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Part II, Section 2
The Half Plane Model of the Hyperbolic Plane

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Contents

2 The Half Plane Model of the Hyperbolic Plane

2.1 Möbius Transformations of the Upper Half Plane

We investigate those Möbius transformations of the Riemann sphere that map the subset $\mathbb{R} \cup \{\infty\}$ of the Riemann sphere onto itself. This subset, obtained on adjoining the point ∞ "at infinity" to the real line $\mathbb{R}$ may be regarded as great circle on the Riemann sphere. Indeed each complex number is representable in the form $x + i\sqrt{y}$, where x and y are real numbers and $i = \sqrt{-1}$, and is mapped, under stereographic projection, to the point of the unit sphere S^2 in $\mathbb{R}^3$ with coordinates

$$\left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2}\right)$$

(see Proposition 1.2). The element ∞ adjoined to the Riemann sphere corresponds, under stereographic projection, to the point $(0,0,-1)$ of the unit sphere S^2 in $\mathbb{R}^2$. Accordingly the subset $\mathbb{R} \cup \{\infty\}$ of the Riemann sphere obtained on adding the point ∞ "at infinity" to the real line $\mathbb{R}$ corresponds, under stereographic projection, to the great circle on the sphere S^2 consisting of those points (u, v, w) of S^2 for which $v = 0$.

We recall that, given three distinct points of the Riemann sphere, which we may denote by p_∞, p_0 and p_1, the unique Möbius transformation μ of the Riemann sphere that satisfies the conditions $\mu(p_\infty) = \infty$, $\mu(p_0) = 0$ and $\mu(p_1) = p_1$ is that Möbius transformation determined so that

$$\mu(p) = (p_\infty, p_0; p_1, p)$$

for all points p of the Riemann sphere (see Proposition 1.13), where $(p_\infty, p_0; p_1, p)$ denotes the cross-ratio of the points p_∞, p_0, p_1 and p. We recall also that cross-ratios of appropriate quadruples of complex numbers are defined such that Let z_1, z_2, z_3 and z_4 be complex numbers. We recall that

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \quad \text{provided that} \quad z_1 \neq z_4 \text{ and } z_2 \neq z_3,$$

$$(z_4, z_2; z_3, z_4) = \infty \quad \text{provided that} \quad z_2 \neq z_4 \text{ and } z_3 \neq z_4,$$

$$(z_1, z_3; z_3, z_4) = \infty \quad \text{provided that} \quad z_1 \neq z_3 \text{ and } z_3 \neq z_4,$$

$$(\infty, z_2; z_3, z_4) = \frac{z_4 - z_2}{z_3 - z_2} \quad \text{provided that} \quad z_2 \neq z_3,$$

$$(\infty, z_3; z_3, z_4) = \infty \quad \text{provided that} \quad z_3 \neq z_4,$$

$$(z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1} \quad \text{provided that} \quad z_1 \neq z_4,$$

$$(z_4, \infty; z_3, z_4) = \infty \quad \text{provided that} \quad z_3 \neq z_4,$$

$$(z_1, z_2; \infty, z_4) = \frac{z_4 - z_2}{z_4 - z_1} \quad \text{provided that} \quad z_1 \neq z_4,$$

$$(z_4, z_2; \infty, z_4) = \infty \quad \text{provided that} \quad z_2 \neq z_4,$$

$$(z_1, z_2; z_3, \infty) = \frac{z_3 - z_1}{z_3 - z_2} \quad \text{provided that} \quad z_2 \neq z_3,$$

$$(z_1, z_3; z_3, \infty) = \infty \quad \text{provided that} \quad z_1 \neq z_3.$$

We refer to the subset of the complex plane consisting of those complex numbers z for which $\text{Im}[z] > 0$ as the *upper half plane*; we refer to the subset of the complex plane consisting of those complex numbers z for which $\text{Im}[z] < 0$ as the *lower half plane*.

The observation encapsulated in the following lemma follows directly from the identities set out above.

Lemma 2.1 *Let z_1, z_2, z_3 and z_4 be complex numbers, where* $\text{Im}[z_1] > 0$, $\text{Im}[z_2] > 0$, $\text{Im}[z_3] < 0$, $\text{Im}[z_4] < 0$ *(so that z_1 and z_2 belong to the upper half plane and z_3 and z_4 belong to the lower half plane. Then the cross ratio $(z_1, z_2; z_3, z_4)$ is well-defined and is a complex number. (Thus $(z_1, z_2; z_3, z_4)$ is distinct from the point ∞ of the Riemann sphere.)*

Let us define the *real circle* in the Riemann sphere $\mathbb{P}^1$ to be the subset $\mathbb{R} \cup \{\infty\}$ of the Riemann sphere consisting of those elements of the Riemann sphere represented by real numbers together with the point of the Riemann sphere represented by the point denoted by the symbol ∞.

Lemma 2.2 *Let p_∞, p_0 and p_1 be distinct elements of the real circle $\mathbb{R} \cup \{\infty\}$ in the Riemann sphere. Then there exist real numbers a, b, c and d, where $ad - bc \neq 0$, such that*

$$(p_\infty, p_0; p_1, z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$; $(p_\infty, p_0; p_1, -d/c) = \infty$ and $(p_\infty, p_0; p_1, \infty) = a/c$. The real numbers a, b, c and d are thus coefficients of the unique Möbius transformation μ for which $\mu(p_\infty) = \infty$, $\mu(p_0) = 0$ and $\mu(p_1) = 1$.

Proof Suppose that s_∞, s_0 and s_1 are distinct real numbers. Then

$$(s_\infty, s_0; s_1, z) = \frac{(s_1 - s_\infty)(z - s_0)}{(s_1 - s_0)(z - s_\infty)} = \frac{az + b}{cz + d},$$

where

$$a = s_1 - s_\infty, \quad b = -s_0(s_1 - s_\infty), \quad c = s_1 - s_0, \quad d = -s_\infty(s_1 - s_0).$$

It follows that

$$ad - bc = (s_1 - s_\infty)(s_1 - s_0)(s_0 - s_\infty) \neq 0.$$

We have thus found real numbers that are the coefficients of a Möbius transformation that sends s_∞, s_0 and s_1 to ∞, 0 and 1 respectively.

However there exists only one Möbius transformation that sends p_∞, p_0 and p_1 to ∞, 0 and 1 (see Proposition 1.15). Thus the real numbers a, b, c and d are indeed coefficients of the unique Möbius transformation for which $\mu(p_\infty) = \infty$, $\mu(p_0) = 0$ and $\mu(p_1) = 1$, as required. ∎

We recall that, in situations where four complex numbers z_1, z_2, z_3 and z_4 are distinct, the cross-ratio $(z_1, z_2; z_3, z_4)$ of these complex numbers is defined so that

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

The cross-ratio is also defined as discussed previously in various situations where the point ∞ replaces one of the complex numbers, and in situations when two of the four complex numbers involved in the cross-ratio are equal. In particular the cross-ratio is given by the above formula in all cases where $z_3 \neq z_2$ and $z_4 \neq z_1$,

Proposition 2.3 *Let H be the open upper half of the complex plane, bounded by the real axis, so that*

$$H = \{z \in \mathbb{C} : Im[z] > 0\},$$

and $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ be a Möbius transformation. Then the Möbius transformation μ maps the upper half plane H onto itself, so that $\mu(H) = H$, if and only if there exist real numbers a, b, c and d satisfying $ad - bc = 1$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z satisfying $cz + d \neq 0$.

Proof First let a, b, c and d be real numbers satisfying $ad - bc = 1$, and let μ be the Möbius transformation of the Riemann sphere defined so that $\mu(z) = (az+b)/(cz+d)$ for all complex numbers z for which $cz+d \neq 0$. Now c and d are real numbers and therefore, given any complex number z, the complex conjugate of $cz+d$ is $c\overline{z}+d$, where $\overline{z}$ denotes the complex conjugate of z. It follows that $(cz+d)(c\overline{z}+d) = |cz+d|^2$, and therefore

$$\begin{aligned} \mu(z) &= \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} \\ &= \frac{ac|z|^2 + bd + (ac+bd)\operatorname{Re}[z] + i(ac-bd)\,[Im](z)}{|cz+d|^2} \end{aligned}$$

for all complex numbers z for which $cz+d \neq 0$. Now the coefficients a, b, c and d are real numbers for which $ad - bc = 1$. It follows that

$$\operatorname{Im}[\mu(z)] = \frac{(ac-bd)\operatorname{Im}[z]}{|cz+d|^2} = \frac{\operatorname{Im}[z]}{|cz+d|^2}$$

for all complex numbers z for which $cz+d \neq 0$, and thus $\operatorname{Im}[\mu(z)] > 0$ for all complex number z for which $\operatorname{Im}[z] > 0$. This shows that $\mu(H) \subset H$, and thus the Möbius transformation μ maps the open upper half plane H into itself.

Also all Möbius transformations are invertible mappings from the Riemann sphere to itself, and moreover the condition $ad - bc = 1$ satisfied by the coefficients a, b,c and d ensures that

$$\mu^{-1}(w) = \frac{dw - b}{-c + aw}$$

for all complex numbers w for which $aw - c \neq 0$ (Corollary 1.5). It follows that if w is an element of the open upper half plane H then $\mu^{-1}(w) \in H$ and $w = \mu(\mu^{-1}(w))$, and therefore $w \in \mu(H)$. We can now conclude that $\mu(H) = H$.

Now let μ be any Möbius transformation that satisfies $\mu(H) = H$. We must prove the existence of real numbers a, b, c and d with the property that $\mu(z) = (az+b)/(cz+d)$ for all complex numbers z for which $cz+d \neq 0$. Now the Möbius transformation μ has an inverse μ^{-1}, and the Möbius transformations μ and μ^{-1} map the open upper half plane H onto itself. A straightforward continuity argument shows that they must map the subset $\mathbb{R} \cup \{\infty\}$ of the Riemann sphere onto itself, as this subset constitutes the boundary of the upper half plane in the Riemann sphere.

Now it follows, on applying Lemma 2.2, that there exist real coefficients a', b', c' and d', where $a'd' - b'c' \neq 0$, such that

$$\mu(z) = \frac{a'z + b'}{c'z + d'}$$

for all complex numbers z for which $c'z + d' \neq 0$.

Now $i \in H$ and $\mu(H) = H$. It follows that $\mathrm{Im}[\mu(i)] > 0$. But

$$\mu(i) = \frac{(a'i + b')(d' - c'i)}{(c'i + d')(d' - c'i)} = \frac{a'c' + b'd' + (a'd' - b'c')i}{|c'i + d|^2}.$$

It follows that $a'd' - b'c' > 0$. Let

$$a = \frac{a'}{\sqrt{a'd' - b'c'}}, \quad b = \frac{b'}{\sqrt{a'd' - b'c'}},$$

$$c = \frac{c'}{\sqrt{a'd' - b'c'}}, \quad d = \frac{d'}{\sqrt{a'd' - b'c'}}.$$

Then $ad - bc = 1$ and

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z for which $cz + d \neq 0$. The result follows. ∎

Corollary 2.4 *let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$. Then a Möbius transformation μ of the Riemann sphere satisfies $\mu(H) = H$ if and only if μ maps at least one element of H into H and $\mu(\overline{z}) = \overline{\mu(z)}$ for all complex numbers z.*

Proof If a Möbius transformation μ maps the upper half plane H onto itself then there exist real numbers a, b, c and d satisfying $ad - bc = 1$ such that $\mu(z) = (az + b)/(cz + d)$ for all complex numbers z for which $cz + d \neq 0$ (Proposition 2.3). It then follows directly from basic properties of complex conjugation that $\overline{\mu(z)} = \mu(\overline{z})$ for all complex numbers z.

Conversely let μ be a Möbius transformation with the property that $\overline{\mu(z)} = \mu(\overline{z})$ for all complex numbers z. Then there exist complex numbers a, b, c and d satisfying $ad - bc = 1$ such that

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z with $cz + d \neq 0$. Then

$$\frac{a\overline{z} + b}{c\overline{z} + d} = \mu(\overline{z}) = \overline{\mu(z)} = \frac{\overline{a}\,\overline{z} + \overline{b}}{\overline{c}\,\overline{z} + \overline{d}}$$

for all complex numbers z. This ensures the existence of some complex number λ such that

$$\overline{a} = \lambda a, \quad \overline{b} = \lambda b, \quad \overline{c} = \lambda c, \quad \overline{d} = \lambda d$$

(Lemma 1.8). Moreover

$$1 = \overline{a}\,\overline{d} - \overline{b}\,\overline{d} = \lambda^2(ad - bc) = \lambda^2,$$

and thus $\lambda = \pm 1$. In the case where $\lambda = 1$ let $a' = a$, $b' = b$, $c' = c$ and $d' = d$. In the case where $\lambda = -1$ let $a' = -ia$, $b' = -ib$, $c' = ic$ and $d' = id$. In both cases let μ_0 be the Möbius transformation of the Riemann sphere defined so that $\mu_0(z) = (a'z+b)/(c'z+d')$ for all complex numbers z for which $c'z + d' \neq 0$. Then a', b', c' and d' are real numbers satisfying $a'd' - b'c' = 1$. It follows that $\mu_0(H) = H$ (Proposition 2.3).

Now $\mu(z) = \mu_0(z)$ for all complex numbers z in the case where $\lambda = 1$, and $\mu(z) = -\mu_0(z)$ for all complex numbers z in the case where $\lambda = -1$. It follows that the Möbius transformation μ maps the open upper half plane H onto itself in the case when $\lambda = 1$, but maps the open upper half plane onto the open lower half plane $\{z \in Z : \operatorname{Im} z < 0\}$ in the case when $\lambda = -1$. The result follows. ∎

2.2 The Poincaré Half Plane Model of the Hyperbolic Plane

Lemma 2.5 *Let z_1 and z_2 be complex numbers with $\operatorname{Im}[z_1] > 0$ and $\operatorname{Im}[z_2] > 0$. Then $|z_1 - z_2| < |z_1 - \overline{z}_2|$ and*

$$(z_1, \overline{z}_1; z_2, \overline{z}_2) = \frac{|z_1 - z_2|^2}{|z_1 - \overline{z}_2|^2},$$

and therefore

$$0 \leq (z_1, \overline{z}_1; z_2, \overline{z}_2) < 1.$$

Proof Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where x_1, y_1, x_2 and y_2 are real numbers and $i = \sqrt{-1}$. Then $y_1 > 0$ and $y_2 > 0$. It follows that

$$|z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 < (x_1 - x_2)^2 + (y_1 + y_2)^2 = |z_1 - \overline{z}_2|^2,$$

and thus $|z_1 - z_2| < |z_1 - \overline{z}_2|$.

Evaluating the cross-ratio, we find that

$$(z_1, \overline{z}_1; z_2, \overline{z}_2) = \frac{(z_1 - z_2)(\overline{z}_1 - \overline{z}_2)}{(z_1 - \overline{z}_2)(\overline{z}_1 - z_2)} = \frac{|z_1 - z_2|^2}{|z_1 - \overline{z}_2|^2}.$$

This value of this cross-ratio must satisfy $0 \leq (z_1, \overline{z}_1; z_2, \overline{z}_2) < 1$, as required. ∎

Proposition 2.6 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, and let z_1, z_2, w_1 and w_2 be complex numbers belonging to the open upper half plane H. Then there exists a Möbius transformation μ of the Riemann sphere with the properties that $\mu(H) = H$, $\mu(z_1) = w_1$ and $\mu(z_2) = w_2$ if and only if*

$$(z_1, \overline{z}_1; z_2, \overline{z}_2) = (w_1, \overline{w}_1; w_2, \overline{w}_2).$$

Proof Suppose that there exists a Möbius transformation μ with the required properties. Then $\mu(\overline{z}) = \overline{\mu(z)}$ for all complex numbers z (Corollary 2.4). In particular

$$\mu(\overline{z}_1) = \overline{\mu(z_1)} = \overline{w}_1 \quad \text{and} \quad \mu(\overline{z}_2) = \overline{\mu(z_2)} = \overline{w}_2$$

Thus the four complex numbers z_1, $\overline{z}_1$, z_2 and $\overline{z}_2$ are mapped by μ to w_1, $\overline{w}_1$, w_2 and $\overline{w}_2$. The invariance of cross-ratio under the action of Möbius transformations therefore ensures that

$$(z_1, \overline{z}_1, z_2; \overline{z}_2) = (w_1, \overline{w}_1; w_2, \overline{w}_2)$$

(see Proposition 1.16).

Conversely suppose that relevant cross-ratios determined by the complex numbers z_1, z_2, w_1 and w_2 and their complex conjugates satisfy

$$(z_1, \overline{z}_1; z_2, \overline{z}_2) = (w_1, \overline{w}_1; w_2, \overline{w}_2).$$

Then there exists a Möbius transformation μ of the Riemann sphere that sends z_1, z_2, $\overline{z}_1$ and $\overline{z}_2$ to w_1, w_2, $\overline{w}_1$ and $\overline{w}_2$ respectively. Let $\mu_0 \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the mapping from the Riemann sphere to itself determined so that $\overline{\mu_0(z)} = \mu(\overline{z})$ for all complex numbers z. Then $\mu_0 \colon \mathbb{P}^1 \to \mathbb{P}^1$ is also a Möbius transformation of the Riemann sphere. Indeed if

$$\mu(z) = \frac{az + b}{cz + d}$$

for all complex numbers z satisfying $cz + d \neq 0$, where a, b, c and d are complex constants satisfying $ad - bc = 1$, then

$$\mu_0(z) = \frac{\overline{a}\, z + \overline{b}}{\overline{c}z + \overline{d}}$$

for all complex numbers z satisfying $\overline{c}z + \overline{d} \neq 0$. Moreover the distinct complex numbers z_1, z_2, $\overline{z}_1$ and $\overline{z}_2$ get mapped to w_1, w_2, $\overline{w}_1$ and $\overline{w}_2$ respectively under each of the Möbius transformations μ and μ_0. Thus there are at least three complex numbers z for which $\mu(z) = \mu_0(z)$. It follows that the Möbius transformations μ and μ_0 must coincide (Proposition 1.15), and therefore $\mu(\overline{z}) = \overline{\mu(z)}$ for all complex numbers z. It follows that $\mu(H) = H$ (see Corollary 2.4). Thus the Möbius transfomation μ maps the open upper half plane H onto itself and maps z_1 and z_2 to w_1 and w_2 respectively, as required. ∎

Definition Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, and let z_1 and z_2 be complex numbers belonging to the open upper half plane H. We define the *Poincaré distance* $\sigma(z_1, z_2)$ from z_1 to z_2 by the formula

$$\sigma(z_1, z_2) = \log\left(\frac{|z_1 - \overline{z}_2| + |z_1 - z_2|}{|z_1 - \overline{z}_2| - |z_1 - z_2|}\right).$$

Remark The formula given above is but one of many that might be employed to specify the value of the Poincaré distance between two complex numbers lying in the upper half plane H. In particular, in the context of differential geometry, one can specify, through an appropriate line integral, a hyperbolic length assigned to any continuous and piecewise continuously differentiable curve in the upper half plane. Specifically let $\gamma\colon [a, b] \to H$ be a continuous and piecewise continuously differentiable curve in H parameterized by a closed interval $[a, b]$, so that $\gamma(t)$ is defined for all real numbers t satisfying $a \leq t \leq b$. Then the hyperbolic length of γ is given by the formula

$$\int_a^b \frac{1}{\mathrm{Im}[\gamma(t)]}\,|\gamma'(t)|\,dt.$$

The Poincaré distance between two complex numbers z_1 and z_2 in the upper half plane is then the greatest lower bound of the hyperbolic lengths of all continuous and piecewise continuously differentiable curves $\gamma\colon [a, b] \to H$ in the upper half plane H for which $\gamma(a) = z_1$ and $\gamma(b) = z_2$.

Lemma 2.7 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, and let z_1 and z_2 be complex numbers belonging to the open upper half plane H. Then the Poincaré distance $\sigma(z_1, z_2)$ from z_1 and z_2 has the properties that $\sigma(z_1, z_2) \geq 0$ and*

$$\sigma(z_1, z_2) = \sigma(z_2, z_1).$$

Moreover $\sigma(z_1, z_2) = 0$ if and only if $z_1 = z_2$.

Proof The inequality $\sigma(z_1, z_2) \geq 0$ follows from the inequality

$$|z_1 - \overline{z}_2| + |z_1 - z_2| \geq |z_1 - \overline{z}_2| - |z_1 - z_2|$$

which results from the basic inequality $|z_1 - z_2| \geq 0$. Moreover $\sigma(z_1, z_2) = 0$ if and only if the left hand side of the above inequality is equal to the right hand side. This is the case if and only if $z_1 = z_2$. The identity $\sigma(z_1, z_2) = \sigma(z_2, z_1)$ follows from the fact that $z_1 - \overline{z}_2$ and $z_2 - \overline{z}_1$ are complex conjugates of one another and therefore $|z_1 - \overline{z}_2| = |z_2 - \overline{z}_1|$. ∎

Lemma 2.8 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, and let z_1 and z_2 be complex numbers belonging to the open upper half plane H. Then the Poincaré distance $\sigma(z_1, z_2)$ from z_1 and z_2 satisfies*

$$\sigma(z_1, z_2) = \log\left(\frac{1 + \sqrt{(z_1, \overline{z}_1; z_2, \overline{z_2})}}{1 - \sqrt{(z_1, \overline{z}_1; z_2, \overline{z_2})}}\right).$$

Proof The definition of the cross-ratio ensures that

$$(z_1, \overline{z}_1; z_2, \overline{z}_2) = \frac{|z_1 - z_2|^2}{|z_1 - \overline{z}_2|^2}$$

(Lemma 2.5). It follows that

$$\begin{aligned}
\sigma(z_1, z_2) &= \log\left(\frac{|z_1 - \overline{z}_2| + |z_1 - z_2|}{|z_1 - \overline{z}_2| - |z_1 - z_2|}\right) \\
&= \log\left(\frac{1 + \dfrac{|z_1 - z_2|}{|z_1 - \overline{z}_2|}}{1 - \dfrac{|z_1 - z_2|}{|z_1 - \overline{z}_2|}}\right) \\
&= \log\left(\frac{1 + \sqrt{(z_1, \overline{z}_1; z_2, \overline{z_2})}}{1 - \sqrt{(z_1, \overline{z}_1; z_2, \overline{z_2})}}\right),
\end{aligned}$$

as required. ∎

Proposition 2.9 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, and let z_1, z_2, w_1 and w_2 be complex numbers belonging to the open upper half plane H. Then there exists a Möbius transformation μ of the Riemann sphere with the properties that $\mu(H) = H$, $\mu(z_1) = w_1$ and $\mu(z_2) = w_2$ if and only if $\sigma(z_1, z_2) = \sigma(w_1, w_2)$, where $\sigma(z_1, z_2)$ denotes the Poincaré distance from z_1 and z_2 and $\sigma(w_1, w_2)$ denotes the Poincaré distance from w_1 and w_2.*

Proof This result follows directly on taking into account the formula of Lemma 2.8, expressing the Poincaré distance between two points z_1, z_2 of the upper half plane H in terms of the cross-ratio $(z_1, \overline{z}_1; z_2, \overline{z}_2)$, and applying the result of Proposition 2.6. ∎

Lemma 2.10 *Let y_1 and y_2 be positive real numbers, and let $i = \sqrt{-1}$. Then the Poincaré distance $\sigma(iy_1, iy_2)$ from iy_1 to iy_2 is given by the formula*

$$\sigma(iy_1, iy_2) = |\log y_1 - \log y_2|.$$

Proof We may suppose, without loss of generality, that $y_1 > y_2$. Then

$$\begin{aligned}\sigma(iy_1, iy_2) &= \log\left(\frac{|iy_1 + iy_2| + |iy_1 - iy_2|}{|iy_1 + iy_2| - |iy_1 - iy_2|}\right)\\ &= \log\left(\frac{(y_1 + y_2) + (y_1 - y_2)}{(y_1 + y_2) - (y_1 - y_2)}\right)\\ &= \log\left(\frac{y_1}{y_2}\right) = \log y_1 - \log y_2.\end{aligned}$$

The result follows. ∎

Proposition 2.11 *Let x_1, y_1, x_2 and y_2 be real numbers where $y_1 > 0$ and $y_2 > 0$, and let Let z_1 and z_2 be complex numbers belonging to the upper half plane H, where*

$$H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}.$$

Then the Poincaré distance $\sigma(z_1, z_2)$ from z_1 to z_2 satisfies the inequality

$$\sigma(z_1, z_2) \geq |\log \mathrm{Im}[z_1] - \log \mathrm{Im}[z_2]|.$$

Moreover

$$\sigma(z_1, z_2) = |\log \mathrm{Im}[z_1] - \log \mathrm{Im}[z_2]|.$$

if and only if $\mathrm{Re}[z_1] = \mathrm{Re}[z_2]$.

Proof Let $x_1 = \mathrm{Re}[z_1]$, $x_2 = \mathrm{Re}[z_2]$, $y_1 = \mathrm{Im}[z_1]$ and $y_2 = \mathrm{Im}[z_2]$, and let

$$\rho = \frac{|z_1 - z_2|}{|z_1 - \overline{z}_2|}.$$

Then $y_1 > 0$, $y_2 > 0$ and

$$\begin{aligned}\rho^2 &= \frac{|z_1 - z_2|^2}{|z_1 - \overline{z}_2|^2}\\ &= \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{(x_1 - x_2)^2 + (y_1 + y_2)^2}\\ &= \frac{(x_1 - x_2)^2 + y_1^2 + y_2^2 - 2y_1y_2}{(x_1 - x_2)^2 + y_1^2 + y_2^2 + 2y_1y_2}\end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{4y_1y_2}{(x_1-x_2)^2 + y_1^2 + y_2^2 + 2y_1y_2} \\
&\geq 1 - \frac{4y_1y_2}{y_1^2 + y_2^2 + 2y_1y_2} \\
&= \frac{(y_1-y_2)^2}{(y_1+y_2)^2}.
\end{aligned}
$$

It follows that $\rho \geq \rho_0$, where

$$\rho_0 = \frac{|y_1 - y_2|}{y_1 + y_2},$$

Also $\rho < 1$. Consequently

$$
\begin{aligned}
\frac{|z_1 - \overline{z}_2| + |z_1 - z_2|}{|z_1 - \overline{z}_2| - |z_1 - z_2|} &= \frac{1+\rho}{1-\rho} = \frac{2}{1-\rho} - 1 \\
&\geq \frac{2}{1-\rho_0} - 1 = \frac{1+\rho_0}{1-\rho_0} \\
&= \frac{y_1 + y_2 + |y_1 - y_2|}{y_1 + y_2 - |y_1 - y_2|}.
\end{aligned}
$$

Considering separately the cases when $y_1 \geq y_2$ and $y_2 \geq y_1$, we conclude that

$$\sigma(z_1, z_2) \geq |\log y_1 - \log y_2| = |\log \mathrm{Im}[z_1] - \log \mathrm{Im}[z_2]|.$$

Moreover if $x_1 \neq x_2$ then

$$1 - \frac{4y_1y_2}{(x_1-x_2)^2 + y_1^2 + y_2^2 + 2y_1y_2} > 1 - \frac{4y_1y_2}{y_1^2 + y_2^2 + 2y_1y_2}.$$

But then $\rho > \rho_0$, and consequently

$$\sigma(z_1, z_2) > |\log \mathrm{Im}[z_1] - \log \mathrm{Im}[z_2]|.$$

The result follows. ∎

Corollary 2.12 *Let z_1, z_2 and z_3 be complex numbers belonging to the upper half plane H, where*

$$H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\},$$

and let $\sigma(z_1, z_3)$, $\sigma(z_1, z_2)$ and $\sigma(z_2, z_3)$ denote the Poincaré distances between the respective pairs of points. Suppose that $\mathrm{Re}[z_1] = \mathrm{Re}[z_3]$. *Then*

$$\sigma(z_1, z_3) \leq \sigma(z_1, z_2) + \sigma(z_2, z_3).$$

Moreover $\sigma(z_1, z_3) = \sigma(z_1, z_2) + \sigma(z_2, z_3)$ if and only if z_2 lies on the line segment in the upper half plane H with endpoints represented by the complex numbers z_1 and z_2.

Proof Let $x_j = \text{Re}[z_j]$ and $y_j = \text{Im}[z_j]$ for $j = 1, 2, 3$. Then $x_1 = x_3$. Now it follows from Proposition 2.11 that

$$\sigma(z_1, z_2) \geq |\log y_1 - \log y_2| \quad \text{and} \quad \sigma(z_2, z_3) \geq |\log y_2 - \log y_3|.$$

Moreover the above inequalities are strict unless $x_1 = x_2 = x_3$. Applying these inequalities, we find that

$$\begin{aligned} \sigma(z_1, z_3) &= |\log y_1 - \log y_3| \\ &\leq |\log y_1 - \log y_2| + |\log y_2 - \log y_3| \\ &\leq \sigma(z_1, z_2) + \sigma(z_2, z_3). \end{aligned}$$

Moreover $\sigma(z_1, z_3) < \sigma(z_1, z_2) + \sigma(z_2, z_3)$ unless $x_1 = x_2 = x_3$ and either $y_1 \leq y_2 \leq y_3$ or else $y_1 \geq y_2 \geq y_3$. It follows that $\sigma(z_1, z_3) = \sigma(z_1, z_2) + \sigma(z_2, z_3)$ if and only if unless z_2 lies on the line in the upper half plane whose endpoints are represented by z_1 and z_3, as required. ∎

Proposition 2.13 (Triangle Inequality for Poincaré Distance) *Let z_1, z_2 and z_3 be complex numbers belonging to the upper half plane H, where*

$$H = \{z \in \mathbb{C} : \text{Im}[z] > 0\},$$

and let $\sigma(z_1, z_3)$, $\sigma(z_1, z_2)$ and $\sigma(z_2, z_3)$ denote the Poincaré distances between the respective pairs of points. Then

$$\sigma(z_1, z_3) \leq \sigma(z_1, z_2) + \sigma(z_2, z_3).$$

Proof Positive real numbers v_1 and v_3 can be found such that

$$\sigma(z_1, z_3) = |\log v_1 - \log v_3| = \sigma(iv_1, iv_3).$$

A Möbius transformation $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ can then be found for which $\mu(H) = H$, $\mu(z_1) = iv_1$ and $\mu(z_3) = iv_3$ (Proposition 2.9). Let $w_2 = \mu(z_2)$. Möbius transformations preserve Poincaré distance (Proposition 2.9). Therefore

$$\sigma(z_1, z_3) = \sigma(iv_1, iv_3), \quad \sigma(z_1, z_2) = \sigma(iv_1, w_2)$$

and

$$\sigma(z_2, z_3) = \sigma(w_2, iv_3).$$

Applying Corollary 2.12, it follows that

$$\begin{aligned} \sigma(z_1, z_3) &= \sigma(iv_1, iv_3) = |\log v_3 - \log v_1| \\ &\leq \sigma(iv_1, w_2) + \sigma(w_2, iv_3) \\ &= \sigma(z_1, z_2) + \sigma(z_2, z_3), \end{aligned}$$

as required. ∎

2.3 Geodesics in the Poincaré Half Plane Model

Definition Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, and, for all complex numbers z_1 and z_2 belonging to the upper half plane H let $\sigma(z_1, z_2)$ denote the Poincaré distance from z_1 to z_2. A (connected) continuous curve in H is said to be a (length-minimizing) *geodesic* with respect to the Poincaré distance function if

$$\sigma(z_1, z_3) = \sigma(z_1, z_2) + \sigma(z_2, z_3)$$

for all triples z_1, z_2, z_3 of points on the curve for which z_2 is located on the curve between z_1 and z_3.

Remark In the context of differential geometry, and specifically Riemannian geometry, geodesics arise as solutions of appropriate systems of second order ordinary differential equations determined by the geometry of the space to which they belong. Each continuous and piecewise continuously differentiable curve in a connected Riemannian manifold has a length determined by the geometry of that manifold, and the distance between two points of that manifold is defined to be the greatest lower bound of the lengths of all curves in the manifold that join those two points. A sufficiently short geodesic segment in a Riemannian manifold minimizes distance amongst all continuous and continuously piecewise differentiable curves joining the endpoints of the geodesic segment, and the length of such a geodesic segment is therefore equal to the distance between the endpoints of the segment. It follows that if points P_1 and P_3 are the endpoints of a sufficiently short geodesic segment in a Riemannian manifold, and if P_2 is a point on that geodesic segment lying between the endpoints P_1 and P_3 of the segment, then the distance from P_1 to P_3 is the sum of the distances from P_1 to P_2 and from P_2 to P_3. Moreover this property characterizes length-minimizing geodesics in Riemannian manifolds. When the hyperbolic plane is studied using the methods of Riemannian geometry it can be shown that all geodesics (when defined in accordance with usual conventions within Riemannian geometry) minimize length between their endpoints. Accordingly the definition of geodesics for the Poincaré distance function given above, specifically in the context of the geometry of the hyperbolic plane, is consistent with the usage of the term *geodesic* in the context of differential geometry.

Lemma 2.14 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$. Given any real number u, the half-line in the upper half plane consisting of those $z \in H$ for which* $\mathrm{Re}[z] = u$ *is a geodesic with respect to the Poincaré distance function.*

Proof In the case when $u = 0$, the result follows directly on applying Corollary 2.12.

In the case where u is a non-zero real number the function sending each complex number z to $z - u$ is a Möbius transformation that maps the upper half plane H onto itself. This Möbius transformation and its inverse both preserve Poincaré distance and therefore map geodesics to geodesics. The result follows. ∎

Lemma 2.15 *Let* $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, *Then any circular arc in the upper half plane* H *that forms part of the circle centred on a real number is a geodesic with respect to the Poincaré distance function.*

Proof let u be a real number, let R be a positive real number, and let μ be the Möbius transformation that maps the real numbers $u - R$, u and $u + R$ to 0, 1 and ∞ respectively. Then

$$\mu(z) = (u + R, u - R; u, z) = \frac{z - u + R}{u + R - z}$$

for all complex numbers z distinct from $u + R$. Then

$$\begin{aligned}
&\mu(u + R(\cos\theta + i\,\sin\theta)) \\
&\quad = \frac{1 + \cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta} \\
&\quad = \frac{(1 + \cos\theta + i\sin\theta)(1 - \cos\theta + i\sin\theta)}{(1 - \cos\theta - i\sin\theta)(1 - \cos\theta + i\sin\theta)} \\
&\quad = \frac{(1 + i\sin\theta)^2 - \cos^2\theta}{(1 - \cos\theta)^2 + \sin^2\theta} \\
&\quad = \frac{1 + 2i\sin\theta - \sin^2\theta - \cos^2\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \\
&\quad = \frac{i\sin\theta}{1 - \cos\theta}
\end{aligned}$$

for all real numbers θ. This calculation shows that $\mathrm{Re}[\mu(z)] = 0$ for all complex numbers z for which $|z - u| = R$.

Now it follows from Corollary 2.12. any half-line or line segment in the upper half plane that is contained in the imaginary axis is a geodesic. Also Möbius transformations and their inverses preserve the Poincaré distance function and therefore map geodesics to geodesics. It follows that any circular arc in the upper half plane forming part of the circle of radius R centred on the real number u is mapped under the Möbius transformation μ to a geodesic, and must therefore itself be a geodesic. The result follows. ∎

Remark Let H be the open upper half of the complex plane, defined so that $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$. The calculations undertaken in the proof of

Lemma 2.15 can be used to obtain an expression for the Poincaré distance between points on a semicircle in the upper half plane H centred on a point lying on the real axis. Indeed let u be a real number, let R be a positive real number, and let $\mu: \mathbb{P}^1 \to \mathbb{P}^1$ of the Riemann sphere that maps each complex number z distinct from $u - R$ to $(z - u + R)/(u + R - z)$ Then μ maps points on the semicircle in the upper half plane of radius R centred on the real number y to points on the imaginary axis. Now $\mu(H) = H$, and therefore the Möbius transformation μ preserves Poincaré distance σ. (Proposition 2.9). Also Poincaré distance along the imaginary axis is given by the logarithm function (Lemma 2.10). In the proof of Lemma 2.15 it was shown that

$$\mu(u + R(\cos\theta + i\,\sin\theta)) = \frac{i\sin\theta}{1 - \cos\theta}.$$

Putting these results together, and noting that $\mu(u + Ri) = i$, we find that

$$\begin{aligned} \sigma(u + R(\cos\theta + i\,\sin\theta), u + iR) &= \log\left(\frac{\sin\theta}{1 - \cos\theta}\right) \\ &= \frac{1}{2}\log\left(\frac{\sin^2\theta}{(1 - \cos\theta)^2}\right) \\ &= \frac{1}{2}\log\left(\frac{1 + \cos\theta}{1 - \cos\theta}\right) \end{aligned}$$

for all real numbers θ satisfying $0 < \theta < \pi$.

Proposition 2.16 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$. Then a continuous curve between two points of the upper half plane H is a geodesic with respect to the Poincaré distance function if and only if either it is a line segment whose direction is perpendicular to the real axis or else it is a circular arc whose centre lies on the real axis.*

Proof Let z_1 and z_2 be complex numbers in the upper half plane H. Then there exist real numbers v_1 and v_2 and a Möbius transformation μ of the Riemann sphere such that $\mu(H) = H$, $\mu(z_1) = iv_1$ and $\mu(z_2) = iv_2$. Now the Möbius transformation μ preserves Poincaré distance (Proposition 2.9), and therefore maps geodesics to geodesics. It follows that a continuous curve A joining z_1 to z_2 is a geodesic from z_1 to z_2 if and only if $\mu(A)$ is a geodesic from iv_1 to iv_2. It follows from Corollary 2.12 that the curve A is a geodesic for the Poincaré distance function if and only if $\mu(A)$ is the line segment joining iv_1 to iv_2.

Now Möbius transformations map lines and circles in the complex plane to lines and circles. (Thus the image of a line under a Möbius transformation must be a line or a circle, and the same is true of inverse images because all Möbius transformations are invertible. It follows that the geodesic A must be either a line segment or a circular arc.

Suppose that the the curve A is both a geodesic and a segment of a line L. Then a complex number z lies on the line L if and only if either $\mu(z) = \infty$ or $\mathrm{Re}[z] = 0$. Also $\mu(\overline{z}) = \overline{\mu(z)}$ for all complex numbers z, because $\mu(H) = H$ (Corollary 2.4). It follows that $\overline{z} \in L$ for all $z \in L$. The line L is thus perpendicular to the real axis, and thus A is, in this case, a line segment whose direction is perpendicular to the real axis. Conversely any such line segment is a geodesic (Lemma 2.14).

If A is a geodesic but is not a line segment then it must be a circular arc. Let Z be the whole circle of which it forms part. The circle Z then consists of those complex numbers z for which either either $\mu(z) = \infty$ or else $\mathrm{Re}[\mu(z)] = 0$. Also Also $\mu(\overline{z}) = \overline{\mu(z)}$ for all complex numbers z, because $\mu(H) = H$. It follows that $\overline{z} \in Z$ for all $z \in Z$, and therefore the centre of the circle Z must lie on the real axis. Conversely if the arc A forms part of a circle Z whose centre lies on the real axis then it is a geodesic (Lemma 2.15). The result follows. ∎