MAU23302 Euclidean and Non-Euclidean Geometry
Hilary Term 2021
Some Sample Problems relating to the Hyperbolic Plane

1. Let μ be a Möbius transformation of the Riemann sphere, and let $\hat{\mu}$ be the transformation of the Riemann sphere defined so that $\hat{\mu}(\infty) = \Omega(\mu(\infty))$ and

$$\hat{\mu}(z) = \Omega(\mu(\overline{z}))$$

for all complex z, where Ω is the transformation of the Riemann sphere defined so that $\Omega(0) = \infty$, $\Omega(\infty) = 0$ and $\Omega(z) = 1/\overline{z}$ for all non-zero complex number z. Also let the coefficients a, b, c and d of the Möbius transformation be complex constants satisfying $ad - bc \neq 0$ that are chosen so that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$.

Problem.

(a) Show that $\hat{\mu}$ is also a Möbius transformation of the Riemann sphere by finding complex constants $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ that ensure that

$$\hat{\mu}(z) = \frac{\hat{a}z + \hat{b}}{\hat{c}z + \hat{d}}$$

for all complex numbers z for which $\hat{c}z + \hat{d} \neq 0$.

Problem.

(b) Now let the coefficients a, b, c and d of the Möbius transformation μ be chosen so that $c = 1$, and let $w = -\overline{d}$. Suppose also that $\mu(\infty) = \Omega(\mu(\infty))$ and $\mu(\overline{z}) = \Omega(\mu(z))$ for all complex numbers z. Explain why $\mathrm{Im}[w] \neq 0$, and show that there exists some complex constant η satisfying $|\eta| = 1$ that is determined so that $\mu(\overline{w}) = \infty$, $\mu(\infty) = 1$ and

$$\mu(z) = \eta \, \frac{z - w}{z - \overline{w}}$$

for all complex numbers z distinct from $\overline{w}$.

2. Let w be a complex number satisfying $\mathrm{Im}[w] > 0$, let K be a real number satisfying $0 < K < 1$, and let

$$Q = \{z \in \mathbb{C} : |z - w| = K\,|z - \overline{w}|\}.$$

Problem.

Show that the set Q is a circle in the complex plane of radius $\dfrac{2K\,\mathrm{Im}[w]}{1 - K^2}$ centred on $\dfrac{w - K^2\overline{w}}{1 - K^2}$.

3. Let H be the upper half-plane within the complex plane, defined so that

$$H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}.$$

Hyperbolic motions of the upper half-plane, in the context of the half-plane model of hyperbolic geometry, are transformations of the upper half plane that are angle-preserving and are isometries of the Poincaré metric σ on the upper half-plane, where

$$\sigma(z, w) = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}$$

for all complex numbers z and w belonging to the upper half-plane H. The orientation-preserving hyperbolic motions of the upper half-plane H are the restrictions to H of Möbius transformations of the Riemann sphere that map the upper half-plane H onto itself. A Möbius transformations μ of the Riemann sphere maps the upper half plane onto itself if and only if there exist real numbers a, b, c and d satisfying $ad - bc = 1$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. Then $\mu(\infty) = \infty$ in cases where $c = 0$. Also $\mu(\infty) = a/c$ and $\mu(-d/c) = \infty$ in cases where $c \neq 0$.

There are also orientation-reversing hyperbolic motions of the upper half plane. These can be expressed as the restriction to the upper half-plane H of compositions of transformations of the Riemann sphere of the form $\mu \circ \hat{\kappa}$, where μ is some Möbius transformation that maps the upper half-plane onto itself and where τ is the transformation of the Riemann sphere defined so that $\tau(\infty) = \infty$ and $\tau(z) = -\overline{z}$ for all complex numbers z. Accordingly, given an orientation-reversing hyperbolic motion φ of the upper half-plane H, there exist real numbers a, b, c and d satisfying $ad - bc = 1$ such that

$$\varphi(z) = \frac{b - a\overline{z}}{d - c\overline{z}}.$$

Problem.

Let φ be an orientation-preserving hyperbolic motion φ of the upper half-plane H, where φ is determined by real coefficients a, b, c and d as described above, and let

$$\Gamma = \{z \in H : \varphi(z) = z\}.$$

Suppose that $c \neq 0$, $a = d$, and accordingly $a^2 - bc = 1$. Show that Γ is a geodesic in the upper half-plane H and determine the centre and the radius of the circle in the complex plane whose intersection with the upper half-plane H is the geodesic Γ.