Module MAU23302: Hilary Semester Examination 2021 Worked solutions

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1. From the details of the specification, we see that, in the triangles ABC and ABD, the sides BA and BC are respectively equal in length to the sides BA and BD, and the included angles ABC and ABD are equal to one another. Applying the SAS Congruence Rule (*Elements*, Proposition 4), we conclude that the triangles ABC and ABD are congruent, and thus, in particular, the line sides AC and AD are equal to one another in length.

   We now note that, in the triangles ACE and ADE, the sides AC, CE and AE are respectively equal to the sides AD, DE and AE. Applying the SSS Congruence Rule (*Elements*, Proposition 8), we conclude that the triangles ACE and ADE are congruent, and thus, in particular, the angles BCE and BDE are equal.

   We now note that, in the triangles BCE and BDE, the sides CB and CE are respectively equal in length to the sides DB and DE, and the included angles BCE and BDE are equal to one another. Consequently the triangles BCE and BDE are congruent to one another, and thus, in particular, the angles CBE and DBE are equal to one another.

   Now, considering the angles that meet at the vertex B, the angles ABC and ABD are equal to one another, and also the angles CBE and DBE are equal to one another. Moreover the sum of the four angles is equal to four right angles. Consequently the sum of the angles ABC and CBE is equal to two right angles. It follows that the line segments AB and BE form part of a single straight line (*Elements*, Proposition 14), and consequently the points A, B and E are collinear, as required.

2. The line passing through the points B and C is a transversal cutting the two parallel line segments EC and BD. It follows that the alternating angles ECB and CBD are equal to one another (*Elements*, Proposition 29). Also the line segment passing through the points C and D is a transversal cutting those two parallel lines segments EC and BD. It follows the corresponding angles ACE and ADB are equal to one another (*Elements*, Proposition 29). But the angles ACE and ECB are equal to one another, because CE bisects the angles ACB. Consequently the four angles CBD, ECB, ACE and ADB are equal to one another (*Elements*, Common Notion 1). Also the angles ADB and CDB are identical to one another. We conclude therefore that the angles CBD and CDB are equal to one another. It now follows, (by *Elements*, Proposition 6), that the triangle CBD is an isoceles triangle with equal sides CB and CD. This completes the proof.

3. (a) The definition of cross-ratio ensures that

$$(-i, i; \overline{w}, z) = \frac{(\overline{w}+i)(z-i)}{(\overline{w}-i)(z+i)}.$$

Then

$$\overline{(-i, i; \overline{w}, z)} = \frac{(w-i)(\overline{z}+i)}{(w+i)(\overline{z}-i)}.$$

Now $\mathrm{Im}[(-i, i; \overline{w}, z)] = 0$ if and only if

$$(-i, i; \overline{w}, z) = \overline{(-i, i; \overline{w}, z)}.$$

Consequently $\mathrm{Im}[(-i, i; \overline{w}, z)] = 0$ if and only if

$$(\overline{w}+i)(z-i)(w+i)(\overline{z}-i) = (w-i)(\overline{z}+i)(\overline{w}-i)(z+i).$$

Now

$$\begin{aligned}
&(\overline{w}+i)(z-i)(w+i)(\overline{z}-i) \\
&\quad = (\overline{w}\,w + i(\overline{w}+w) - 1)(\overline{z}\,z - i(\overline{z}+z) - 1) \\
&\quad = (|w|^2 + 2i\ \mathrm{Re}[w] - 1)(|z|^2 - 2i\ \mathrm{Re}[z] - 1) \\
&\quad = (|w|^2 - 1))(|z|^2 - 1) + 4\ \mathrm{Re}[w]\ \mathrm{Re}[w]) \\
&\qquad + 2i\left((|w|^2-1)\mathrm{Re}[z] - (|z|^2-1)\mathrm{Re}[w]\right).
\end{aligned}$$

and

$$\begin{aligned}
&(w-i)(\overline{z}+i)(\overline{w}-i)(z+i) \\
&\quad = (\overline{w}\,w - i(\overline{w}+w) - 1)(\overline{z}\,z + i(\overline{z}+z) - 1) \\
&\quad = (|w|^2 - 2i\,\mathrm{Re}[w] - 1)(|z|^2 + 2i\,\mathrm{Re}[z] - 1) \\
&\quad = (|w|^2 - 1))(|z|^2 - 1) + 4\ \mathrm{Re}[w]\ \mathrm{Re}[w]) \\
&\qquad - 2i\left((|w|^2-1)\,\mathrm{Re}[z] - (|z|^2-1)\,\mathrm{Re}[w]\right).
\end{aligned}$$

Consequently $\mathrm{Im}[(-i, i; \overline{w}, z)] = 0$ if and only if

$$(|w|^2 - 1)\,\mathrm{Re}[z] = (|z|^2 - 1)\,\mathrm{Re}[w].$$

(b) The equation $\mathrm{Im}[(-i, i; \overline{w}, z)] = 0$ is that of the unique circle in the complex plane that passes through the points $-i$, i and $\overline{w}$. The point w lies on this circle, and accordingly the circle is mapped onto itself by complex conjugation. Accordingly the circle passes through i and w, the centre of the circle lies on the real axis, and consequently the intersection of the circle with the upper half plane is the geodesic in question. Now it follows from (a) the circle is also represented by the equation

$$(|w|^2 - 1)\,\mathrm{Re}[z] = (|z|^2 - 1)\,\mathrm{Re}[w].$$

(Note that, from the form of this equation, it is clearly a circle with the stated properties.) Setting $z = x$, where x is a real number, we conclude that p and q are the roots of the quadratic polynomial

$$ux^2 - (|w|^2 - 1)x - u.$$

Consequently

$$\begin{aligned}
p &= \frac{|w|^2 - 1 - \sqrt{(|w|^2-1)^2 + 4u^2}}{2u}, \\
&= \frac{|w|^2 - 1 - \sqrt{|w|^4 - 2|w|^2 + 1 + 4u^2}}{2u}, \\
q &= \frac{|w|^2 - 1 + \sqrt{(|w|^2-1)^2 + 4u^2}}{2u}, \\
&= \frac{|w|^2 - 1 + \sqrt{|w|^4 - 2|w|^2 + 1 + 4u^2}}{2u}.
\end{aligned}$$