MAU23302 Euclidean and Non-Euclidean Geometry
Hilary Term 2021
Worked Solutions to some Sample Problems relating to the Hyperbolic Plane

1. Let μ be a Möbius transformation of the Riemann sphere, and let $\hat{\mu}$ be the transformation of the Riemann sphere defined so that $\hat{\mu}(\infty) = \Omega(\mu(\infty))$ and
$$\hat{\mu}(z) = \Omega(\mu(\overline{z}))$$
for all complex z, where Ω is the transformation of the Riemann sphere defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex number z. Also let the coefficients a, b, c and d of the Möbius transformation be complex constants satisfying $ad - bc \neq 0$ that are chosen so that
$$\mu(z) = \frac{az + b}{cz + d}$$
for all complex numbers z for which $cz + d \neq 0$.

Problem.

(a) Show that $\hat{\mu}$ is also a Möbius transformation of the Riemann sphere by finding complex constants $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ that ensure that
$$\hat{\mu}(z) = \frac{\hat{a}z + \hat{b}}{\hat{c}z + \hat{d}}$$
for all complex numbers z for which $\hat{c}z + \hat{d} \neq 0$.

Solution.

$$\hat{\mu} z = \Omega\left(\frac{a\,\overline{z} + b}{c\,\overline{z} + d}\right) = \frac{\overline{c}\,z + \overline{d}}{\overline{a}\,z + \overline{b}}.$$

Consequently we can take $\hat{a} = \overline{c}$, $\hat{b} = \overline{d}$, $\hat{c} = \overline{a}$, $\hat{d} = \overline{b}$.

Problem.

(b) Now let the coefficients a, b, c and d of the Möbius transformation μ be chosen so that $c = 1$, and let $w = -\overline{d}$. Suppose also that $\mu(\infty) = \Omega(\mu(\infty))$ and $\mu(\overline{z}) = \Omega(\mu(z))$ for all complex numbers z. Explain why $\mathrm{Im}[w] \neq 0$, and show that there exists some complex constant η satisfying $|\eta| = 1$ that is determined so that $\mu(\overline{w}) = \infty$, $\mu(\infty) = 1$ and

$$\mu(z) = \eta\,\frac{z - w}{z - \overline{w}}$$

for all complex numbers z distinct from $\overline{w}$.

Solution.

Using the result of (a), and applying Proposition 1.8 we see that there must exist some non-zero complex number η such that $a = \eta\hat{a}$, $b = \eta\hat{b}$, $c = \eta\hat{c}$ and $d = \eta\hat{d}$. Accordingly

$$a = \eta\overline{c}, \quad b = \eta\overline{d}, \quad c = \eta\overline{a}, \quad d = \eta\overline{b}.$$

Now we have chosen coefficients so that $c = 1$. Then $\overline{c} = 1$. It follows that $a = \eta$ and

$$1 = \eta\overline{a} = \eta\,\overline{\eta} = |\eta|^2.$$

Consequently $|\eta| = 1$. Also $d = -\overline{w}$ and $b = -\eta w$. Consequently

$$\mu(z) = \eta\,\frac{z - w}{z - \overline{w}}$$

whenever $z \neq \overline{w}$. Standard properties of Möbius transformations then ensure that $\mu(\overline{w})$ and $\mu(\infty)$ have the values stated. Now if w were a real number then it would follow that $ad - bc = 0$. But the coefficients a, b, c and d of a Möbius transformation are required to satisfy the condition $ad - bc \neq 0$. Consequently it must be the case that $\mathrm{Im}[w] \neq 0$.

2. Let w be a complex number satisfying $\operatorname{Im}[w] > 0$, let K be a real number satisfying $0 < K < 1$, and let

$$Q = \{z \in \mathbb{C} : |z - w| = K\,|z - \overline{w}|\}.$$

Problem.

Show that the set Q is a circle in the complex plane of radius $\dfrac{2K\operatorname{Im}[w]}{1 - K^2}$ centred on $\dfrac{w - K^2\overline{w}}{1 - K^2}$.

Solution.

Let z be a complex number. Then z lies on the curve Q if and only if

$$(z - w)(\overline{z} - \overline{w}) = K^2(z - \overline{w})(\overline{z} - w).$$

Now

$$\begin{aligned}
&(z - w)(\overline{z} - \overline{w}) - K^2(z - \overline{w})(\overline{z} - w) \\
&\qquad = (1 - K^2)(|z|^2 + |w|^2) - \overline{w}z - w\overline{z} - K^2(wz + \overline{wz}) \\
&\qquad = (1 - K^2)(|z|^2 + |w|^2) - 2\operatorname{Re}[\overline{w}z] + 2K^2\operatorname{Re}[wz] \\
&\qquad = (1 - K^2)(|z|^2 + |w|^2) - 2\operatorname{Re}[(\overline{w} - K^2 w)z]
\end{aligned}$$

It follows that $z \in Q$ if and only if

$$\begin{aligned}
0 &= |z|^2 - 2\operatorname{Re}\left[\frac{\overline{w} - K^2 w}{1 - K^2}\,z\right] + |w|^2 \\
&= \left(z - \frac{w - K^2\overline{w}}{1 - K^2}\right)\left(\overline{z} - \frac{\overline{w} - K^2 w}{1 - K^2}\right) + |w|^2 - \left|\frac{w - K^2\overline{w}}{1 - K^2}\right|^2 \\
&= \left|z - \frac{w - K^2\overline{w}}{1 - K^2}\right|^2 + |w|^2 - \left|\frac{w - K^2\overline{w}}{1 - K^2}\right|^2.
\end{aligned}$$

Now

$$\begin{aligned}
&|w - K^2\overline{w}|^2 - (1 - K^2)^2|w|^2 \\
&\qquad = (w - K^2\overline{w})(\overline{w} - K^2 w) \\
&\qquad\qquad - (1 - 2K^2 + K^4)|w|^2 \\
&\qquad = (1 + K^4)|w|^2 - K^2(w^2 + \overline{w}^2) \\
&\qquad\qquad - (1 - 2K^2 + K^4)|w|^2 \\
&\qquad = 2K^2(|w|^2 - \operatorname{Re}[w^2]) \\
&\qquad = 2K^2(|w|^2 - \operatorname{Re}[w]^2 + \operatorname{Im}[w]^2) \\
&\qquad = 4K^2\operatorname{Im}[w]^2).
\end{aligned}$$

Consequently $z \in Q$ if and only if

$$\left| z - \frac{w - K^2\overline{w}}{1 - K^2} \right|^2 = \frac{4K^2\,\mathrm{Im}[w]^2)}{(1 - K^2)^2}.$$

Consequently the curve Q is a circle of radius $\dfrac{2K\,\mathrm{Im}[w]}{1 - K^2}$ centred on $\dfrac{w - K^2\overline{w}}{1 - K^2}$.

3. Let H be the upper half-plane within the complex plane, defined so that

$$H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}.$$

Hyperbolic motions of the upper half-plane, in the context of the half-plane model of hyperbolic geometry, are transformations of the upper half plane that are angle-preserving and are isometries of the Poincaré metric σ on the upper half-plane, where

$$\sigma(z, w) = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}$$

for all complex numbers z and w belonging to the upper half-plane H. The orientation-preserving hyperbolic motions of the upper half-plane H are the restrictions to H of Möbius transformations of the Riemann sphere that map the upper half-plane H onto itself. A Möbius transformations μ of the Riemann sphere maps the upper half plane onto itself if and only if there exist real numbers a, b, c and d satisfying $ad - bc = 1$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then $\mu(\infty) = \infty$ in cases where $c = 0$. Also $\mu(\infty) = a/c$ and $\mu(-d/c) = \infty$ in cases where $c \neq 0$.

There are also orientation-reversing hyperbolic motions of the upper half plane. These can be expressed as the restriction to the upper half-plane H of compositions of transformations of the Riemann sphere of the form $\mu \circ \hat{\kappa}$, where μ is some Möbius transformation that maps the upper half-plane onto itself and where τ is the transformation of the Riemann sphere defined so that $\tau(\infty) = \infty$ and $\tau(z) = -\overline{z}$ for all complex numbers z. Accordingly, given an orientation-reversing hyperbolic motion φ of the upper half-plane H, there exist real numbers a, b, c and d satisfying $ad - bc = 1$ such that

$$\varphi(z) = \frac{b - a\overline{z}}{d - c\overline{z}}.$$

Problem.

Let φ be an orientation-preserving hyperbolic motion φ of the upper half-plane H, where φ is determined by real coefficients a, b, c and d as described above, and let

$$\Gamma = \{z \in H : \varphi(z) = z\}.$$

Suppose that $c \neq 0$, $a = d$, and accordingly $a^2 - bc = 1$. Show that Γ is a geodesic in the upper half-plane H and determine the centre and the radius of the circle in the complex plane whose intersection with the upper half-plane H is the geodesic Γ.

Solution.

In the case where $a = d$, the curve Γ is the intersection of the upper half-plane H with the set

$$\{z \in \mathbb{C} : (a - c\overline{z})z = b - a\overline{z}\}.$$

Consequently a complex number z in the upper half-plane lies on the curve Γ if and only if

$$c|z|^2 - a\overline{z} - az + b = 0.$$

Now $\overline{z} + z = 2\,\mathrm{Re}[z]$. Thus a complex number z lies on the curve Γ if and only if

$$\begin{aligned} 0 &= c|z|^2 - a\overline{z} - az + b \\ &= c|z|^2 - 2a\,\mathrm{Re}[z] + b \\ &= c\left(\left(z - \frac{a}{c}\right)\left(\overline{z} - \frac{a}{c}\right) + \frac{bc - a^2}{c}\right) \\ &= c\left(\left|z - \frac{a}{c}\right|^2 - \frac{1}{c^2}\right) \end{aligned}$$

Consequently

$$\Gamma = \left\{ z \in H : \left|z - \frac{a}{c}\right| = \frac{1}{c} \right\}.$$

It follows that Γ is a geodesic, being the intersection of the upper half-plane with a circle in the complex plane centred on the real number a/c and with radius equal to $1/c$.