1, p_4)$, $(p_2, p_3; p_4, p_1)$ *are all equal to* $\dfrac{q}{q-1}$*;*

Proof Let u_1, v_1, u_2, v_2, u_3, v_3, u_4 and v_4 be complex numbers with the properties that u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$ (where $u_j/v_j = \infty$ in cases where $u_j \neq 0$ and $v_j = 0$). Then

$$q = (p_1, p_2; p_3, p_4) = \frac{(u_1v_3 - u_3v_1)(u_2v_4 - u_4v_2)}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)}.$$

It follows directly that

$$(p_1, p_2; p_3, p_4), \quad (p_2, p_1; p_4, p_3), \quad (p_3, p_4; p_1, p_2) \text{ and } (p_4, p_3; p_2, p_1)$$

are all equal to q. Also

$$(p_1, p_2; p_4, p_3) = \frac{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)}{(u_1v_3 - u_3v_1)(u_2v_4 - u_4v_2)} = \frac{1}{q}.$$

Next we note that

$$(p_4, p_2; p_3, p_1) = \frac{(u_4v_3 - u_3v_4)(u_2v_1 - u_1v_2)}{(u_2v_3 - u_3v_2)(u_4v_1 - u_1v_4)}.$$

It follows that

$$\begin{aligned} &1 - (p_4, p_2; p_3, p_1) \\ &\quad = \frac{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1) + (u_4v_3 - u_3v_4)(u_2v_1 - u_1v_2)}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{u_1u_2v_3v_4 - v_1u_2v_3u_4 - u_1v_2u_3v_4 + v_1v_2u_3u_4}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} \\
&\qquad + \frac{v_1u_2v_3u_4 - v_1u_2u_3v_4 - u_1v_2v_3u_4 + u_1v_2u_3v_4}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} \\
&= \frac{u_1u_2v_3v_4 + v_1v_2u_3u_4 - v_1u_2u_3v_4 - u_1v_2v_3u_4}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} \\
&= \frac{(u_1v_3 - u_3v_1)(u_2v_4 - u_4v_2)}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} \\
&= q.
\end{aligned}$$

Consequently

$$(p_4, p_2; p_3, p_1) = 1 - q.$$

It then follows that

$$(p_4, p_2; p_1, p_3) = \frac{1}{1-q}.$$

Furthermore

$$(p_3, p_2; p_1, p_4) = 1 - (p_4, p_2; p_1, p_3) = 1 - \frac{1}{1-q} = \frac{q}{q-1},$$

and therefore

$$(p_3, p_2; p_4, p_1) = \frac{q-1}{q}.$$

The remaining identities follow directly. ∎

Lemma 1.14 *Let z_1, z_2 and, z_3 be distinct complex numbers. Then*

$$(z_1, z_2; z_3, \infty) = \frac{z_1 - z_3}{z_2 - z_3}$$

Proof Let $u_1 = z_1$, $u_2 = z_2$, $u_3 = z_3$, $u_4 = 1$, $v_1 = v_2 = v_3 = 1$ and $v_4 = 0$. Then $z_j = u_j/v_j$ for $j = 1, 2, 3$ and $\infty = u_4/v_4$. It follows from the definition of cross-ratios that

$$(z_1, z_2; z_3, \infty) = \frac{(u_1v_3 - u_3v_1)(u_2v_4 - u_4v_2)}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} = \frac{z_1 - z_3}{z_2 - z_3},$$

as required. ∎

Lemma 1.15 *Let p_1, p_2, p_3, p_4 be a quadruple of points of the Riemann sphere satisfying the condition that no three of the points all coincide with one another. Then the following identities hold when two of the points coincide with one another:*

$$(p_1, p_2; p_3, p_4) = \infty \text{ whenever } p_2 = p_3 \text{ or } p_1 = p_4;$$

$$(p_1, p_2; p_3, p_4) = 0 \text{ whenever } p_1 = p_3 \text{ or } p_2 = p_4;$$

$$(p_1, p_2; p_3, p_4) = 1 \text{ whenever } p_1 = p_2 \text{ or } p_3 = p_4.$$

Proof Let complex numbers u_j and v_j be chosen for $j = 1, 2, 3, 4$ such that u_j and v_j are not both zero and $p_j = u_j/v_j$ for $j = 1, 2, 3, 4$. The definition of cross-ratios ensures that

$$(p_1, p_2; p_3, p_4) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v_4 - u_4 v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v_4 - u_4 v_1)}.$$

Now, for distinct integers j and k between 1 and 4, $p_j = p_k$ if and only if $u_j v_k = u_k v_j$. Also there exists a non-zero complex number w for which $u_2 = w u_1$ and $v_2 = w v_1$ if and only if $p_1 = p_2$, and there exists a non-zero complex number w for which $u_4 = w u_3$ and $v_4 = w v_3$ if and only if $p_3 = p_4$. The required identities therefore follow directly. ∎

Lemma 1.16 *Let p_1, p_2 and p_3 be distinct elements of the Riemann sphere, and let $\mu^*_{p_1,p_2,p_3}\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the unique Möbius transformation of the Riemann sphere for which $\mu^*_{p_1,p_2,p_3}(p_1) = \infty$, $\mu^*_{p_1,p_2,p_3}(p_2) = 0$ and $\mu^*_{p_1,p_2,p_3}(p_3) = 1$. Then*

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p)$$

for all points p of the Riemann sphere.

Proof The Möbius transformation $\mu^*_{p_1,p_2,p_3}$ is characterized by the property that

$$\mu^*_{p_1,p_2,p_3}\left(\frac{u}{v}\right) = \frac{(u_1 v_3 - u_3 v_1)(u_2 v - u v_2)}{(u_2 v_3 - u_3 v_2)(u_1 v - u v_1)}$$

for all complex numbers u and v that are not both zero (as noted in the proof of Proposition 1.8). The result therefore follows on comparing this expression characterizing the Möbius transformation $\mu^*_{p_1,p_2,p_3}$ with the definition of cross-ratios of quadruples of points on the Riemann sphere. ∎

Proposition 1.17 *Let p_1, p_2 and p_3 be distinct elements of the Riemann sphere, and let q be a point of the Riemann sphere. Then there exists a unique element p_4 of the Riemann sphere for which $(p_1, p_2; p_3, p_4) = q$.*

Proof Möbius transformations of the Riemann sphere are invertible functions from the Riemann sphere to itself (see Corollary 1.6). Let $p_4 = \mu^{*-1}_{p_1,p_2,p_3}(q)$, where $\mu^*_{p_1,p_2,p_3}$ denotes the unique Möbius transformation of the Riemann sphere for which

$$\mu^*_{p_1,p_2,p_3}(p_1) = \infty, \quad \mu^*_{p_1,p_2,p_3}(p_2) = 0 \quad \text{and} \quad \mu^*_{p_1,p_2,p_3}(p_3) = 1.$$

It then follows (applying the identity established in Lemma 1.16) that

$$q = \mu^*_{p_1,p_2,p_3}(p_4) = (p_1, p_2; p_3, p_4),$$

as required. ∎

Proposition 1.18 *Let p_1, p_2, p_3, p_4 be distinct elements of the Riemann sphere $\mathbb{P}^1$, and let q_1, q_2, q_3, q_4 also be distinct elements of $\mathbb{P}^1$. Then a necessary and sufficient condition for the existence of a Möbius transformation $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere with the property that $\mu(p_j) = q_j$ for $j = 1, 2, 3, 4$ is that*

$$(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4).$$

Proof Let $\mu^*_{p_1,p_2,p_3}: \mathbb{P}^1 \to \mathbb{P}^1$ and $\mu^*_{q_1,q_2,q_3}: \mathbb{P}^1 \to \mathbb{P}^1$ be the unique Möbius transformations of the Riemann sphere for which

$$\mu^*_{p_1,p_2,p_3}(p_1) = \infty, \quad \mu^*_{p_1,p_2,p_3}(p_2) = 0, \quad \mu^*_{p_1,p_2,p_3}(p_3) = 1,$$

$$\mu^*_{q_1,q_2,q_3}(q_1) = \infty, \quad \mu^*_{q_1,q_2,q_3}(q_2) = 0 \quad \text{and} \quad \mu^*_{q_1,q_2,q_3}(q_3) = 1.$$

Then

$$\mu^*_{p_1,p_2,p_3}(p) = (p_1, p_2; p_3, p)$$

and

$$\mu^*_{q_1,q_2,q_3}(p) = (q_1, q_2; q_3, p)$$

for all points p of the Riemann sphere. Let $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined that is the composition function $\mu^{*-1}_{q_1,q_2,q_3} \circ \mu_{p_1,p_2,p_3}$ obtained on following the Möbius transformation $\mu^*_{p_1,p_2,p_3}$ with the inverse of the Möbius transformation $\mu^*_{q_1,q_2,q_3}$. Then the Möbius transformation μ is the unique Möbius transformation that satisfies $\mu(p_j) = q_j$ for $j = 1, 2, 3$ (see Proposition 1.8). Now $\mu(p_4) = \mu(q_4)$ if and only if $\mu^*_{p_1,p_2,p_3}(p_4) = \mu^*_{q_1,q_2,q_3}(q_4)$, and this is the case if and only if

$$(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4).$$

The result follows. ∎

Proposition 1.19 *Four distinct complex numbers z_1, z_2, z_3 and z_4 lie on a single line or circle in the complex plane if and only if their cross-ratio $(z_1, z_2; z_3, z_4)$ is a real number.*

Proof Let $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ be the Möbius transformation of the Riemann sphere defined such that $\mu(p) = (z_1, z_2; z_3, p)$ for all $p \in \mathbb{P}^1$. Then $\mu(z_1) = \infty$, $\mu(z_2) = 0$ and $\mu(z_3) = 1$. Möbius transformations map lines and circles to lines and circles (Propostion 1.11). It follows that a complex number z distinct from z_1, z_2 and z_3 lies on the circle in the complex plane passing through the points z_1, z_2 and z_3 if and only if $\mu(z)$ lies on the unique line in the complex plane that passes through 0 and 1, in which case $\mu(z)$ is a real number. The result follows. ∎

1.7 Cross-Ratios and Angles

We recall some basic properties of the algebra of complex numbers. Any complex number z can be written in the form

$$z = |z|\,(\cos\theta + \sqrt{-1}\sin\theta)$$

where $|z|$ is the modulus of z and θ is the angle in radians, measured anticlockwise, between the positive real axis and the line segment whose endpoints are represented by the complex numbers 0 and z. Moreover

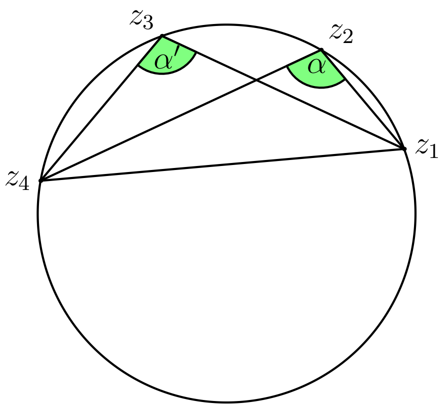
$$\frac{1}{\cos\alpha + \sqrt{-1}\,\sin\alpha} = \cos\alpha - \sqrt{-1}\,\sin\alpha$$

and

$$\begin{aligned}(\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\beta + \sqrt{-1}\,\sin\beta)\\ = \cos(\alpha+\beta) + \sqrt{-1}\,\sin(\alpha+\beta)\end{aligned}$$

for all real numbers α and β.

Proposition 1.20 *Let z_1, z_2, z_3 and z_4 be distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle. Then the angle between the lines joining z_2 to z_4 and z_1 is equal to the angle between the lines joining z_3 to z_4 and z_1.*



Proof Let α denote the angle between the lines joining z_2 to z_4 and z_1, and let α' be the angle between the lines joining z_3 to z_4 and z_1. We must show that $\alpha = \alpha'$. Now it follows from the standard properties of complex numbers that

$$\begin{aligned}\frac{z_1 - z_2}{z_4 - z_2} &= \frac{|z_1 - z_2|}{|z_4 - z_2|}(\cos\alpha + \sqrt{-1}\sin\alpha),\\ \frac{z_1 - z_3}{z_4 - z_3} &= \frac{|z_1 - z_3|}{|z_4 - z_3|}(\cos\alpha' + \sqrt{-1}\sin\alpha').\end{aligned}$$

It now follows from the definition of cross-ratio that

$$
\begin{aligned}
(z_2, z_3; z_1, z_4) &= \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)} = \frac{z_1 - z_2}{z_4 - z_2} \div \frac{z_1 - z_3}{z_4 - z_3} \\
&= \frac{|z_1 - z_2|\,|z_4 - z_3|}{|z_1 - z_3|\,|z_4 - z_2|} \times \frac{\cos\alpha + \sqrt{-1}\,\sin\alpha}{\cos\alpha' + \sqrt{-1}\,\sin\alpha'}.
\end{aligned}
$$

Now

$$
\frac{1}{\cos\alpha' + \sqrt{-1}\sin\alpha'} = \cos\alpha' - \sqrt{-1}\sin\alpha',
$$

and therefore

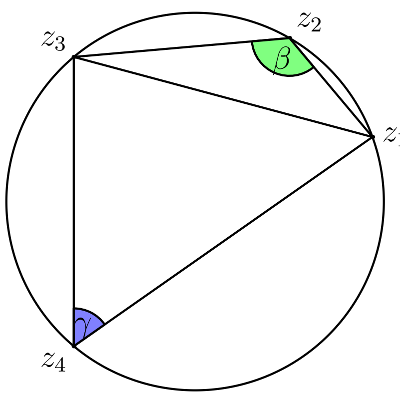
$$
\begin{aligned}
\frac{\cos\alpha + \sqrt{-1}\,\sin\alpha}{\cos\alpha' + \sqrt{-1}\,\sin\alpha'} &= (\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\alpha' - \sqrt{-1}\,\sin\alpha') \\
&= \cos(\alpha - \alpha') + \sqrt{-1}\,\sin(\alpha - \alpha').
\end{aligned}
$$

Consequently

$$
(z_2, z_3; z_1, z_4) = |(z_2, z_3; z_1, z_4)|(\cos(\alpha - \alpha') + \sqrt{-1}\,\sin(\alpha - \alpha')).
$$

But the cross ratio $(z_2, z_3; z_1, z_4)$ is a real number, because the complex numbers z_1, z_2, z_3 and z_4 lie on a circle (see Proposition 1.19), and consequently $\alpha - \alpha'$ must be an integer multiple of π. Also $0 < \alpha < \pi$ and $0 < \alpha' < \pi$, and therefore $-\pi < \alpha - \alpha' < \pi$. It follows that $\alpha - \alpha' = 0$, and thus $\alpha = \alpha'$, as required. ∎

Proposition 1.21 *Let z_1, z_2, z_3 and z_4 be distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle, let β be the angle between the lines joining z_2 to z_3 and z_1, and let γ be the angle between the lines joining z_4 to z_1 and z_3. Then $\beta + \gamma = \pi$.*



Proof It follows from the standard properties of complex numbers that

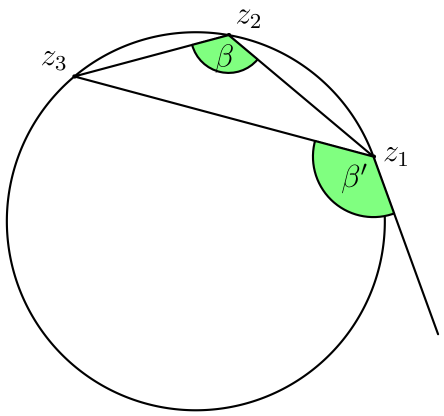
$$\begin{aligned}\frac{z_1 - z_2}{z_3 - z_2} &= \frac{|z_1 - z_2|}{|z_3 - z_2|}(\cos\beta + \sqrt{-1}\sin\beta),\\ \frac{z_3 - z_4}{z_1 - z_4} &= \frac{|z_3 - z_4|}{|z_1 - z_4|}(\cos\gamma + \sqrt{-1}\sin\gamma).\end{aligned}$$

It now follows from the definition of cross-ratio that

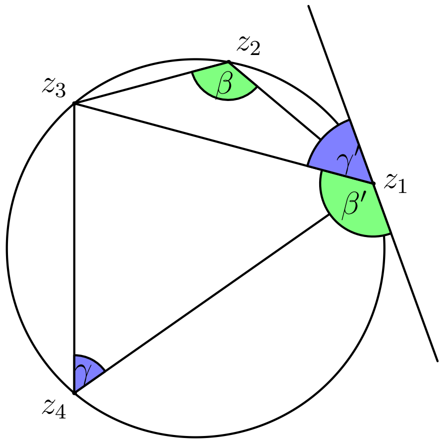
$$\begin{aligned}&(z_2, z_4; z_1, z_3)\\ &= \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{z_1 - z_2}{z_3 - z_2} \times \frac{z_3 - z_4}{z_1 - z_4}\\ &= \frac{|z_1 - z_2|\,|z_3 - z_4|}{|z_1 - z_4|\,|z_3 - z_2|}\,(\cos\beta + \sqrt{-1}\,\sin\beta)(\cos\gamma + \sqrt{-1}\,\sin\gamma)\\ &= |(z_2, z_4; z_1, z_3)|\,(\cos(\beta+\gamma) + \sqrt{-1}\,\sin(\beta+\gamma)).\end{aligned}$$

But the cross ratio $(z_2, z_4; z_1, z_3)$ is a real number, because the complex numbers z_1, z_2, z_4 and z_3 lie on a circle (see Proposition 1.19), and consequently $\beta + \gamma$ must be an integer multiple of π. Also $0 < \beta < \pi$ and $0 < \gamma < \pi$, and therefore $0 < \beta + \gamma < 2\pi$. It follows that $\beta + \gamma = \pi$, as required. ∎

Proposition 1.22 *Let z_1, z_2 and z_3 distinct complex numbers lying on a circle in the complex plane, listed in anticlockwise around the circle. Then the angle between the lines joining z_2 to z_3 and z_1 is equal to the angle between the line joining z_3 to z_1 and the ray tangent to the circle at z_1 that is directed so that the point z_2 and the tangent ray lie on opposite sides of the line that passes through the points z_1 and z_3.*

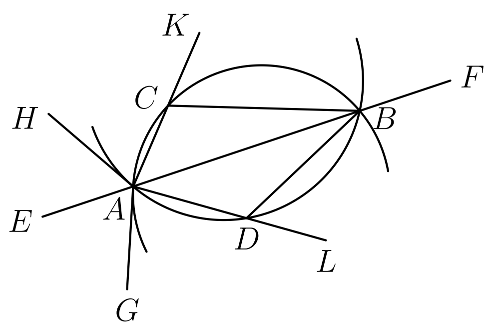


Proof Let β denote the angle between the lines joining z_2 to z_3 and z_1. Also let a point z_4 be taken on the circle so that z_1, z_2, z_3 and z_4 are distinct and moreover the points z_1 and z_4 lie on opposite sides of the line that passes through z_1 and z_3, and let γ denote the angle between the lines joining z_4 to z_1 and z_3. It follows from Proposition 1.21 that $\beta + \gamma = \pi$.



Now suppose that the point z_4 moves along the circle towards the point z_1. As the point z_4 approaches z_1 the direction of the chord of the circle from z_4 to z_1 approaches the direction of the ray tangent to the circle at z_1 that points into the side of the line through z_1 and z_3 in which z_2 lies. But the angle between the rays joining z_4 to z_1 and z_3 remains constant as z_4 approaches z_1. Consequently the angle γ' between the tangent ray at z_1 pointing into the side of the chord joining z_1 to z_3 and that chord itself is equal to the angle γ. The angle β' between the chord joining z_1 and z_3 and the tangent ray pointing into the side of that chord opposite to z_2 is then the supplement of the angle γ', where $\gamma' = \gamma$, and therefore $\beta' + \gamma = \pi = \beta + \gamma$. Consequently $\beta' = \beta$. The result follows. ∎

Proposition 1.23 *Let a geometrical configuration be as depicted in the accompanying figure. Thus let ACB and ADB be circular arcs that cut at the points A and B. Let the line joining points A and B be produced beyond A and B to E and F respectively. Let AG and AH be tangent to the circular arcs BCA and BDA respectively at A, where C and H lie on one side of AB and D and G lie on the other. Also let the lines AC and AD be produced to K and L respectively. Then the angle GAH is the sum of the angles KCB and LDB.*

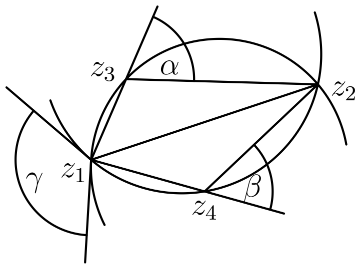


Proof Applying results of previous propositions, together with standard geometrical results, we find that

$$\begin{array}{rcll} \angle GAB & = & \angle ACB & \text{(Proposition 1.22)} \\ \Rightarrow \angle EAG & = & \angle KCB & \text{(supplementary angles)} \\ \angle HAB & = & \angle ADB & \text{(Proposition 1.22)} \\ \Rightarrow \angle EAH & = & \angle LDB & \text{(supplementary angles)} \\ \Rightarrow \angle GAH & = & \angle EAG + \angle EAH & \\ & = & \angle KCB + \angle LDB, & \end{array}$$

as required. ∎

Proposition 1.24 *Let two circles in the complex plane intersect at points represented by complex numbers z_1 and z_2, and let points represented by complex numbers z_3 and z_4 be taken on arcs of the respective circles joining z_1 and z_2 so that the point representing z_3 lies on the left hand side of the directed line from z_1 and z_2 and the point represented by the point z_4 lies on the right hand side of that line (as depicted in the accompanying figure).*



Then

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}\,(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent lines to the two circles at the intersection point represented by the complex number z_1.

Proof The configuration of the points z_1, z_2, z_3 and z_4 ensures that direction of the line from z_1 to z_3 is transformed into the direction of the line from z_3 to z_2 by rotation clockwise through an angle α less than two right angles. Similarly the direction of the line from z_1 to z_4 is transformed into the direction of the line from z_4 to z_2 by rotation anticlockwise through an angle β less than two right angles. Basic properties of complex numbers therefore ensure that

$$\begin{aligned}
\frac{z_2 - z_3}{z_3 - z_1} &= \frac{|z_2 - z_3|}{|z_3 - z_1|}(\cos\alpha - \sqrt{-1}\,\sin\alpha).\\
\frac{z_2 - z_4}{z_4 - z_1} &= \frac{|z_2 - z_4|}{|z_4 - z_1|}(\cos\beta + \sqrt{-1}\,\sin\beta).
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{\cos\beta + \sqrt{-1}\,\sin\beta}{\cos\alpha - \sqrt{-1}\,\sin\alpha}\\
&\quad= (\cos\alpha + \sqrt{-1}\,\sin\alpha)(\cos\beta + \sqrt{-1}\,\sin\beta)\\
&\quad= \cos(\alpha + \beta) + \sqrt{-1}\,\sin(\alpha + \beta).
\end{aligned}$$

Moreover the geometry of the configuration ensures that $\alpha + \beta = \gamma$ (Proposition 1.23). Thus

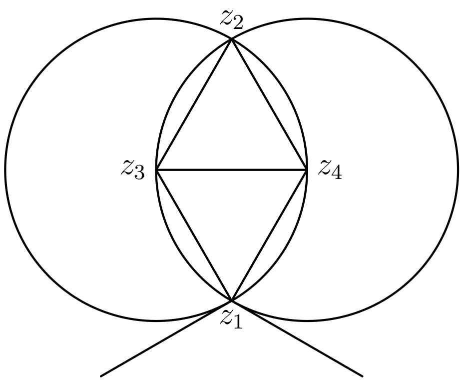
$$\begin{aligned} & \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ & \quad = \frac{|z_2 - z_4|\,|z_3 - z_1|}{|z_4 - z_1||z_2 - z_3|} (\cos\gamma + \sqrt{-1}\sin\gamma). \end{aligned}$$

But

$$\frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} = (z_1, z_2; z_3, z_4).$$

The result follows. ∎

Example The circles in the complex plane of radius 2 centred on -1 and 1 intersect at the points $\pm\sqrt{3}\,i$, where $i = \sqrt{-1}$. In this situation, take $z_1 = -\sqrt{3}\,i$, $z_2 = \sqrt{3}\,i$, $z_3 = -1$ and $z_4 = 1$. Then



$$\begin{aligned} (z_1, z_2; z_3, z_4) &= \frac{(-1 + \sqrt{3}\,i)(1 - \sqrt{3}\,i)}{(-1 - \sqrt{3}\,i)(1 + \sqrt{3}\,i)} = \frac{2 + 2\sqrt{3}\,i}{2 - 2\sqrt{3}i} \\ &= \frac{(2 + 2\sqrt{3}\,i)^2}{(2 - 2\sqrt{3}i)(2 + 2\sqrt{3}i)} \\ &= \frac{1}{2}(-1 + \sqrt{3}\,i) \end{aligned}$$

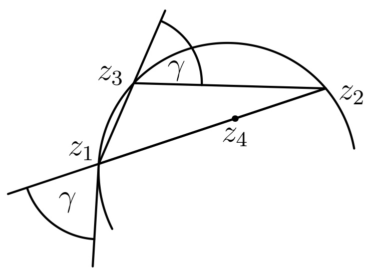
It follows that $(z_1, z_2; z_3, z_4) = \cos\gamma + \sqrt{-1}\,\sin\gamma$, where $\gamma = \frac{2}{3}\pi$. Thus the angle between the tangent lines to the circles at the intersection point z_1 is thus $\frac{4}{3}$ of a right angle. This is what one would expect from the basic geometry of the configuration, given that the triangle with vertices z_1, z_3 and z_4 is equilateral and the tangent lines to the circles are perpendicular to the lines joining the point of intersection to the centres of those circles.

Proposition 1.25 *Let z_1 and z_2 be complex numbers representing the end-points of a circular arc in the complex plane. Also, in the case where the circular arc lies on the left hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the circular arc and the straight line segment respectively, and, in the case where the circular arc lies on the right hand side of the directed line from z_1 to z_2, let points z_3 and z_4 be taken between z_1 and z_2 on the straight line segment and the the circular arc respectively. Then*

$$(z_1, z_2; z_3, z_4) = \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma),$$

where γ is the angle between the tangent line to the circle at the intersection point represented by the complex number z_1 and the line obtained by producing the chord joining z_2 and z_1 beyond z_1.

Proof We consider the configuration in which the circular arc lies on the left hand side of the directed line from z_1 to z_2. In that case the configuration is as depicted in the accompanying figure. In this configuration the angle made



at z_3 by the lines from z_1 and z_2 is equal to the angle between the chord from z_1 to z_2 and the depicted tangent line. The complements of those angles are then also equal to one another; these equal complements have been labelled γ in the figure.

Also the direction of the line from z_3 to z_2 is obtained from the direction of the line from z_1 to z_3 by rotation clockwise through an angle γ less than two right angles. It follows that

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|}(\cos\gamma - \sqrt{-1}\,\sin\gamma).$$

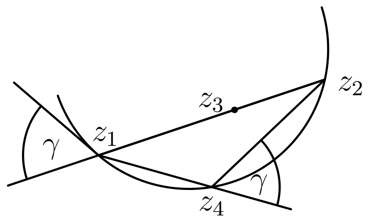
Also the direction of $z_2 - z_4$ is the same as that of $z_4 - z_1$, and therefore

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|}.$$

It follows that

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\ &= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ &= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma).\end{aligned}$$

We consider now the case in which the circular arc from z_1 to z_2 lies on the right hand side of the directed line from z_1 to z_2. In this case the complex numbers z_3 and z_4 represent points between z_1 and z_2 on the line and the circular arc respectively, as depicted in the following figure.



In this configuration, the angle sought is the angle γ, which in this case is equal both to the angle between the depicted tangent line to the circle at z_1 and the line that produces the chord joining z_2 to z_1 beyond z_1. Moreover, in this case

$$\frac{z_2 - z_4}{z_4 - z_1} = \frac{|z_2 - z_4|}{|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma)$$

and

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{|z_2 - z_3|}{|z_3 - z_1|}.$$

It follows in this case also that

$$\begin{aligned}(z_1, z_2; z_3, z_4) &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \\ &= \frac{z_2 - z_4}{z_4 - z_1} \times \frac{z_3 - z_1}{z_2 - z_3} \\ &= \frac{|z_3 - z_1|\,|z_4 - z_2|}{|z_3 - z_2|\,|z_4 - z_1|}(\cos\gamma + \sqrt{-1}\,\sin\gamma).\end{aligned}$$

This completes the proof. ∎

Proposition 1.26 *Let two lines in the complex plane intersect at at point represented by the complex number z_1, and let points represented by z_3 and z_4*

be taken distinct from z_1, one on each of the two lines, where these points are labelled so that the direction of $z_3 - z_1$ is obtained from the direction of $z_4 - z_1$ by rotation anticlockwise through an angle γ less than two right angles. Then

$$(z_1, \infty; z_3, z_4) = \frac{|z_3 - z_1|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Proof The cross-ratio in this situation is defined so that

$$(z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1}.$$

Furthermore

$$\frac{z_3 - z_1}{z_4 - z_1} = \frac{|z_3 - z_1|}{|z_4 - z_1|} (\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

The result follows directly. ∎

Lines in the complex plane correspond to circles on the Riemann sphere that pass through the point at infinity. With that in mind, it can seen that Propositions 1.24, 1.25 and 1.26 conform to a common pattern, and show that, where two curves intersect at a point, each of those curves being either a circle or a straight line, the angle between the tangent lines to those curves at the point of intersection may be expressed in terms of the argument of an appropriate cross-ratio.

Indeed, to determine the angle the tangent lines to two circles on the Riemann sphere at a point p_1 where they intersect, one can determine the other point of intersection p_2, a point p_3 on one circular arc between p_1 to p_2, and a point p_4 on the other circular arc between p_1 and p_2. A positive real number R and a real number γ satisfying $-\pi < \gamma < \pi$ can then be determined so that

$$(p_1, p_2; p_3, p_4) = R(\cos\gamma + \sqrt{-1}\,\sin\gamma).$$

Then the angle between the tangent lines to those circles at the point p_1 of intersection, measured in radians, is then the absolute value $|\gamma|$ of γ.

Proposition 1.27 *Möbius transformations of the Riemann sphere $\mathbb{P}^1$ are angle-preserving. Thus if two circles on the Riemann sphere intersect at a point p of the Riemann sphere, and if a Möbius transformation μ maps p to a point q of the Riemann sphere, then the angle between the tangent lines to the original circles at the point p is equal to the angle between the tangent lines to the corresponding circles at the point q, the corresponding circles being the images of the original circles under the Möbius transformation.*

Proof The angle between the tangent lines to the original circles at p is determined by the value of a cross ratio of the form $(p_1, p_2; p_3, p_4)$, where p_1 and p_2 are the points of intersection of the original circles, and p_3 and p_4 lie on the circular arcs joining p_1 to p_2, with p_4 on the right hand side as the circle through p_3 is traversed in the direction from p_1 through p_3 to p_2. The angle between the tangent lines to the corresponding circles at q is determined in the analogous fashion by the value of the cross ratio $(q_1, q_2; q_3, q_4)$, where q_j is the image of p_j under the Möbius transformation sending the original circles to the corresponding circles. Proposition 1.18 ensures that $(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4)$. The result follows. ∎

1.8 The Orientation-Preserving Property of Möbius Transformations

Proposition 1.28 *Let μ be a Möbius transformation of the Riemann sphere, let w be a complex number for which $\mu(w)$ is also a complex number, let s be a positive real number, and let $\alpha\colon [0,1] \to \mathbb{R}$ be the path in the complex plane defined such that*

$$\alpha(t) = w + s(\cos 2\pi t + \sqrt{-1}\,\sin 2\pi t)$$

for all real numbers t satisfying $0 \leq t \leq 1$, so that the point $\alpha(t)$ moves round a circle of radius s about w in the anticlockwise direction as t increases from 0 to 1. Then, provided that s is sufficiently close to zero, the point $\mu(\alpha(t))$ will move in an anticlockwise direction around $\mu(w)$ as t increases from 0 to 1.

Proof There exist complex coefficients a, b, c and d satisfying $ad - bc \neq 0$ that are such as to ensure that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z that are distinct from $-d/c$. Then

$$\begin{aligned} \mu(z) - \mu(w) &= \frac{az+b}{cz+d} - \frac{aw+b}{cw+d} \\ &= \frac{(az+b)(cw+d) - (aw+b)(cz+d)}{(cz+d)(cw+d)} \\ &= \frac{(ad-bc)(z-w)}{(cz+d)(cw+d)} \\ &= \frac{ad-bc}{(cw+d)^2} \times (z-w) \times \frac{cw+d}{cz+d} \end{aligned}$$

Now the quotient $(cz+d)/(cw+d)$ approaches the value 1 as the complex number z approaches w. Consequently a positive real number s_0 can be found such that $\mu(z) \in \mathbb{C}$ and

$$\operatorname{Re}\left[\frac{cz+d}{cw+d}\right] > 0$$

whenever $|z-w| \leq s_0$. Let the real number s be chosen such that $0 < s \leq s_0$, and let

$$\alpha(t) = w + s(\cos 2\pi t + \sqrt{-1}\,\sin 2\pi t)$$

for all real numbers t satisfying $0 \leq t \leq 1$. Then, for each real number t between 0 and 1 there exists a unique real number $\eta(t)$ satisfying $-\frac{1}{4} < \eta(t) < \frac{1}{4}$ such that

$$\frac{c\alpha(t)+d}{cw+d} = \left|\frac{c\alpha(t)+d}{cw+d}\right| (\cos(2\pi\eta(t)) + \sqrt{-1}\sin(2\pi\eta(t)))$$

We obtain in this fashion a continuous real-valued function $\eta\colon [0,1] \to \mathbb{R}$ that sends each real number t satisfying $0 \leq t \leq 1$ between zero and one to the unique real number $\eta(t)$ in the range $-\frac{1}{4} < \eta(t) < \frac{1}{4}$ for which the above equation is satisfied. Moreover $\alpha(0) = \alpha(1)$, and therefore $\eta(0) = \eta(1)$. A real number m can also be found such that

$$\frac{ad-bc}{(cw+d)^2} = \left|\frac{ad-bc}{(cw+d)^2}\right| (\cos(2\pi m) + \sqrt{-1}\sin(2\pi m)).$$

Well-known trigonometrical identies involving sine and cosine functions then ensure that

$$\frac{\mu(\alpha(t)) - \mu(w)}{|\mu(\alpha(t)) - \mu(w)|} = \cos(2\pi\psi(t)) + \sqrt{-1}\sin(2\pi\psi(t))$$

for all real numbers t lying between 0 and 1, where

$$\psi(t) = m + t - \eta(t)$$

for all real numbers t between 0 and 1. (We are here using the fact that the argument of a product of complex numbers is the sum of the arguments of those complex numbers.) Now $\psi(1) - \psi(0) = 1$, because $\eta(0) = \eta(1)$. Consequently the point $\mu(\alpha(t))$ moves once round the point $\mu(w)$ in the complex plane in an anticlockwise direction as t increases from 0 to 1, as required. ∎

Proposition 1.28 ensures that Möbius transformations of the Riemann sphere are *orientation-preserving*.

A subset X of the complex plane $\mathbb{C}$ is said to be *open* if, given any any complex number w belonging to X, some open disk in the complex plane of sufficiently small radius centred on w is wholly contained within the set X.

Definition An invertible function $\varphi: X \to Y$ between open subsets X and Y of the complex plane is said to be *orientation-preserving* if, given any point w of X, paths that traverse circles of sufficiently small radius centred on w once in the anticlockwise direction are mapped by φ to paths that wind around $\varphi(w)$ once in the anticlockwise direction.

Definition An invertible function $\varphi: X \to Y$ between open subsets X and Y of the complex plane is said to be *orientation-reversing* if, given any point w of X, paths that traverse circles of sufficiently small radius centred on w once in the anticlockwise direction are mapped by φ to paths that wind around $\varphi(w)$ once in the clockwise direction.

The transformation of the complex plane that maps each complex number to its complex conjugate is an example of an orientation-reversing transformation of the complex plane.

The composition of two orientation-preserving transformations between open subsets of the complex plane is orientation-preserving, as is the composition of two orientation-reversing transformations between such subsets. A transformation obtained on composing an orientation-preserving transformation with an orientation-reversing transformation is orientation-reversing, as is a transformation obtained on composing an orientation-reversing transformation with an orientation-preserving transformation.

2 The Disk Model of the Hyperbolic Plane

2.1 Inversion of the Riemann Sphere in the Unit Circle

Let D denote the open unit disk in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let S denote the unit circle in the complex plane $\mathbb{C}$, and in the Riemann sphere, defined so that

$$S = \{z \in \mathbb{C} : |z| = 1\}$$

We define the *inversion* Ω of the Riemann sphere in the circle S bounding the open unit disk D to be the transformation of the Riemann sphere defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Then $\Omega(z) = z$ for all $z \in S$, and the composition $\Omega \circ \Omega$ of the inversion Ω with itself is the identity transformation of the Riemann sphere. Moreover Ω maps the open unit disk D into the region of the Riemann sphere that lies outside the unit circle S.

Lemma 2.1 *Let μ be a Möbius transformation of the Riemann sphere, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Also let a, b, c and d be complex coefficients determined so that*

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then $\Omega \circ \mu \circ \Omega$ is also a Möbius transformation, and moreover

$$\Omega(\mu(\Omega(z))) = \frac{\overline{c} + \overline{d}z}{\overline{a} + \overline{b}z}$$

for all complex numbers $z \in \mathbb{C}$ for which $\overline{a} + \overline{b}z \neq 0$ and $\overline{c} + \overline{d}z \neq 0$.

Proof It follows from the definition of Möbius transformations that there exist complex numbers a, b, c and d such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then

$$\Omega(\mu(\Omega(z))) = \Omega\left(\frac{a\dfrac{1}{\overline{z}} + b}{c\dfrac{1}{\overline{z}} + d}\right) = \Omega\left(\frac{a + b\overline{z}}{c + d\overline{z}}\right) = \frac{\overline{c} + \overline{d}z}{\overline{a} + \overline{b}z}$$

for all $z \in \mathbb{C}$ for which $z \neq 0$, $\overline{c} + \overline{d}z \neq 0$ and $\overline{a} + \overline{b}z \neq 0$. Also

$$\Omega(\mu(\Omega(0))) = \Omega(\mu(\infty)) = \Omega\left(\frac{a}{c}\right) = \frac{\overline{c}}{\overline{a}}$$

provided that $a \neq 0$ and $c \neq 0$,

$$\Omega(\mu(\Omega(0))) = \Omega(\mu(\infty)) = \Omega(\infty) = 0 \quad \text{when} \quad c = 0,$$

$$\Omega(\mu(\Omega(0))) = \Omega(\mu(\infty)) = \Omega(0) = \infty \quad \text{when} \quad a = 0,$$

$$\Omega\left(\mu\left(\Omega\left(-\frac{\overline{c}}{\overline{d}}\right)\right)\right) = \Omega\left(\mu\left(-\frac{d}{c}\right)\right) = \Omega(\infty) = 0,$$

provided that $c \neq 0$ and $d \neq 0$, and

$$\Omega\left(\mu\left(\Omega\left(-\frac{\overline{a}}{\overline{b}}\right)\right)\right) = \Omega\left(\mu\left(-\frac{b}{a}\right)\right) = \Omega(0) = \infty$$

provided that $a \neq 0$ and $b \neq 0$.

Now the definition of Möbius transformations requires that $ad - bc \neq 0$. Consequently $c \neq 0$ when $d = 0$, and $a \neq 0$ when $b = 0$. We have therefore determined the image of each element of the Riemann sphere under the composition map $\Omega \circ \mu \circ \Omega$, and can now conclude that this composition map $\Omega \circ \mu \circ \Omega$ is indeed a Möbius transformation, and that it is characterized by the property that

$$\Omega(\mu(\Omega(z)) = \frac{\overline{c} + \overline{d}z}{\overline{a} + \overline{b}z}$$

for all complex numbers $z \in \mathbb{C}$ for which $\overline{a} + \overline{b}z \neq 0$ and $\overline{c} + \overline{d}z \neq 0$, as required. ∎

Proposition 2.2 *Let μ be a Möbius transformation of the Riemann sphere, let D be the open unit disk in the complex plane, where*

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and let Ω be the inversion of the Riemann sphere in the unit circle that is defined so that

$$\Omega(0) = \infty, \quad \Omega(\infty) = 0 \quad \textit{and} \quad \Omega(z) = \frac{1}{\overline{z}} \textit{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Then the Möbius transformation μ maps the unit disk D onto itself if and only if both of the following two conditions are satisfied:

(i) $\Omega \circ \mu = \mu \circ \Omega$;

(ii) there exists at least one $z \in D$ *for which* $\mu(z) \in D$.

Proof First suppose that the Möbius transformation μ maps the unit disk D onto itself. Let z be a complex number satisfying $|z| = 1$. Then $tz \in D$ for all real numbers t satisfying $0 \leq t < 1$, and consequently $|\mu(tz)| < 1$ for all real numbers t satisfying $0 \leq t < 1$. The continuity of Möbius transformations then ensures that $|\mu(z)| \leq 1$. Now if it were the case that $|\mu(z)| < 1$ then there would exist $w \in D$ for which $\mu(w) = \mu(z)$, because the Möbius transformation μ maps the unit disk D onto itself. But this is not possible, because if it were, then two distinct z and w complex numbers would be mapped by μ to the same complex number, contradicting the fact that Möbius transformations are invertible transformations of the Riemann sphere. Thus the Möbius transformation μ maps the unit circle into itself.

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1). Moreover $\Omega(z) = z$ and $|\mu(z)| = 1$ for all complex numbers z satisfying $|z| = 1$, and therefore $\hat{\mu}(z) = \mu(z)$ for all complex numbers z satisfying $|z| = 1$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Proposition 1.9). It follows therefore that $\hat{\mu} = \mu$. Consequently $\Omega \circ \mu = \mu \circ \Omega$. It now follows directly that any Möbius transformation that maps the unit disk D onto itself must satisfy conditions (i) and (ii) in the statement of the proposition.

Conversely, suppose that Möbius transformation μ of the Riemann sphere satisfies conditions (i) and (ii) in the statement of the proposition. Then $\Omega \circ \mu = \mu \circ \Omega$. Let z be a complex number satisfying $|z| \neq 1$. Then $\Omega(z) \neq z$. It follows that $\mu(\Omega(z)) \neq \mu(z)$, because Möbius transformations are invertible transformations of the Riemann sphere, and therefore $\Omega(\mu(z)) \neq \mu(z)$, from which it follows that $|\mu(z)| \neq 1$. Consequently no complex number belonging to the open unit disk D is mapped by the Möbius transformation D to a point that lies on the unit circle. It follows that if one endpoint of a straight line segment or circular arc contained in the open disk D is mapped by μ into D, then the same must be true of the other endpoint of that straight line segment or circular arc.

Now the complex numbers belonging to the unit disk D can be joined to one another by straight line segments. Moreover condition (ii) in the statement of the proposition ensures that at least one complex number belonging to the unit disk D is mapped by the Möbius transformation μ into the unit disk D. Consequently the unit disk is mapped into itself by the Möbius transformation μ.

Moreover if the Möbius transformation μ has the property that $\Omega \circ \mu =$

$\mu \circ \Omega$ then

$$\Omega \circ \mu^{-1} = \mu^{-1} \circ \mu \circ \Omega \circ \mu^{-1} = \mu^{-1} \circ \Omega \circ \mu \circ \mu^{-1} = \mu^{-1} \circ \Omega,$$

and consequently the inverse μ^{-1} of the Möbius transformation μ also satisfies (i) and (ii) in the statement of the proposition, and therefore maps the open unit disk D into itself. It follows that if the Möbius transformation μ satisfies conditions (i) and (ii) then it must map the open unit disk D onto itself, as required. ∎

Corollary 2.3 *Let μ be a Möbius transformation of the Riemann sphere, and let S be the unit circle consisting of all complex numbers z for which $|z| = 1$. Suppose that $\mu(S) \subset S$ and that $|\mu(0)| < 1$. Then the Möbius transformation μ maps the open unit disk onto itself. Moreover $\Omega \circ \mu = \mu \circ \Omega$, where Ω is the inversion of the Riemann sphere in the unit circle S, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.*

Proof Let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is a Möbius transformation of the Riemann sphere (Lemma 2.1), and moreover $\hat{\mu}(z) = \mu(z)$ for all $z \in S$, because $\mu(S) \subset S$ and $\Omega(z) = z$ for all $z \in S$. Now two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Proposition 1.9). It follows that $\hat{\mu} = \mu$, and therefore $\Omega \circ \mu = \mu \circ \Omega$. The required result now follows on applying Proposition 2.2. ∎

Lemma 2.4 *Given distinct complex numbers z_1 and z_2, where $|z_1| = |z_2| = 1$, there exists a Möbius transformation μ of the Riemann sphere mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$.*

Proof Choose a complex number z_3 distinct from z_1 and z_2 for which $|z_3| = 1$. Then there exists a unique Möbius transformation μ_1 with the properties that $\mu_1(z_1) = -1$, $\mu_1(z_2) = 1$ and $\mu_1(z_3) = i$. Möbius transformations map circles to circles, and, given any three distinct complex numbers that are not collinear, there exists exactly one circle in the complex plane passing through all three of these complex numbers. Consequently the Möbius transformation μ_1 must map the unit circle onto itself. If $|\mu_1(0)| < 0$ let the Möbius transformation μ be identical to μ_1; if $|\mu_1(0)| > 1$ or $\mu_1(0) = \infty$ let the Möbius transformation μ be defined so that $\mu(z) = 1/\mu_1(z)$ for all complex numbers z for which $\mu_1(z) \neq 0$. Then μ maps the unit circle onto itself, $\mu(z_1) = -1$, $\mu(z_2) = 1$ and $|\mu(0)| < 1$. Then $\mu(D)$ must map the open unit disk onto itself (see Corollary 2.3). The Möbius transformation μ then has the required properties. ∎

Proposition 2.5 *Let a and b be complex numbers satisfying $|b| < |a|$, and let μ be the Möbius transformation of the Riemann sphere defined so that*

$$\mu(z) = \frac{az + b}{\overline{b}\, z + \overline{a}} \quad \text{whenever } \overline{b}\, z + \overline{a} \neq 0,$$

$\mu(-\overline{a}/\overline{b}) = \infty$ and $\mu(\infty) = a/\overline{b}$ in cases where $b \neq 0$ and $\mu(\infty) = \infty$ in cases where $b = 0$. Then $|\mu(z)| < 1$ whenever $|z| < 1$, $|\mu(z)| = 1$ whenever $|z| = 1$, and $|\mu(z)| > 1$ whenever $|z| > 1$ and $\overline{b}z + \overline{a} \neq 0$. Moreover the Möbius transformation μ maps the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself.

Proof Calculating, we find that

$$\begin{aligned} |\overline{b}z + \overline{a}|^2 - |az + b|^2 &= (\overline{b}z + \overline{a})(b\overline{z} + a) - (az + b)(\overline{a}\,\overline{z} + \overline{b}) \\ &= |b|^2|z|^2 + |a|^2 + a\,\overline{b}z + \overline{a}b\,\overline{z} \\ &\qquad - |a|^2|z|^2 - |b|^2 - a\,\overline{b}z - \overline{a}b\,\overline{z} \\ &= (|a|^2 - |b|^2)(1 - |z|^2) > 0. \end{aligned}$$

Consequently $|\mu(z)| < 1$ whenever $|z| < 1$, $|\mu(z)| = 1$ whenever $|z| = 1$ and $|\mu(z) > 1$ whenever $|z| > 1$ and $\overline{b}z + \overline{a} \neq 0$.

Now the inverse μ^{-1} of the Möbius transformation μ is characterized by the property that

$$\mu^{-1}(z) = \frac{\overline{a}z - b}{-\overline{b}z + a}$$

for all complex numbers z for which $-\overline{b}z + a \neq 0$ (see Corollary 1.6). Because the coefficients of this Möbius transformation μ^{-1} have properties analogous to those of the Möbius transformation μ, we can conclude that μ^{-1} maps the open unit disk into itself, and therefore μ maps the open unit disk onto itself, as required. ∎

Corollary 2.6 *Let w be a complex number satisfying $|w| < 1$, and let μ_w be the Möbius transformation of the Riemann sphere defined so that $\mu_w(1/\overline{w}) = \infty$, $\mu(\infty) = -1/\overline{w}$ and*

$$\mu_w(z) = \frac{z - w}{1 - \overline{w}\, z}$$

for all complex numbers z distinct from $1/\overline{w}$. Then the Möbius transformation μ_w maps the open unit disk onto itself. Moreover

$$\mu_w(tw) = \frac{t - 1}{1 - |w|^2 t}\, w$$

for all real numbers t distinct from $1/|w|^2$, and consequently the diameter of the unit circle passing through 0 and w is mapped onto itself by the Möbius transformation μ_w. In particular $\mu_w(w) = 0$ and $\mu_w(0) = -w$.

Proposition 2.7 *Let μ be a Möbius transformation of the Riemann sphere that maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ into itself, whilst mapping the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ into itself. Then there exist complex numbers a and b, where $|b| < |a|$, such that*

$$\mu(z) = \frac{az + b}{\overline{b}z + \overline{a}} \quad \textit{for all } z \in \mathbb{C} \textit{ for which } \overline{a}z + \overline{b} \neq 0.$$

Proof The Möbius transformation μ maps the unit circle into itself. It follows from Proposition 2.2 that $\Omega \circ \mu = \mu \circ \Omega$, where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Consequently $\mu = \Omega \circ \Omega \circ \mu = \Omega \circ \mu \circ \Omega$ because the composition of the inversion Ω with itself is the identity transformation of the Riemann sphere. Let a_1, b_1, c_1 and d_1 be complex coefficients determined so that

$$\mu(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{whenever } c_1 z + d_1 \neq 0.$$

Then the identity $\mu = \Omega \circ \mu \circ \Omega$ ensures that

$$\frac{a_1 z + b_1}{c_1 z + d_1} = \frac{\overline{d}_1 z + \overline{c}_1}{\overline{b}_1 z + \overline{a}_1}$$

for all complex numbers z for which $a_1 z + b_1 \neq 0$, $\overline{a}_1 + \overline{b}_1 z \neq 0$, $c_1 z + d_1 \neq 0$, and $\overline{c}_1 + \overline{d}_1 z \neq 0$ (see Lemma 2.1). Consequently there exists some non-zero complex number ω with the property that $\overline{a}_1 = \omega d_1$, $\overline{b}_1 = \omega c_1$, $\overline{c}_1 = \omega b_1$ and $\overline{d}_1 = \omega a_1$ (see Proposition 1.10). It then follows that

$$\overline{a}_1 \, \overline{d}_1 = \omega^2 a_1 d_1.$$

But

$$|\overline{a}_1 \, \overline{d}_1| = |a_1 d_1|.$$

It follows that $|\omega^2| = 1$, and therefore $|\omega| = 1$. Accordingly a real number θ can be found so that

$$\omega = \cos 2\theta + \sqrt{-1} \, \sin 2\theta.$$

Let

$$\eta = \cos\theta + \sqrt{-1} \, \sin\theta.$$

It then follows from De Moivre's Theorem that $\eta^2 = \omega$. Now $\overline{\eta}^2 \, \eta^2 = |\eta|^4 = 1$. It follows that $\overline{\eta}^2 \omega = 1$. Let $a = \eta a_1$ and $b = \eta b_1$, $c = \eta c_1$ and $d = \eta d_1$. Then

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{whenever } cz + d \neq 0.$$

Also $a_1 = \overline{\eta} a$, $b_1 = \overline{\eta} b$, $c_1 = \overline{\eta} c$ and $d_1 = \overline{\eta} d$. Consequently

$$\overline{d} = \overline{\eta}\, \overline{d}_1 = \overline{\eta} \omega a_1 = \overline{\eta}^2 \omega a = a$$

and

$$\overline{c} = \overline{\eta}\, \overline{c}_1 = \overline{\eta} \omega b_1 = \overline{\eta}^2 \omega b = b.$$

Accordingly

$$\mu(z) = \frac{az + b}{\overline{b}\, z + \overline{a}} \quad \text{whenever } \overline{b}\, z + \overline{a} \neq 0.$$

Moreover $|\mu(0)| < 1$, because μ maps the unit disk into itself. consequently $|b| < |a|$, as required. ∎

2.2 The Poincaré Distance Function on the Unit Disk

Definition Let D be the open unit disk in the complex plane $\mathbb{C}$, defined so that

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The *Poincaré distance function* ρ on D is defined so that

$$\rho(z, w) = \log \left(\frac{|1 - \overline{w}\, z| + |z - w|}{|1 - \overline{w}\, z| - |z - w|} \right)$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$.

Note that

$$\frac{|z - w|}{|1 - \overline{w}\, z|} < 1$$

for all complex numbers z and w satisfying $|z| < 1$ and $|w| < 1$. (This follows directly from Corollary 2.6). Consequently the Poincaré distance $\rho(z, w)$ between any two points z and w of the unit disk is a well-defined positive real-number.

Lemma 2.8 *Let s and t be real numbers satisfying $-1 < s < t < 1$. Then the Poincaré distance, in the unit disk, between s and t is given by the formula*

$$\rho(s, t) = \log \left(\frac{1 + t}{1 - t} \right) - \log \left(\frac{1 + s}{1 - s} \right).$$

Proof Evaluating, and noting that $1 - st > 0$ (because $|s| < 1$ and $|t| < 1$) and $|t - s| = t - s$ (since $s < t$ by assumption), we find that

$$\begin{aligned} \rho(s,t) &= \log\left(\frac{|1-st|+|t-s|}{|1-st|-|t-s|}\right) \\ &= \log\left(\frac{1-st+t-s}{1-st+s-t}\right) \\ &= \log\left(\frac{(1-s)(1+t)}{(1+s)(1-t)}\right) \\ &= \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right), \end{aligned}$$

as required. ∎

Proposition 2.9 *Let ρ be the Poincaré distance function on the open unit disk D, and let δ be a positive real number. Then*

$$\{z \in D : \rho(z,0) = \delta\} = \{z \in D : |z| = R\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof It follows from the definition of Poincaré distance function that all complex numbers z satisfying $\rho(z,0) = \delta$ are equidistant from zero. They therefore constitute a circle centred on zero. It remains to determine the radius of that circle. Now it follows, on applying Lemma 2.8, that

$$\delta = \log\left(\frac{1+R}{1-R}\right).$$

Consequently

$$e^\delta - 1 = \frac{2R}{1-R}, \quad e^\delta + 1 = \frac{2}{1-R},$$

and therefore

$$R = \frac{e^\delta - 1}{e^\delta + 1},$$

as required. ∎

The Poincaré distance function ρ on the unit disk D has the property that $\rho(z,w) = \rho(w,z)$ for all $z, w \in D$. It therefore follows immediately from Lemma 2.8 that

$$\rho(s,t) = \left|\log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right)\right|$$

for all real numbers s and t satisfying $-1 < s < 1$ and $-1 < t < 1$.

Lemma 2.10 *Let z and w be complex numbers, and let Ω be the inversion of the Riemann sphere in the unit circle, defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z. Then*

$$(z, \Omega(z); w, \Omega(w)) = \left|\frac{z - w}{1 - \overline{w}z}\right|^2$$

for all complex numbers z and w with the exception of those pairs z, w for which $|z| = 1$ and $z = w$.

Proof Let z and w be complex numbers. Suppose that it is not the case that $|z| = 1$ and $z = w$. Examination of possible cases shows that it is not then possible for three of the complex numbers z, $\Omega(z)$, w and $\Omega(w)$ to coincide with one another. Indeed if $|z| \neq 1$ and $|w| \neq 1$ then exactly two of the points $z, \Omega(z), w, \Omega(w)$ will lie in the unit disk consisting of those complex numbers whose modulus is less than one, and therefore it is not possible for any three of the four points to coincide with one another. If $|z| = 1$, it would only be possible for three of the points $z, \Omega(z), w, \Omega(w)$ to coincide with one another if it were also the case that $w = z$. Consequently the cross-ratio $(z, \Omega(z), w, \Omega(w)$ is defined in all cases with the exception of those where $|z| = 1$ and $w = z$.

Now let $u_1 = z$, $v_1 = 1$, $u_2 = 1$, $v_2 = \overline{z}$, $u_3 = w$, $v_3 = 1$, $u_4 = 1$, $v_4 = \overline{w}$. Then $u_1/v_1 = z$, $u_2/v_2 = \Omega(z)$, $u_3/v_3 = w$ and $u_4/v_4 = \Omega(w)$. The definition of cross-ratio then ensures that

$$\begin{aligned}
(z, \Omega(z); w, \Omega(w)) &= \frac{(u_1v_3 - u_3v_1)(u_2v_4 - u_4v_2)}{(u_2v_3 - u_3v_2)(u_1v_4 - u_4v_1)} \\
&= \frac{(z - w)(\overline{w} - \overline{z})}{(1 - w\overline{z})(z\overline{w} - 1)} \\
&= \left|\frac{z - w}{1 - \overline{w}z}\right|^2,
\end{aligned}$$

as required. ∎

Proposition 2.11 *Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z, w)$ denote the Poincaré distance between z and w. Then*

$$\rho(z, w) = \log\left(\frac{1 + \sqrt{(z, \Omega(z); w, \Omega(w))}}{1 - \sqrt{(z, \Omega(z); w, \Omega(w))}}\right),$$

where $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex numbers z.

Proof Evaluating, and applying the result of Lemma 2.10, we find that

$$\begin{aligned}\rho(z,w) &= \log\left(\frac{|1-\overline{w}\,z|+|z-w|}{|1-\overline{w}\,z|-|z-w|}\right)\\ &= \log\left(\frac{1+\dfrac{|z-w|}{|1-\overline{w}\,z|}}{1-\dfrac{|z-w|}{|1-\overline{w}\,z|}}\right)\\ &= \log\left(\frac{1+\sqrt{(z,\Omega(z);w,\Omega(w))}}{1-\sqrt{(z,\Omega(z);w,\Omega(w))}}\right),\end{aligned}$$

as required. ∎

Corollary 2.12 *Let z and w be complex numbers satisfying $|z| < 1$ and $|w| < 1$, and let $\rho(z,w)$ denote the Poincaré distance between z and w. Then the cross-ratio $(z,\Omega(z);w,\Omega(w))$ is expressed in terms of the Poincaré distance according to the formula*

$$(z,\Omega(z);w,\Omega(w)) = \left(\frac{e^{\rho(z,w)}-1}{e^{\rho(z,w)}+1}\right)^2.$$

Proof Let $q = (z,\Omega(z);w,\Omega(w))$ and $s = \rho(z,w)$. It follows from Proposition 2.11 that

$$s = \log\left(\frac{1+\sqrt{q}}{1-\sqrt{q}}\right).$$

Consequently

$$e^s - 1 = \frac{2\sqrt{q}}{1-\sqrt{q}}, \quad e^s + 1 = \frac{2}{1-\sqrt{q}},$$

and thus

$$q = \left(\frac{e^s-1}{e^s+1}\right)^2.$$

The result follows. ∎

Definition A transformation φ that maps the open unit disk D in the complex plane onto itself is said to be an *isometry* (with respect to Poincaré distance) if

$$\rho\Big(\varphi(z),\varphi(w)\Big) = \rho(z,w)$$

for all complex numbers z and w in the open unit disk D, where ρ denotes the Poincaré distance function on D.

Proposition 2.13 *Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$. Then every Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself is an isometry with respect to the Poincaré distance function on D.*

Proof The Möbius transformation μ has the property that $\mu \circ \Omega = \Omega \circ \mu$, because it maps the unit disk onto itself (see Proposition 2.2). Moreover the values of cross-ratios are preserved under the action of Möbius transformations (Proposition 1.18). Consequently

$$\begin{aligned} \Bigl(\mu(z), \Omega(\mu(z)); \mu(w), \Omega(\mu(w))\Bigr) &= \Bigl(\mu(z), \mu(\Omega(z)); \mu(w), \mu(\Omega(w))\Bigr) \\ &= \Bigl(z, \Omega(z); w, \Omega(w)\Bigr). \end{aligned}$$

The required result therefore follows immediately from an identity previously established (Proposition 2.11) expressing the Poincaré distance $\rho(z, w)$ in terms of the cross-ratio $(z, \Omega(z); w, \Omega(w))$. ∎

Proposition 2.14 *Let z_1, w_1, z_2 and w_2 be elements of the open unit disk D, where*

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, where ρ denotes the Poincaré distance function on D. Then there exists a Möbius transformation μ mapping the open unit disk D onto itself with the property that $\mu(z_1) = z_2$ and $\mu(w_1) = w_2$.

Proof The values of the cross-ratios

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) \quad \text{and} \quad (z_2, \Omega(z_2); w_2, \Omega(w_2))$$

are determined by the values of the Poincaré distances $\rho(z_1, w_1)$ and $\rho(z_2, w_2)$ respectively (see Corollary 2.12). Now $\mu(z_1) = z_2$ and $\mu(w_1) = w_2$. Consequently

$$(z_1, \Omega(z_1); w_1, \Omega(w_1)) = (z_2, \Omega(z_2); w_2, \Omega(w_2)).$$

It follows from this that there exists a unique Möbius transformation μ with the properties that $\mu(z_1) = z_2$, $\mu(\Omega(z_1)) = \Omega(z_2)$, $\mu(w_1) = w_2$ and $\mu(\Omega(w_1)) = \Omega(w_2)$, (see Proposition 1.18).

Now let $\hat{\mu} = \Omega \circ \mu \circ \Omega$. Then $\hat{\mu}$ is itself a Möbius transformation (Lemma 2.1) Then

$$\begin{aligned} \hat{\mu}(z_1) &= \Omega(\mu(\Omega(z_1))) = \Omega(\Omega(z_2)) = z_2, \\ \hat{\mu}(\Omega(z_1)) &= \Omega(\mu(\Omega(\Omega(z_1)))) = \Omega(\mu(z_1)) = \Omega(z_2), \\ \hat{\mu}(w_1) &= \Omega(\mu(\Omega(w_1))) = \Omega(\Omega(w_2)) = w_2, \\ \hat{\mu}(\Omega(w_1)) &= \Omega(\mu(\Omega(\Omega(w_1)))) = \Omega(\mu(w_1)) = \Omega(w_2). \end{aligned}$$

Consequently the Möbius transformations μ and $\hat{\mu}$ both map z_1, $\Omega(z_1)$, w_1 and $\Omega(w_1)$ to z_2, $\Omega(z_2)$, w_2 and $\Omega(w_2)$ respectively. But two distinct Möbius transformations cannot coincide at three or more points of the Riemann sphere. (see Proposition 1.9). Consequently $\hat{\mu} = \mu$, and thus $\Omega \circ \mu = \mu \circ \Omega$. Moreover elements z_1 and z_2 of the open unit disk D are mapped into D. Applying Proposition 2.2, we conclude that the Möbius transformation μ maps the open unit disk D onto itself. This completes the proof. ∎

Lemma 2.15 *Let ρ be the Poincaré distance function on the open unit disk D in the complex plane, let t be a real number satisfying $0 < t < 1$, and let w be a complex number distinct from 0 and t for which $|w| < 1$. Then*

$$\rho(0, w) \leq \rho(0, t) + \rho(t, w).$$

Moreover $\rho(0, w) = \rho(0, t) + \rho(t, w)$ if and only if the complex number w is a positive real number for which $t < w < 1$.

Proof We first note that

$$\rho(0, t) = \log\left(\frac{1+t}{1-t}\right)$$

(see Lemma 2.8).

Given a complex number w in the unit disk that is distinct from 0 and t, let real numbers s and u between -1 and 1 be determined so that

$$\log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+s}{1-s}\right) = \rho(t, w)$$

and

$$\log\left(\frac{1+u}{1-u}\right) - \log\left(\frac{1+t}{1-t}\right) = \rho(t, w).$$

Then $-1 < -u < s < t < u < 1$ and

$$\rho(s, t) = \rho(t, u) = \rho(t, w)$$

(again applying Lemma 2.8).

Let μ_0 be the Möbius transformation of the Riemann sphere defined such that $\mu_0(-1/t) = \infty$, $\mu_0(\infty) = 1/t$ and $\mu_0(z) = (z+t)/(1+tz)$ for all complex numbers z distinct from $-1/t$. Then the Möbius transformation μ_0 maps the unit disk onto itself (Corollary 2.6), is an isometry of the Poincaré distance function (Proposition 2.13), cannot map a circle contained within the unit disk onto any straight line, and therefore maps circles contained within the unit disk onto circles within that disk (Proposition 1.11). Moreover the Möbius transformation μ_0 has the property that $\mu_0(\overline{z}) = \overline{\mu_0(z)}$ for all complex numbers z and therefore must map circles within the unit disk that are centred on points of the real line to circles that are also centred on points of the real line.

Let

$$C_0 = \{z \in D : \rho(0, z) = \rho(t, w)\}.$$

Then C_0 is a circle contained in the unit disk (Proposition 2.9), and $\mu_0(C_0) = C$, where

$$C = \{z \in D : \rho(t, z) = \rho(t, w)\}.$$

Consequently the subset C of the unit disk D, being the image of a circle centred on zero under the Möbius transformation μ_0, must be a circle contained within the unit disk and centred on a point of the complex plane that belongs to the open interval in the real line bounded by -1 and -1.

Now $s \in C$ and $u \in C$. It follows that the centre of the circle C is $\frac{1}{2}(u+s)$, and the radius of the circle C is circle C is $\frac{1}{2}(u - s)$. Consequently all points of the circle C other than u lie inside the circle centred on the origin that passes through the point u. The latter circle is the circle

$$\{z \in \mathbb{C} : \rho(0, z) = \rho(0, t) + \rho(t, w)\}.$$

Moreover w lies on the circle C. It follows that

$$\rho(0, w) \leq \rho(0, t) + \rho(t, w).$$

Moreover

$$\rho(0, w) = \rho(0, t) + \rho(t, w)$$

if and only if w is a real number satisfying $t < w < 1$, as required. ∎

Proposition 2.16 (Triangle Inequality for Poincaré Distance) *The Poincaré distance function ρ on the open unit disk D has the property that*

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3)$$

for all complex numbers z_1, z_2 and z_3 belonging to the disk D.

Proof This inequality follows directly in cases where any two of z_1, z_2 and z_3 coincide with one another. Accordingly it remains to prove that the inequality holds in cases where these three complex numbers are distinct.

Accordingly let z_1, z_2 and z_3 be any three distinct points of the unit disk D. Then there exists a Möbius transformation μ that maps the unit disk onto itself and satisfies $\mu(z_1) = 0$ and $\mu(z_2) = t$ for some real number t satisfying the inequalities $0 < t < 1$ (see Proposition 2.14). Let $w = \mu(z_3)$. We have already shown that

$$\rho(0, w) \leq \rho(0, t) + \rho(t, w).$$

But the Möbius transformation μ is an isometry of the Poincaré distance function (Proposition 2.13). Consequently

$$\rho(z_1, z_3) \leq \rho(z_1, z_2) + \rho(z_2, z_3).$$

as required. ∎

2.3 Geodesics in the Open Unit Disk

Definition We say that an (open) straight line segment or circular arc within the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is *complete* if it is the intersection of the open unit disk with the full circle or straight line in the complex plane of which it forms part.

A complete straight line segment or circular arc in the open unit disk has no endpoints in the open unit disk itself. However its closure has endpoints that lie on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ that constitutes the boundary of the open unit disk: the complete straight line segment or circular arc may be said to *join* the endpoints of its closure in the complex plane.

Definition A straight line segment or circular arc Γ in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is said to be a *geodesic* if it has the property that

$$\rho(z_1, z_3) = \rho(z_1, z_2) + \rho(z_2, z_3)$$

for all complex numbers z_1, z_2 and z_3 positioned on the straight line segment or circular arc Γ so that z_2 occurs between z_1 and z_3.

Definition A *complete geodesic* in the open unit disk is a geodesic in that disk which is the intersection of the open unit disk with a full straight line or circle in the complex plane.

Definition A *geodesic ray* in the open unit disk is a geodesic in that disk which is the intersection of the open unit disk with a closed straight line segment or circular arc in the complex plane for which one endpoint lies in the open unit disk and the other lies outside the open unit disk.

Definition A *geodesic segment* in the open unit disk is a geodesic that is also a closed straight line segment or circular arc contained in the open unit disk both of whose endpoints lie in the open unit disk.

Definition Given a point η on the unit circle in the complex plane, the *diameter* of the unit disk that *joins* $-\eta$ and η is the open straight line segment consisting of those complex numbers that are of the form $t\eta$ for some real number t satisfying the inequalities $-1 < t < 1$.

Proposition 2.17 *Let D be the open unit disk in the complex plane, Then the diameter of the disk D obtained on intersecting the disk D with the real axis of the complex plane is a complete geodesic.*

Proof Let I be the set of real numbers t satisfying $|t| < 1$ and let t_1, t_2 and t_3 be real numbers satisfying $-1 < t_1 < t_2 < t_3 < 1$. It follows from Lemma 2.8 that

$$\begin{aligned}\rho(t_1, t_3) &= \log\left(\frac{1+t_3}{1-t_3}\right) - \log\left(\frac{1+t_1}{1-t_1}\right) \\ &= \log\left(\frac{1+t_3}{1-t_3}\right) - \log\left(\frac{1+t_2}{1-t_2}\right) \\ &\quad\quad + \log\left(\frac{1+t_2}{1-t_2}\right) - \log\left(\frac{1+t_1}{1-t_1}\right) \\ &= \rho(t_1, t_2) + \rho(t_2, t_3).\end{aligned}$$

Thus I is indeed a geodesic in the open unit disk D. ∎

Proposition 2.18 *Given any real number t satisfying $0 < t < 1$, the unique complete geodesic in the open unit disk that passes through both 0 and t is the diameter of the disk obtained on intersecting the disk with the real axis of the complex plane.*

Proof Let Γ be a complete geodesic in the open unit disk D that passes through 0 and t, and let z be chosen on Γ so that t lies between 0 and z. Then $\rho(0, z) = \rho(0, t) + \rho(t, z)$, where ρ denotes the Poincaré distance function on D. Applying Lemma 2.15, we see that z must be a real number between t and 1. Consequently the three points 0, t and z on Γ are real numbers. Now the geodesic Γ must be the intersection of the open unit disk D with a straight line or circle in the complex plane. It follows that Γ must coincide with the intersection of the open unit disk with the real axis of the complex plane. The result follows. ∎

Proposition 2.19 *Möbius transformations mapping the open unit disk onto itself map geodesics onto geodesics.*

Proof Let Γ be a geodesic in the open unit disk D, where $D = \{z \in \mathbb{C} : |z| < 1\}$, and let μ be a Möbius transformation that maps the open unit disk D onto itself. Let w_1, w_2 and w_3 be complex numbers positioned on the image $\mu(\Gamma)$ of the geodesic Γ so that w_2 occurs on $\mu(\Gamma)$ between w_1 and w_3. Then there exist complex numbers z_1, z_2 and z_3 in the open unit disk D lying on the geodesic Γ for which $\mu(z_1) = w_1$, $\mu(z_2) = w_2$ and $\mu(z_3) = w_3$. Moreover z_2 is positioned on Γ between z_1 and z_3. The definition of geodesics then ensures that

$$\rho(z_1, z_3) = \rho(z_1, z_2) + \rho(z_2, z_3)$$

Now $\rho(w_1, w_2)$, $\rho(w_2, w_3)$ and $\rho(w_1, w_3)$ are equal to $\rho(z_1, z_2)$, $\rho(z_2, z_3)$ and $\rho(z_1, z_3)$ respectively, because Möbius transformations that map the open unit disk onto itself are isometries with respect to Poincaré distance (see Proposition 2.13) Consequently

$$\rho(w_1, w_3) = \rho(w_1, w_2) + \rho(w_2, w_3).$$

Thus the line segment or circular arc $\mu(\Gamma)$ is a geodesic, as required. ∎

Proposition 2.20 *Let A be a complete straight line segment or circular arc in the open unit disk D. Suppose that there are complex numbers z_1, z_2 and z_3 on A, where z_2 lies between z_1 and z_3, such that*

$$\rho(z_1, z_3) = \rho(z_1, z_2) + \rho(z_2, z_3).$$

Then A is a complete geodesic in the open unit disk D, and moreover there exists a Möbius transformation μ with the property that $\mu(A)$ is the diameter of the open unit disk that joins -1 and 1.

Proof Let

$$t = \frac{e^\delta - 1}{e^\delta + 1}, \quad \text{where } \delta = \rho(z_1, z_2).$$

Then $\rho(0, t) = \rho(z_1, z_2)$. (see Proposition 2.9). Then there exists a Möbius transformation μ of the Riemann sphere mapping the open unit disk D onto itself which has the properties that $\mu(z_1) = 0$ and $\mu(z_2) = t$. (see Proposition 2.14). Let $w = \mu(z_3)$. Then

$$\rho(0, w) = \rho(0, t) + \rho(t, w),$$

because the Möbius transformation μ is an isometry of the Poincaré distance function ρ. It now follows from Lemma 2.15 that w is a real number and

$t \leq w < 1$. The complex numbers z_1, z_2 and z_3 therefore all lie on the straight line or circle in the complex plane that is the image of the real axis under the inverse μ^{-1} of the Möbius transformation μ. But two distinct straight lines or circles cannot pass through the three points z_1, z_2 and z_3. Consequently the complete arc A is contained in the image of the real axis under μ^{-1}, and therefore the Möbius transformation μ must map the complete arc onto the diameter of the open unit disk that joins -1 and 1. Moreover A must itself be a geodesic, because Möbius transformations that map the open unit disk D onto itself map geodesics onto geodesics Proposition 2.19. This completes the proof. ∎

Corollary 2.21 *A complete straight line segment or circular arc A in the open unit disk D is a complete geodesic if and only if there exists a Möbius transformation μ that maps the straight line segment or circular arc onto a diameter of the unit circle.*

Proof If A is a complete geodesic then a direct application of Proposition 2.20 ensures that existence of a Möbius transformation mapping that complete geodesic onto the diameter of the disk D that joins -1 and 1.

Conversely if some Möbius transformation maps a complete straight line segment or circular arc onto a diameter, then that Möbius transformation can be composed with a rotation of the open unit disk about zero so as to obtain a Möbius transformation mapping the complete straight line segment or circular arc onto the diameter of the disk that is the intersection of the disk with the real axis of the complex plane. That diameter is a geodesic (see Proposition 2.17), and Möbius transformations map geodesics onto geodesics (Proposition 2.19). Consequently A must itself be a geodesic, as required. ∎

Proposition 2.22 *Given two complete geodesics in the open unit disk D, there exists a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and maps one complete geodesic onto the other.*

Proof Let Γ_1 and Γ_2 be complete geodesics in the open unit disk D, and let I be the geodesic joining -1 and 1 that is the intersection of the disk D with the real axis of the complex plane. It follows from Proposition 2.20 that there exist Möbius transformations μ_1 and μ_2 of the Riemann sphere that map the open unit disk onto itself, where μ_1 maps Γ_1 onto I and μ_2 maps Γ_2 onto I. Then $\mu_2^{-1} \circ \mu_1$ is a Möbius transformation of the Riemann sphere that maps the open unit disk D onto itself and also maps the complete geodesic Γ_1 onto the complete geodesic Γ_2, as required. ∎

Proposition 2.23 *Given two distinct complex numbers w_1 and w_2 belonging to the open unit disk in the complex plane, there exists a unique complete geodesic in the open unit disk that passes through both w_1 and w_2.*

Proof Let

$$t = \left| \frac{w_2 - w_1}{1 - \overline{w}_1 w_2} \right|.$$

Then there exists a complex number η satisfying $|\eta| = 1$ for which $t = \mu(w_2)$, where μ is the Möbius transformation of the Riemann sphere that satisfies

$$\mu(z) = \frac{\eta(z - w_1)}{1 - \overline{w}_1 z}.$$

for all complex numbers z satisfying $1 - \overline{w}_1 z \neq 0$. Then μ maps the open unit disk onto itself and also maps w_1 and w_2 to 0 and t respectively. Let $\Gamma = \{z \in D : \mu(z) \in I\}$, where I is the diameter of the open unit disk consisting of all real numbers lying between -1 and 1. The Möbius transformation μ maps Γ onto the diameter I of the disk. Consequently Γ must be a geodesic in the unit disk (Corollary 2.21). This geodesic passes through w_1 and w_2.

We now show that Γ is the unique complete geodesic in the open unit disk that passes through w_1 and w_2. Let Γ' be a complete geodesic in the open unit disk that passes through w_1 and w_2. Then $\mu(\Gamma')$ is also a complete geodesic in the open unit disk, because Möbius transformations that map the open unit disk onto itself map geodesics onto geodesics (Proposition 2.19). But the distinct real numbers 0 and t lie on $\mu(\Gamma')$. It follows from Proposition 2.18 that $\mu(\Gamma')$ is the diameter I of the open unit disk consisting of all real numbers between -1 and 1. Consequently $\Gamma' \subset \Gamma$. The completeness of Γ' then ensures that Γ' coincides with Γ. Thus the complete geodesic Γ is indeed uniquely determined by w_1 and w_2, as required. ∎

Proposition 2.24 *A complete straight line segment or circular arc in the unit disk is a complete geodesic if and only if the straight line or circle in the complex plane of which it forms part intersects the unit circle at right angles.*

Proof A complete straight line segment or circular arc A in the open unit disk D is a complete geodesic if and only if there exists a Möbius transformation μ that maps the arc onto a diameter of the unit circle (see Corollary 2.21).

The diameters of a circle intersect the circle at right angles, and angles between intersecting straight lines and circles are preserved under the action of Möbius transformations (see Proposition 1.27). Consequently if a complete circular arc is a geodesic then it is part of a circle that intersects the unit circle at right angles.

Conversely suppose that a complete circular arc A in the unit circle forms part of a circle that intersects the unit circle at right angles at z_1 and z_2, where $|z_1| = 1$ and $|z_2| = 1$. There then exists a Möbius transformation μ mapping the unit disk D onto itself for which $\mu(z_1) = -1$ and $\mu(z_2) = 1$ (see Lemma 2.4). The image $\mu(A)$ of the circular arc A under μ then intersects the boundary circle at right angles at -1 and 1, because Möbius transformations are angle-preserving. But Möbius transformations map circular arcs to circular arcs or straight lines. It follows that $\mu(A)$ must be the diameter of the unit circle that is the intersection of the open unit disk with the real axis. Consequently the complete circular arc A must be a geodesic (Corollary 2.21). The result follows. ∎

2.4 The Group of Hyperbolic Motions of the Disk

Definition Let X be a subset of the complex plane. A collection of invertible transformations of the set X is said to be a *transformation group* acting on the set X if the following conditions are satisfied:

(i) the identity transformation belongs to the collection;

(ii) any composition of transformations belonging to the collection must itself belong to the collection;

(iii) the inverse of any transformation belonging to the collection must itself belong to the collection.

The collection of all Möbius transformations of the Riemann sphere that map the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ onto itself is a transformation group acting on the open unit disk. Indeed the identity transformation is a Möbius transformation mapping the open unit disk onto itself, the composition of any two Möbius transformations that each map the open unit disk onto itself must also map the open unit disk onto itself, and the inverse of any Möbius transformation that maps the open unit disk onto itself must also map the open unit disk onto itself.

Definition Let D be the open unit disk in the complex plane, defined so that $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $\kappa: D \to D$ be the transformation of the open unit disk defined so that $\kappa(z) = \overline{z}$ for all $z \in D$, where $\overline{z}$ denotes the complex conjugate of the complex number z. A transformation of the open unit disk is said to be a *hyperbolic motion* of the unit disk if either it is a Möbius transformation mapping the unit disk D onto itself or else it expressible as a composition of transformations of the form $\mu \circ \kappa$, where μ is a Möbius transformation mapping the open unit disk onto itself.

Möbius transformations give rise to orientation-preserving transformations of the complex plane (see Proposition 1.28 and the discussion of orientation-preserving and orientation-reversing transformations of the complex plane that follows the proof of that proposition). Also the transformation $\kappa: D \to D$ that maps each complex number z in D to its complex conjugate $\overline{z}$ is orientation-reversing. Consequently a composition of two transformations in which some Möbius transformation follows the complex conjugation transformation κ is orientation-reversing.

Orientation-preserving hyperbolic motions are the analogues, in hyperbolic geometry, of transformations of the flat Euclidean plane that can be represented as the composition of a rotation followed by a translation.

Orientation-reversing hyperbolic motions are the analogues, in hyperbolic geometry, of reflections and glide reflections of the flat Euclidean plane.

Proposition 2.25 *Let D be the open unit disk in the complex plane, consisting of those complex numbers z that satisfy $|z| < 1$. Then, given any orientation-preserving hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$, such that*

$$\varphi(z) = \frac{az + b}{\overline{b}\, z + \overline{a}} \quad \textit{for all } z \in D.$$

Similarly, given any orientation-reversing hyperbolic motion φ of the open unit disk D, there exist complex numbers a and b, where $|b| < |a|$ such that

$$\varphi(z) = \frac{a\, \overline{z} + b}{\overline{b}\, \overline{z} + \overline{a}} \quad \textit{for all } z \in D.$$

Proof This result follows directly on applying Proposition 2.7. ∎

Proposition 2.26 *The collection of all hyperbolic motions of the open unit disk is a transformation group acting on the open unit disk.*

Proof The identity transformation is a Möbius transformation that maps the open unit disk onto itself and is thus a hyperbolic motion. Next let μ_1 and μ_2 be Möbius transformations that map the open unit disk onto itself, Then $\kappa \circ \mu_2 \circ \kappa$ is also a Möbius transformation that maps the open unit disk onto itself. Indeed there exist complex numbers a_2 and b_2, where $|b_2| < |a_2|$, such that

$$\mu_2(z) = \frac{a_2 z + b_2}{\overline{b}_2\, z + \overline{a}_2}$$

for all complex numbers z for which $\overline{b}_2\, z + \overline{a}_2 \neq 0$ (see Proposition 2.7). Then

$$\kappa(\mu_2(\kappa(z))) = \frac{\overline{a}_2 z + \overline{b}_2}{b_2\, z + a_2},$$

and therefore $\kappa \circ \mu \circ \kappa$ is also a Möbius transformation that maps the open unit disk D onto itself. Now

$$\mu_1 \circ (\mu_2 \circ \kappa) = (\mu_1 \circ \mu_2) \circ \kappa, \quad (\mu_1 \circ \kappa) \circ \mu_2 = (\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)) \circ \kappa$$

and

$$(\mu_1 \circ \kappa) \circ (\mu_2 \circ \kappa) = \mu_1 \circ (\kappa \circ \mu_2 \circ \kappa).$$

Moreover $\mu_1 \circ \mu_2$ and $\mu_1 \circ (\kappa \circ \mu_2 \circ \kappa)$, being compositions of Möbius transformations that map the open unit disk onto itself, are themselves Möbius transformations that map the open unit disk onto itself. It follows from this observation that any composition of hyperbolic motions of the open unit disk is itself a hyperbolic motion of the open unit disk. Also

$$(\mu_2 \circ \kappa)^{-1} = \kappa \circ \mu_2^{-1} = (\kappa \circ \mu_2^{-1} \circ \kappa) \circ \kappa,$$

and the inverse of any Möbius transformation that maps the open unit disk onto itself must itself be a Möbius transformation that maps the open unit disk onto itself. Consequently the inverse of any hyperbolic motion is itself a hyperbolic motion. It follows that the collection of all hyperbolic motions of the open unit disk is indeed a transformation group acting on the open unit disk. ∎

Proposition 2.27 *Let Γ be a complete geodesic in the open unit disk D. Then there exists an orientation-reversing hyperbolic motion φ with the property that $\varphi(z) = z$ for all complex numbers z that lie on the geodesic Γ and also those points of the open unit disk D that lie on one side of the geodesic Γ are mapped by points that lie on the other side of Γ.*

Proof Let I be the set of real numbers t that satisfy the inequalities $-1 < t < 1$. Then I is a complete geodesic in the open unit disk D. There then exists a Möbius transformation μ that maps the geodesic I onto the geodesic Γ. (see Proposition 2.20 or Proposition 2.22). Let $\varphi = \mu \circ \kappa \circ \mu^{-1}$, where $\kappa(z) = \overline{z}$ for all $z \in D$. Then the orientation-reversing hyperbolic motion Γ has the required properties. ∎

Proposition 2.28 *Let z_1, w_1, z_2 and w_2 be complex numbers belonging to the open unit disk D. Suppose that $\rho(z_1, w_1) = \rho(z_2, w_2)$, and suppose also that one of the sides of the geodesic Γ_1 in D passing through z_1 and w_1 has been chosen, and that one of the sides of the geodesic Γ_2 in D passing through z_2 and w_2 has also been chosen. Then there exists a hyperbolic motion φ with the following properties: $\varphi(z_1) = z_2$; $\varphi(w_1) = w_2$; φ maps complex numbers on the chosen side of the geodesic Γ_1 to complex numbers on the chosen side of the geodesic Γ_2.*

Proof It follows from Proposition 2.14 there exists a Möbius transformation that maps the open unit disk onto itself and also maps z_1 and w_1 to z_2 and w_2 respectively. If this Möbius transformation does not itself map the chosen side of Γ_1 to the chosen side of Γ_2, then it may be composed with an orientation-reversing hyperbolic motion that fixes all complex numbers of the geodesic Γ_2 whilst mapping complex numbers on one side of Γ_2 to complex numbers on the other side. The result follows. ∎

2.5 The Hyperbolic Centre of a Circle in the Disk

Proposition 2.29 *Let w be a complex number belonging to the open unit disk D in the complex plane, and let ρ denote the Poincaré distance function on D. Let δ be a positive real number. Then*

$$\{z \in D : \rho(z, w) < \delta\} = \left\{ z \in D : \left| \frac{z - w}{1 - \overline{w}\, z} \right| < R \right\},$$

where

$$R = \frac{e^\delta - 1}{e^\delta + 1}.$$

Proof Let

$$\mu_w(z) = \frac{z - w}{1 - \overline{w} z}$$

for all complex numbers z. Then μ_w is a Möbius transformation mapping the open unit disk onto itself for which $\mu_w(w) = 0$ (see Corollary 2.6). Now Möbius transformations mapping the open unit disk onto itself are isometries with regard to the Poincaré distance function (see Proposition 2.13). Consequently

$$\{z \in D : \rho(z, w) < \delta\} = \{z \in D : \rho(\mu_w(z), 0) < \delta\}.$$

The required result now follows on applying Proposition 2.9. ∎

Definition Let D be the open unit disk in the complex plane that consists of those complex numbers z satisfying $|z| < 1$, and let C be a circle in the complex plane that is contained within D. A complex number w is said to be the *hyperbolic centre* of the circle C if the Poincaré distance between z and w is the same for all points z that lie on the circle C.

Proposition 2.30 *Let C be a circle in the complex plane that is contained within the open unit disk D. Suppose that the circle C intersects the real axis at real numbers u and v, where $-1 < u < v < 1$. Suppose also that the hyperbolic centre of the circle C lies on the real axis, and is located at t, where $u < t < v$. Then*

$$\left(\frac{1+t}{1-t}\right)^2 = \frac{(1+u)(1+v)}{(1-u)(1-v)}.$$

Proof Applying Lemma 2.8, we find that t, u and v must satisfy the identity

$$\log\left(\frac{1+v}{1-v}\right) - \log\left(\frac{1+t}{1-t}\right) = \log\left(\frac{1+t}{1-t}\right) - \log\left(\frac{1+u}{1-u}\right).$$

Consequently

$$2\log\left(\frac{1+t}{1-t}\right) = \log\left(\frac{1+u}{1-u}\right) + \log\left(\frac{1+v}{1-v}\right).$$

The required result then follows on taking the exponential of both sides of this identity. ∎