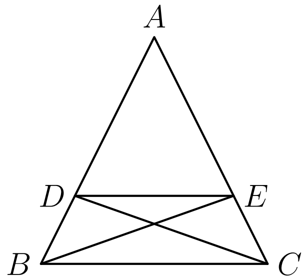
MAU23302 Euclidean and Non-Euclidean Geometry
Hilary Term 2021
Worked Solutions to some Sample Problems relating to Euclidean Geometry

Note. Proofs answering the Euclidean geometry questions below should be based on propositions included in the first book of Euclid's *Elements of Geometry*. Whenever you apply propositions that succeed Proposition 15 in Book I of Euclid's *Elements*, you should cite the book and proposition number. Standard congruence rules may be cited as the SAS, SSS, ASA or SAA *Congruence Rule* as appropriate. Propositions 5 and 15 of Book I of Euclid's *Elements* may be cited as the *Isosceles Triangles Theorem* and the *Vertically-opposite Angles Theorem* respectively.

1. Let a geometric configuration in the Euclidean plane be as depicted below. In this configuration, we suppose that the straight line segments AB and AC are equal to one another in length, and also that the straight line segments BD and CE are equal to one another in length.



Making use of no proposition in Book I of Euclid's *Elements of Geometry* with the exception of the SAS Congruence Rule (Proposition 4), and *only applying Proposition 4 in cases where the triangles being compared represent distinct regions of the plane, and thus not attempting to apply Proposition 4 to compare two triangles where the second of those triangles is obtained from the first by merely relabelling the vertices of the first triangle*, prove that the angles ABC and ACB at the base of the isosceles triangle ABC are equal to one another.

Solution.

The straight line segments AD and AE are equal in length, because they are obtained when equals BD and CE are respectively subtracted from equals AB and AC (*Elements*, Common Notion 3).

In the triangles ABE and ACD the sides AB and AE are respectively equal in length to the sides AC and AE, and the included angle ABC is common. Applying the SAS Congruence Rule (*Elements*, Proposition 4), it follows that the triangles ABE and ACD are congruent, and therefore the remaining sides BE and CD are equal in length and also the angles ABE and AEB are respectively equal to the angles ACD and ADC.

We now note that, in the triangles BDE and CED, the sides BD and BE are respectively equal in length to the sides CE and CD, and the included angle DBE is equal to the included angle ECD. Applying the SAS Congruence Rule (*Elements*, Proposition 4), it follows that the triangles BDE and BED are congruent, and therefore the angle

BDE is equal to the angle CED, and also the angle BED is equal to the angle CDE.

Now the angle BDC is the remainder left when angle CDE is subtracted from angle BDE, and angle CEB is the remainder left when angle BED is subtracted from angle CED. But angles CDE and BED are equal to one another and angles BDE and CED are equal to one another. It follows, on applying Common Notion 3, that the angles BDC and CEB are equal to one another.

Finally we note that, in the triangles DBC and ECB, the sides DB and DC are respectively equal to the sides EC and EB, and the included angles BDC and CEB have been shown to be equal to one another. Applying the SAS Congruence Rule (*Elements*, Proposition 4), it follows that the triangles DBC and ECB are congruent, and therefore the angles DBC and ECB are equal to one another. But angle DBC coincides with angle ABC, and angle ECB coincides with angle ACB. Consequently the angles ABC and ACB are equal to one another, as required.

Presented more symbolically:

Note that $AD = AB - BD$ and $AE = AC - CE$. But $AB = AC$ and $BD = CE$. Consequently $AD = AE$ (Common Notion 3).

In triangles ABE and ACD, $AB = AC$, $AE = AD$ and $\angle BAE = \angle CAD$ (as $\angle BAE$ and $\angle CAD$ both coincide with $\angle BAC$). Consequently $BE = CD$, $\angle ABE = \angle ACD$ and $\angle AEB = \angle ADC$ (SAS, Proposition 4).

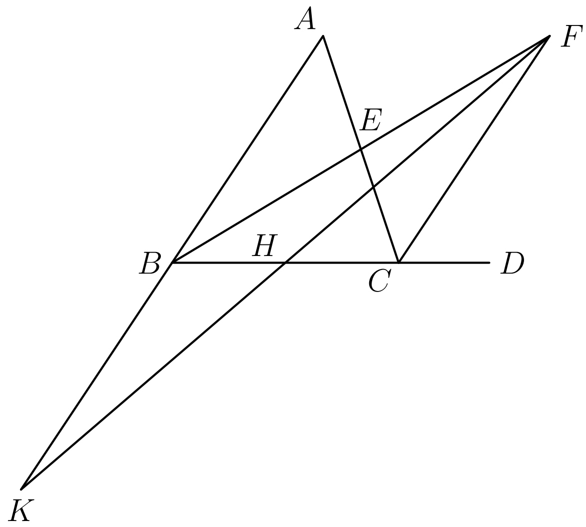
Then, in triangles BDE and CED, $BD = CE$, $BE = CD$ and $\angle DBE = \angle ABE = \angle ACD = \angle ECD$. Consequently $\triangle BDE$ and $\triangle CED$ are congruent, $\angle BDE = \angle CED$ and $\angle BED = \angle CDE$ (SAS, Proposition 4).

Now $\angle BDC = \angle BDE - \angle CDE$, $\angle CEB = \angle CED - \angle BED$. Moreover $\angle BDE = \angle CED$ and $\angle CDE = \angle BED$. Consequently $\angle BDC = \angle CEB$ (Common Notion 3).

Then, in triangles DBC and ECB, $DB = EC$ and $DC = EB$ and $\angle BDC = \angle CEB$. Consequently $\triangle DBC$ and $\triangle ECB$ are congruent, and therefore $\angle DBC = \angle ECB$ (SAS, Proposition 4).

But $\angle DBC = \angle ABC$ and $\angle ECB = \angle ACB$. Consequently $\angle ABC = \angle ACB$, as required.

2. Let a geometric configuration in the Euclidean plane be as depicted below. In this configuration, we suppose that E is the midpoint of the line segment AC and also of the line segment BF, and that H is the midpoint of the line segment BC, and also of the line segment FK.



Using only propositions from Euclid's *Elements of Geometry* that precede Proposition 16 in Book I, without making use of the Fifth Postulate, *and without assuming that the points A, B and K are collinear*, prove that the line segments AB and BK are equal in length.

[Note: it would not be possible to prove that the points A, B and K are collinear without making use of either the Fifth Postulate, Proposition 29, or some proposition that follows Proposition 29 in Book I of Euclid's *Elements of Geometry.* In a typical matching configuration in hyperbolic geometry, no single geodesic would pass through all three of the points A, B and K.]

Solution.

In triangles AEB and CEF, the sides AE and CE are equal in length, and the sides EB and EF are equal in length, because E is the midpoint of both AC and BF. Also the included angles AEB and CEF are equal to one another (Vertically-opposite angles, Proposition 15). Applying the SAS Congruence Rule (*Elements*, Proposition 4), it follows that the triangles AEB and CEF are congruent, and therefore the sides AB and CF are equal to one another in length, the angles EAB and

ECF are equal to one another, and the angle EBA and EFC are equal to one another.

We now consider the triangles CHF and BHK. Now the sides CH and BH are equal in length, and the sides HF and HK are equal in length, because the point H is the midpoint of both BC and FK. (Vertically-opposite angles, Proposition 15). Applying the SAS Congruence Rule (*Elements*, Proposition 4), it follows that the triangles CHF and BHK are congruent, and therefore the sides CF and BK are equal to one another in length.

We have now shown that the sides AB and BK are both equal to the side CF. They are therefore equal to one another (Common Notion 1), as required.

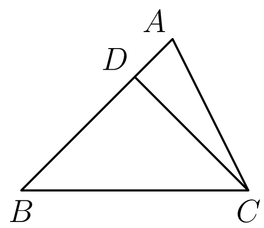
Presented more symbolically:

In triangles AEB and CEF, $AE = CE$ and $EB = EF$. Also $\angle AEB = \angle CEF$ (Vertically-opposite angles, Proposition 15). Consequently $\triangle AEB$ and $\triangle CEF$ are congruent, and therefore $AB = CF$, $\angle EAB = \angle ECF$ and $\angle EBA = \angle EFC$ (SAS, Proposition 4).

Next, in triangles CHF and BHK, $CH = BH$ and $HF = HK$. Also $\angle CHF = \angle BHK$ (Vertically-opposite angles, Proposition 15). Consequently $\triangle CHF$ and $\triangle BHK$ are congruent, and therefore $CF = BK$ (SAS, Proposition 4).

Thus $AB = CF = BK$, as required.

3. Let A, B and C be the vertices of a triangle in the Euclidean plane in which the angle ACB at the vertex C is greater than the angle ABC at the vertex B. Let a point D be taken on the side AB of this triangle so as to ensure that the angles ABC and BCD are equal to one another.



Without using any proposition in Book I of Euclid's *Elements of Geometry* that follows Proposition 15, with the exception of Proposition 20 (which ensures that the sum of the lengths of any two sides of a triangle is greater than the length of the remaining side), and assuming Proposition 20 to be valid and axiomatic, (and without essentially incorporating the proof of any proposition in Book I that follows Proposition 15 so as to avoid explicit citation of that proposition by reproving it,) prove that the side AB of the triangle ABC is in length greater than the side AC of that triangle.

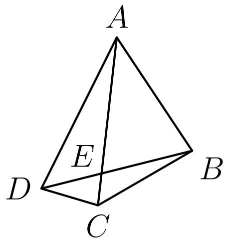
Solution.

The angles ABC and BCD are equal to one another, by construction, and the angles ABC and DBC coincide. Consequently the angles DBC and DCB of the triangle DBC are equal to one another, and consequently that triangle is an isosceles triangles with equal sides DB and DC (Proposition 6). It follows that the side AB of the triangle ABC is equal in length to the sum of the sides AD and DC of the triangle ADC. But the sum of the sides AD and DC is greater in length than the remaining side AC of the triangle ADC (Proposition 20). Consequently the side AB of the triangle ABC is greater in length than the side AC, as required.

Presented more symbolically:

From the construction, $\angle DBC = \angle ABC = \angle DCB$. Consequently $\triangle DBC$ is isosceles, and $DB = DC$ (Proposition 6). It follows that $AB = AD + DB = AD + DC$. But $AD + DC > AC$ (Proposition 20). Consequently $AB > AC$, as required.

4. Let a geometric configuration in the Euclidean plane be as depicted below. In this configuration, we suppose that the point C lies in the interior of the angle DAB, the line segments AD and AC are of equal length, and that the line segments AC and BD intersect at the point E.



Without using any proposition in Book I of Euclid's *Elements of Geometry* that follows Proposition 15, with the exception of Proposition 20 (which ensures that the sum of the lengths of any two sides of a triangle is greater than the length of the remaining side), and assuming Proposition 20 to be valid and axiomatic, (and without essentially incorporating the proof of any proposition in Book I that follows Proposition 15 so as to avoid explicit citation of that proposition by reproving it,) prove that the line segment BD exceeds in length the line segment BC.

[Hint: consider the four triangles that meet at the vertex E and apply the result stated in Proposition 20 to two of those triangles.]

Solution.

Consider triangles EBC and EAD. In the triangle EBC, the sum of the sides BE and EC is greater than the remaining side BC (Proposition 20). Also, in the triangle EAD, the sum of the sides AE and ED is greater than the remaining side AD (Proposition 20). It follows that the sum of the four sides BE, EC, AE and ED exceeds the sum of the two sides BC and AD. But the side AC is equal to the sum of the sides AE and EC in length, because the point E lies between A and C. Also the side BD is equal in length to the sum of the sides BE and ED, because the point E lies between B and D. Consequently the sum of the two sides AC and BD exceeds the sum of the two sides AD and BC. But the sides AC and AD are equal in length (by hypothesis). It therefore follows that the side BD of the triangle ABD exceeds in length the side BC of the triangle ABC, as required.

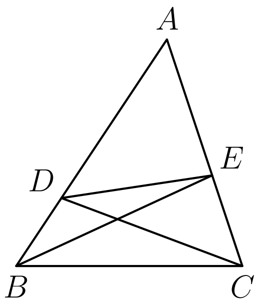
Presented more symbolically:

In $\triangle EBC$, $BC < BE + EC$ (Proposition 20). In $\triangle EAD$, $AD < AE + ED$ (Proposition 20). Consequently

$$BC + AD < BE + EC + AE + ED.$$

But $AE + EC = AC$ and $BE + ED = BD$, because E lies between A and C and also between B and D. It follows that $BC+AD < AC+BD$. But $AC = AD$, by assumption. Consequently $BC < BD$, as required.

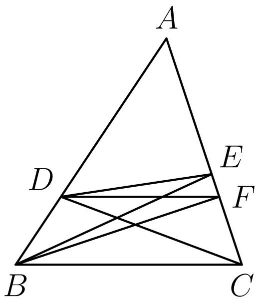
5. Let a geometric configuration in the Euclidean plane be as depicted below. In this configuration, we suppose that the angle ADE is less than the angle ABC.



Prove that the triangle AEB is smaller in area than the triangle ADC.

Solution.

Let a point F be taken on AC so as to ensure that the angles ADF and ABC are equal, join DF and also join BF. It then follows that the lines DF and BC are parallel (Proposition 28).



Now the triangles BCD and BCF are on the same base BC and between the same parallels BC and DF. It follows that the triangles BCD and BCF are equal in area. Now the triangle BCF is a part of the triangle BCE. Consequently the triangle BCE is greater in area than the triangle BCF, and is thus greater in area than the triangle BCD. Now the triangle ABC is in area the sum of the triangles AEB and BCE. It is also in area the sum of the triangles ADC and BCD. Consequently, subtracting the areas of the triangles BCE and BCD from that of the triangle ABC, we conclude that the triangle AEB is smaller in area than the triangle ADC, as required.