Hyperbolic Elements of Geometry (Euclidean and Non-Euclidean Geometry)

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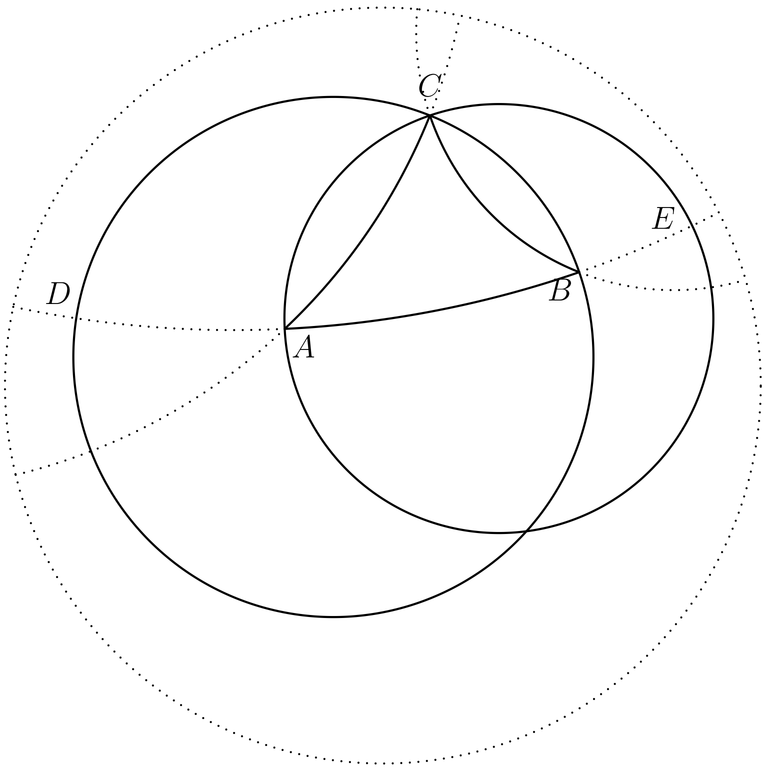
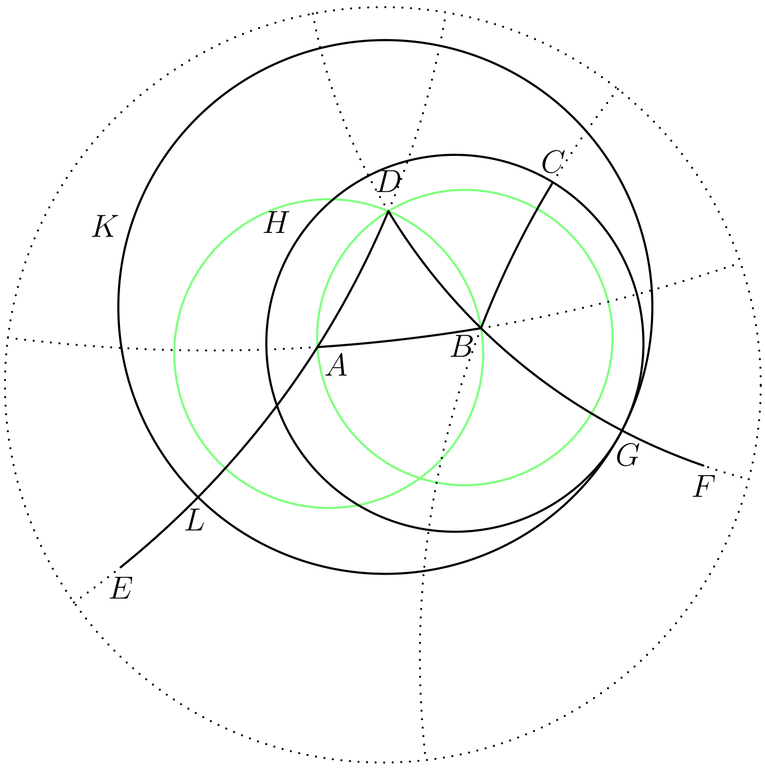
Contents

Hyperbolic Proposition 1 (Construction) *On a geodesic segment AB in the hyperbolic plane, to construct an equilateral geodesic triangle ABC.*



Construction To construct the equilateral geodesic triangle, first draw a circle in the hyperbolic plane passing though the point B with hyperbolic centre at the point A. Also draw a circle in the hyperbolic plane passing through the point A with hyperbolic centre at the point B. Now the second circle passes though the centre of the first circle, but also passes through points that lie outside the first circle. Consequently there are points at which those two circles intersect. Let C be a point at which the two circles intersect, and construct the geodesic triangle with vertices at A, B and C. Then the geodesic segments AB and AC are equal in hyperbolic length because the points B and C lie on a circle whose hyperbolic centre is located at the point A. Similarly the geodesic segments AB and BC are equal in hyperbolic length, because the points A and C lie on a circle whose hyperbolic centre is located at the point B. Consequently the geodesic triangle ABC is the required equilateral geodesic triangle. ∎

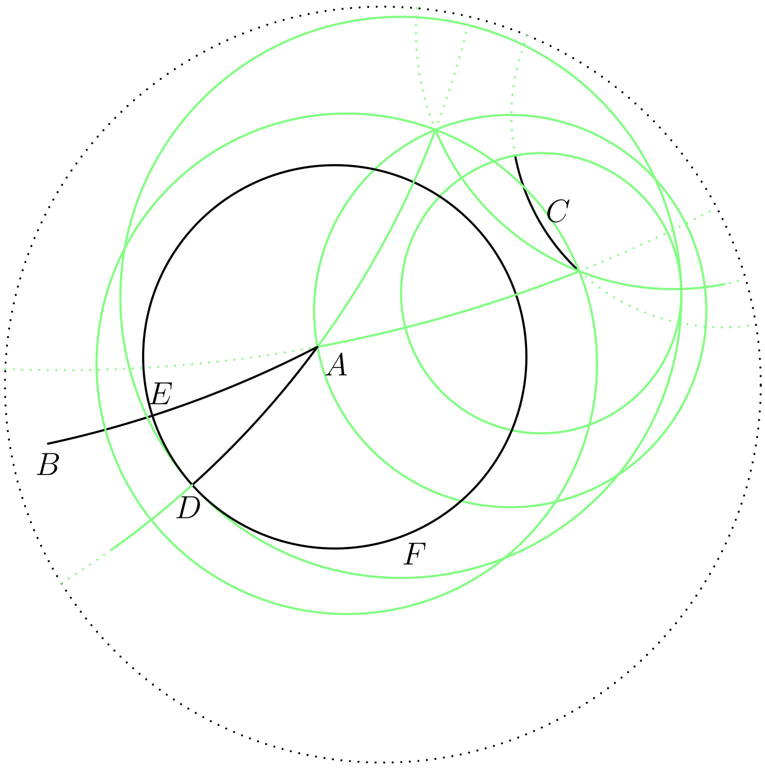
Hyperbolic Proposition 2 (Construction) *To place at a point A of the hyperbolic plane a geodesic segment equal in hyperbolic length to a given geodesic segment BC.*



Construction First construct an equilateral geodesic triangle ABD on the geodesic segment AB (Proposition 1). Then draw a circle in the hyperbolic plane passing though the point C whose hyperbolic centre is located at the point B. Let the geodesic segment DB be produced beyond B to intersect that circle just constructed at the point G. Then draw a circle in the hyperbolic plane passing through the point G whose hyperbolic centre is located at the point D. Then let the geodesic segment DA be produced beyond A to meet this new circle at the point L. Then the geodesic segments DG and DL are equal with respect to hyperbolic length. Also the parts DB and DA of those geodesic segments are equal with respect to hyperbolic length. Consequently the remainders BG and AL are equal with respect to hyperbolic length. But the geodesic segment BC is also equal in hyperbolic length to the geodesic segment BG, because the points C and G are located on a circle in the hyperbolic plane whose hyperbolic centre is located at the point B.

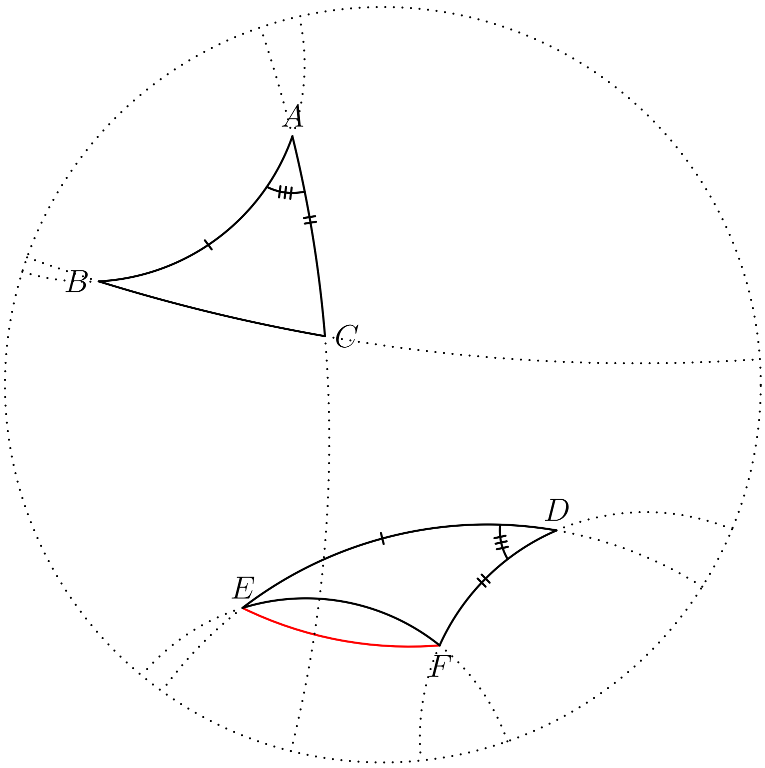
Consequently the geodesic segments BC, BG and AL are all equal to one another with respect to hyperbolic length, and thus AL is the required geodesic segment. ∎

Hyperbolic Proposition 3 (Construction) *Given two unequal geodesic segments AB and C in the hyperbolic plane, to cut off from the longer geodesic segment AB a geodesic segment equal in hyperbolic length to the shorter geodesic segment C.*



Construction First construct a geodesic segment AD, with one endpoint at the point A, so that AD is equal to C with respect to hyperbolic length. Then draw a circle through the point D whose hyperbolic centre is located at the point A, and let that circle intersect the geodesic segment AB at the point E. Then AE is the required geodesic segment cut off from the segment AB. ∎

Hyperbolic Proposition 4 (SAS Congruence Rule) *If two geodesic triangles ABC and DEF in the hyperbolic plane have the two sides AB, AC equal in hyperbolic length to the two sides DE, DF respectively, and have the contained angles BAC and EDF equal to one another, then the geodesic triangles ABC and DEF are congruent, and thus the sides BC and EF are equal to one another in hyperbolic length, and the angles of the geodesic triangle ABC at B and C are equal to the angles of the geodesic triangle DEF at E and F respectively.*



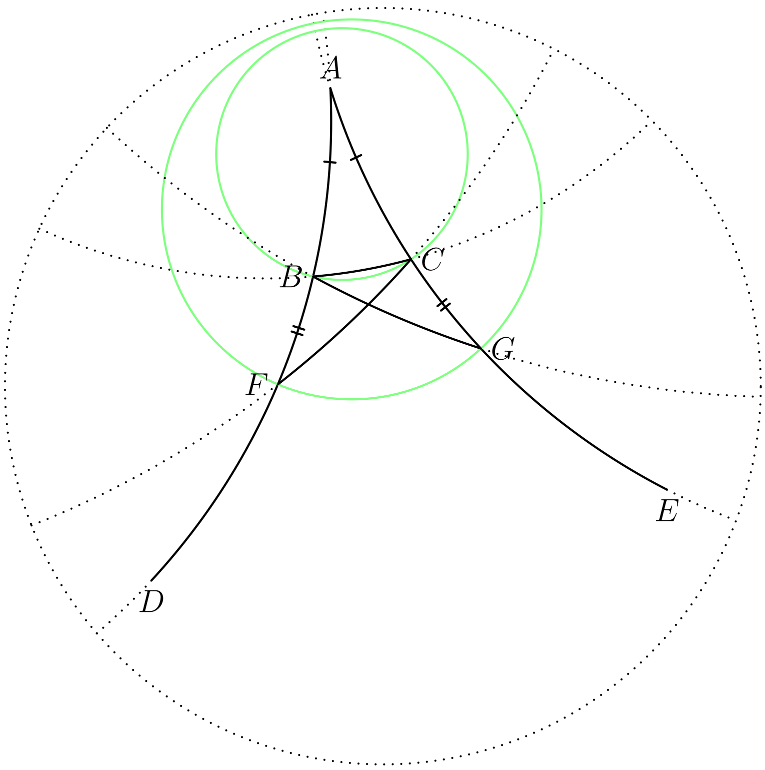
Proof The sides AB and DE of the respective geodesic triangles ABC and DEF are equal in hyperbolic length. Consequently there exists a hyperbolic motion φ of the hyperbolic plane, preserving both the hyperbolic distance between pairs of points and the angles between geodesic segments at the points at which they intersect, where this hyperbolic motion φ maps the points A and B of the hyperbolic plane onto the points D and E of that plane, and also maps any point of the hyperbolic plane that lies on the same side of the geodesic AB as the point C to some point that lies on the same side of the geodesic DE as the point F. The equality of the angles BAC and EDF then ensures that points on the geodesic ray starting from the point A

and passing through the point C are mapped to points of the geodesic ray starting from the point D and passing through the point F. But the geodesic segments AC and DF are also equal in hyperbolic length. It follows that the hyperbolic motion φ must map the point C to the point F. Thus

$$\varphi(A) = D, \quad \varphi(B) = E \quad \text{and} \quad \varphi(C) = F.$$

Now the distance-preserving property of the hyperbolic motion φ ensures that BA, AC and BC are equal to ED, DF and EF respectively with regard to hyperbolic length. Also the angle-preserving property of the hyperbolic motion φ ensures that the angles BAC, ABC and BCA are equal to angles EDF, DEF and EFD respectively. The result follows. ∎

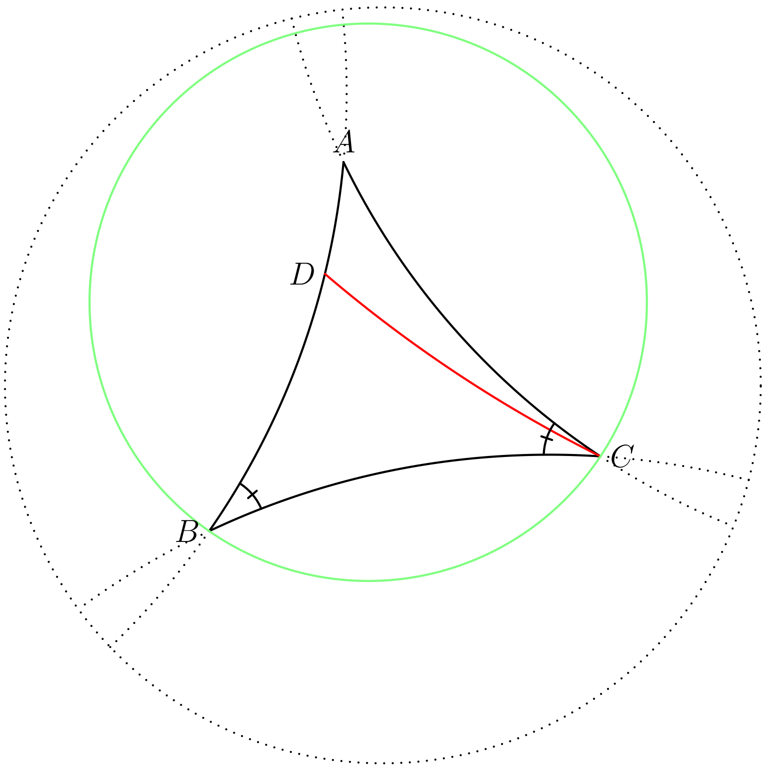
Hyperbolic Proposition 5 (Isosceles Geodesic Triangles) *Let ABC be an isosceles geodesic triangle in the hyperbolic plane, and let the equal sides AB and AC be produced to points D and E. Then the angles CBD and BCE under the base BC are equal to one another, as are the angles ABC and ACB of the isosceles geodesic triangle ABC at the endpoints B and C of the base.*



Proof Let points F and G be constructed on the geodesics AD and AE so that B lies between A and F, C lies between A and G and BF is equal to CG wih respect to hyperbolic length (Proposition 3). Now the geodesic segments AB and AC are equal in hyperbolic length. Consequently the geodesic segments AF and AC are equal in hyperbolic length to the geodesic segments AG and AB respectively. It therefore follows from the SAS Congruence Rule (Proposition 4) that the geodesic triangles AFC and AGB are congruent, and consequently the sides FC and GB are equal to one another in hyperbolic length. Moreover the angles BFC and CGB, being identical to the angles AFC and AGB respectively, are equal to one another. Consequently the geodesic triangles BFC and CGB are congruent. It then follows that the angles CBD and BCE, being identical to the angles CBF and BCG

respectively, are equal to one another. Thus the angles under the base of the isosceles geodesic triangle ABC are equal to one another. Now the congruence of the geodesic triangles BFC and CGB also ensures that the angles BCF and CBG are equal to one another. We previously showed that the geodesic triangles AFC and AGB are congruent, from which it follows that the angles ACF and ABG are equal to one another. Subtracting the equal angles BCF and CBG from the equal angles ACF and ABG, we conclude that the angles ABC and ACB are equal to one another. Thus the angles of the isosceles geodesic triangle ABC at the endpoints B and C of the base are equal to one another, as required. ∎

Hyperbolic Proposition 6 (Converse of Proposition 5) *If in a geodesic triangle ABC in the hyperbolic plane two angles B and C be equal to one another, the sides AC and AB which subtend the equal angles will also be equal to one another.*



Proof Suppose that the two sides AC and AB were not equal to one another with respect to hyperbolic length. Then one would be longer than the other. Suppose therefore that the side AB were longer than the side AC. We would then be able to cut off from AB a geodesic segment BD equal in hyperbolic length to the side AC of the geodesic triangle. Applying the SAS Congruence Rule (Proposition 4), we would then conclude that the geodesic triangles ABC and DCB would be congruent, and therefore the angles ABC and ACD would be equal to one another. But the angles ABC and ACB are equal to one another. Thus, under the assumption that the geodesic segment BD is equal to AC, we would have to conclude that the angles ACB and DCB would be equal to one another. But this is impossible since DCB would also be a proper part of the angle ACB. Consequently neither of the sides AB and AC can be longer than the other, and therefore these two sides are equal in hyperbolic length, as claimed. ∎

Hyperbolic Proposition 7 *Given a geodesic segment AB in the hyperbolic plane, there cannot be constructed two distinct geodesic triangles ABC and ABD on the same side of the geodesic AB with the properties that the sides AC and AD are equal to one another in hyperbolic length and the sides BC and BD are also equal to one another in hyperbolic length.*

Proof Let C be a point of the hyperbolic plane that does not lie on the complete geodesic that passes through the points A and B, where A and B are distinct. Now it is not possible to find any point D distinct from the point C but lying on the geodesic ray starting at the point A and passing through the point C so as to make the geodesic segments AC and AD equal to one another in hyperbolic length and also make the geodesic segments BC and BD equal in hyperbolic length. Similarly it is not possible to find any point D distinct from the point C but lying on the geodesic ray starting at the point A and passing through the point C so as to make the geodesic segments AC and AD equal to one another in hyperbolic length and also make the geodesic segments BC and BD equal in hyperbolic length. We may therefore restrict our attention to cases in which the point D does not lie on the geodesic rays starting at the points A and B that pass through the point C.

Now the removal of these two geodesic rays divides the side of the hyperbolic plane to which the point C belongs into four regions. A complete investigation of all relevant cases therefore needs to consider the following four cases:—

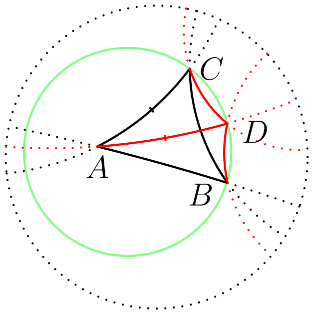
Case (i): this is the case where the point D lies inside the angle BAC but outside the angle ABC;

Case (ii): this is the case where the point D lies inside the angle BAC and also inside the angle ABC;

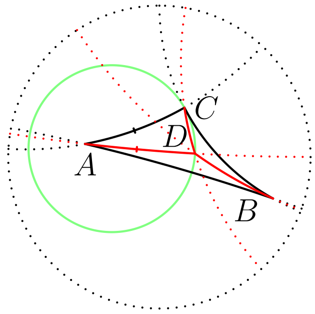
Case (iii): this is the case where the point D lies outside the angle BAC but inside the angle ABC;

Case (iv): this is the case where the point D lies outside the angle BAC and also outside the angle ABC.

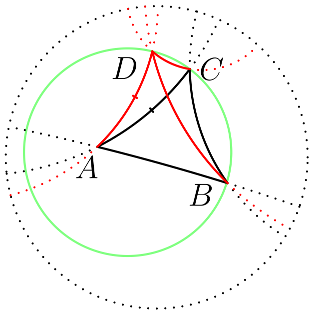
Now it should be noted that the point D lies outside the angle BAC if and only if the point C lies inside the angle BAD, and the point D lies inside the angle BAC if and only if the point C lies outside the angle BAD. Consequently the four cases to consider are the following:



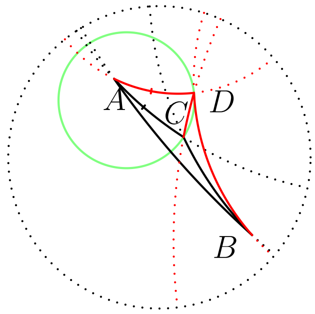
Case (i)



Case (ii)



Case (iii)



Case (iv)

Case (i): this is the case where the point D lies inside the angle BAC but outside the angle ABC;

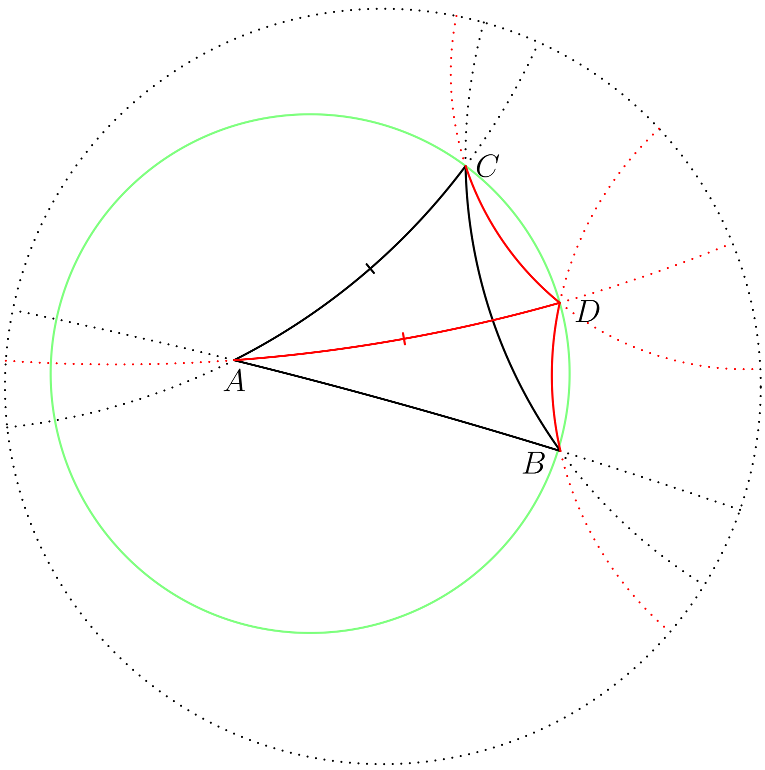
Case (ii): this is the case where the point D lies inside the angle BAC and also inside the angle ABC;

Case (iii): this is the case where the point C lies inside the angle BAD but outside the angle ABD;

Case (iv): this is the case where the point C lies inside the angle BAD and also inside the angle ABD.

A comparison of these characterizations of the cases shows that the result in cases (iii) and (iv) follows from the corresponding result in cases (i) and (ii) on interchanging the roles of the points C and D. Thus to prove the proposition in full generality, it only remains to prove the result in cases (i) and (ii).

Accordingly we first prove the result in case (i). Accordingly suppose that the geodesic segments AC and AD are equal in hyperbolic length and that the point D lies in the angle BAC but outside the geodesic triangle ABC. This is the configuration depicted in the following figure:

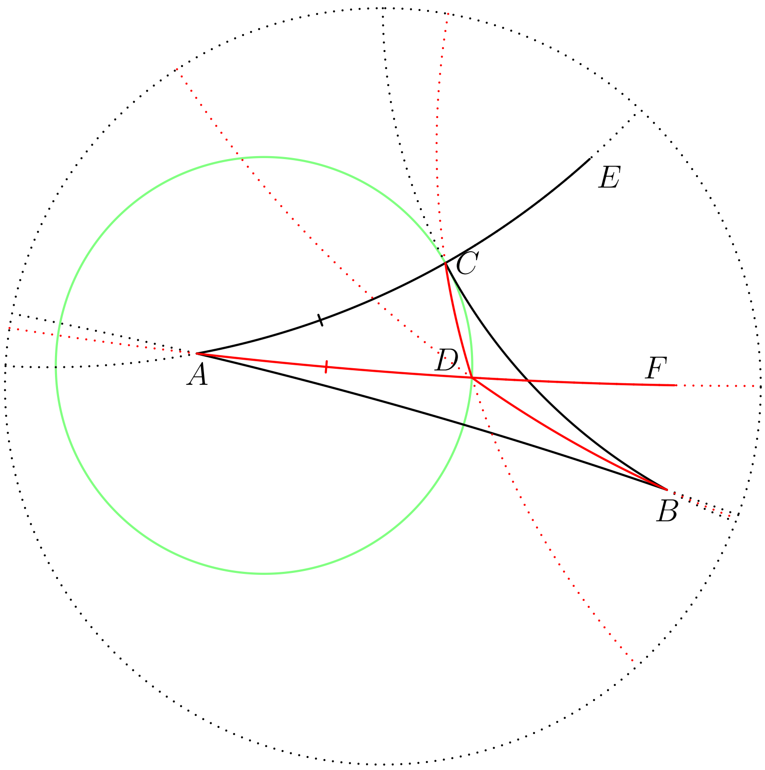


Join the points C and D by a geodesic segment. Then ACD is an isosceles geodesic triangle with equal sides AC and AD, and consequently the angles ACD and ADC at the base of the geodesic triangle are equal. Now the points A and D lie on the opposite sides of the geodesic that passes through the points B and C, and consequently the point B lies in the interior of the angle ACD. The angle BCD between the geodesic segments CB and CD at the point C is therefore less than the angle ACD between the geodesic segments CA and CD at that point.

Also, considering angles at the point D, it can be seen that the points A and C lie on the same side of the geodesic passing through D and B, and the points A and B lie on the same side of the geodesic passing through D and C. Consequently the point A lies in the interior of the angle BDC between the geodesic segments DB and DC, and therefore the angle ADC is less than BCD.

We have now demonstrated the following: the angle BCD is less than the angle ACD; the angle ACD is equal to the angle ADC; the angle ADC is less than the angle BDC. It follows that the angle BCD is less than the angle BDC. Consequently the geodesic triangle BCD is not an isosceles geodesic triangle with equal sides BC and BD, for it it were, we would have arrived at a result contradicting Proposition 5. We have thus shown that, in case (i), if the geodesic segments AC and AD are equal to one another in hyperbolic length, then the geodesic segments BC and BD must by unequal in hyperbolic length.

We now turn our attention to case (ii). In this case the configuration is as depicted in the following diagram:



In this case the point D lies in the interior of the geodesic triangle ABC.

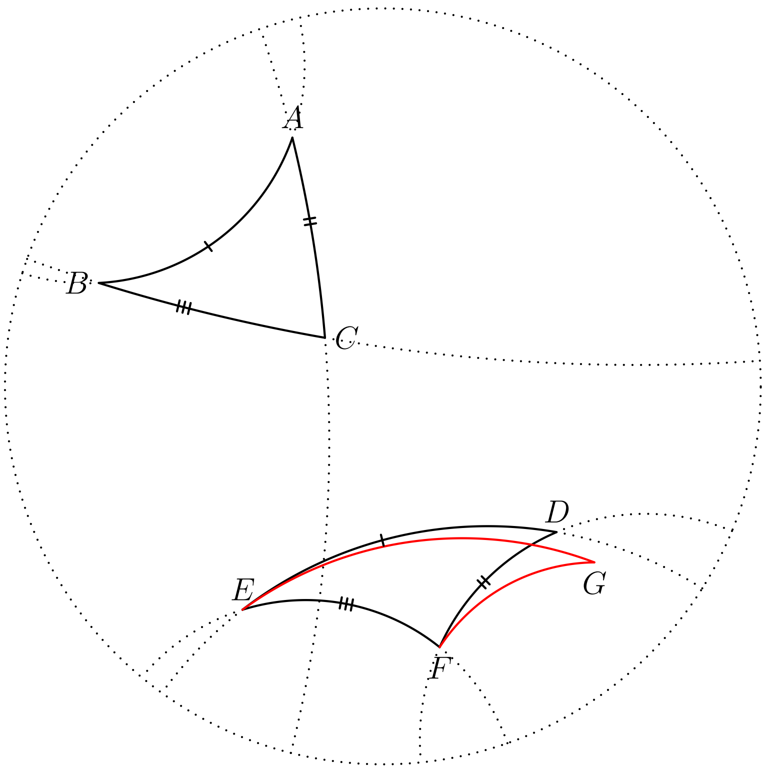
We produce the geodesic segements AC and AD to points E and F of the hyperbolic plane, so that the points E and F are joined to the point A by geodesics passing through the points C and D respectively.

Suppose that the geodesic segments AC and AD are equal in hyperbolic length. It then follows from Proposition 5 that the angles ECD and FDC are equal, because they are the angles under the base of an isosceles geodesic triangle. Now, in this case, the points D and E of the hyperbolic plane lie on opposite sides of the geodesic passing through the points B and C and therefore the angle ECD between the geodesic segments CE and CD is greater than the angle ECB between the geodesic segments CE and CB. Also the point F lies in the interior of the angle CDB between the geodesic segments AD and DB, and therefore the angle CDF between the geodesic segments DC and DF is less than the angle CDB between the geodesic segments DC and DB.

We have now demonstrated the following: the angle BCD is less than the angle ECD; the angle ECD is equal to the angle FDC; the angle FDC is less than the angle BDC. It follows that the angle BCD is less than the angle BDC. Consequently the geodesic triangle BCD is not an isosceles geodesic triangle with equal sides BC and BD, for it it were, we would have arrived at a result contradicting Proposition 5. We have thus shown that, in case (ii), if the geodesic segments AC and AD are equal to one another in hyperbolic length, then the geodesic segments BC and BD must by unequal in hyperbolic length.

As explained previously, the required result in cases (iii) and (iv) follows from the results proved in cases (i) and (ii) on interchanging the roles of the points C and D. Consequently the result of Proposition 7 has been proved in full generality, as required. ∎

Hyperbolic Proposition 8 (SSS Congruence Rule) *If, in geodesic triangles ABC and DEF in the hyperbolic plane, the sides AB, BC, CA of the geodesic triangle ABC are respectively equal to the sides DE, EF and FD in hyperbolic length, then the geodesic triangles are congruent, and consequently the angles between the sides of of the geodesic triangle ABC at the vertices A, B and C are respectively equal to the angles between the sides of the geodesic triangle DEF at the vertices D, E and F.*

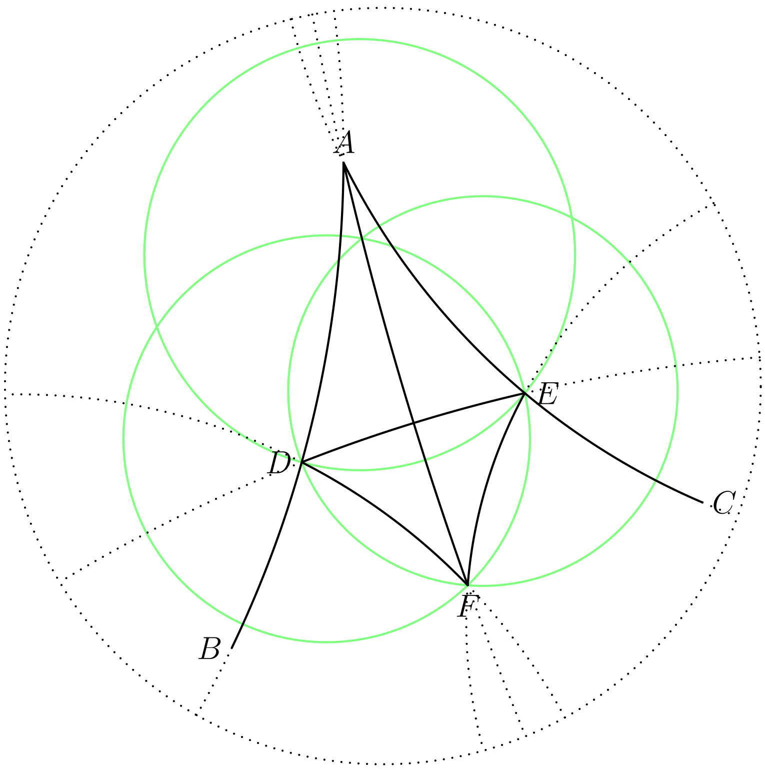


Proof The geodesic segments BC and EF are equal in hyperbolic length, and therefore there exists a hyperbolic motion φ of the hyperbolic plane that maps the points B and C onto the points E and F respectively, where this hyperbolic motion φ preserves both the hyperbolic lengths of geodesic segments and that angles between geodesic segments at their points of intersection. Moreover the hyperbolic motion φ may be chosen so that it maps the point A onto a point G of the hyperbolic plane that lies on the same side of the geodesic through the points E and F as the point D. The sides GE, EF and FG of the geodesic triangle GEF are respectively equal to the sides AB, BC and CA of the geodesic triangle ABC, because

$$\varphi(A) = G, \quad \varphi(B) = E \quad \text{and} \quad \varphi(C) = F.$$

But the sides AB, BC and CA of the geodesic triangle ABC are respectively equal in hyperbolic length to the sides DE, EF, FD of the geodesic triangle DEF. Consequently the sides GE, EF and FG of the geodesic triangle GEF are respectively equal in hyperbolic length to the sides DE, DF and FA of the tringle DEF. It now follows immediately from Proposition 7 that the points D and G coincide. Consequently the geodesic triangles ABC and DEF are congruent, as required. ∎

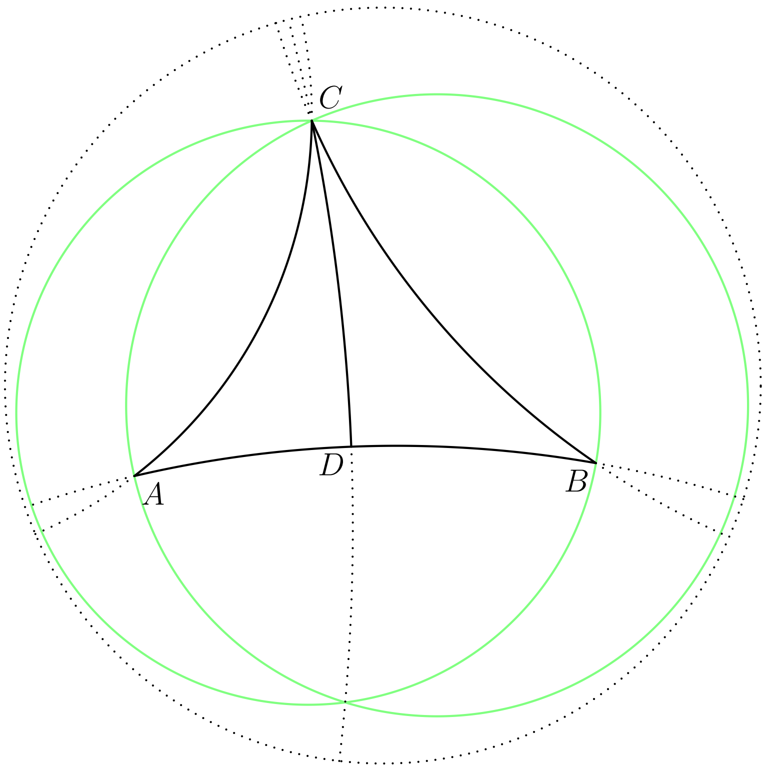
Hyperbolic Proposition 9 (Construction) *To bisect the angle between two geodesics in the hyperbolic plane at a point at which they intersect.*



Construction We seek to bisect the angle between two geodesic segments AB and AC that intersect at a point A of the hyperbolic plane. Take points D and E on the geodesic segments AB and AC respectively, chosen so that AD and AE are equal in hyperbolic length. (The possibility of finding such points D and E is guaranteed by Proposition 3, and can be achieved in practice by choosing a point D and E to be the points where a circle of sufficiently small radius in the hyperbolic plane with hyperbolic centre A intersects the geodesic segments AB and AC.) Then construct the equilateral geodesic triangle DEF on the geodesic segment DE so that the points A and F of the hyperbolic plane lie on opposite sides of the geodesic that passes through the points D and F. Then join the points A and F be a geodesic segment. It can be shown that this geodesic segment bisects the angle BAC.

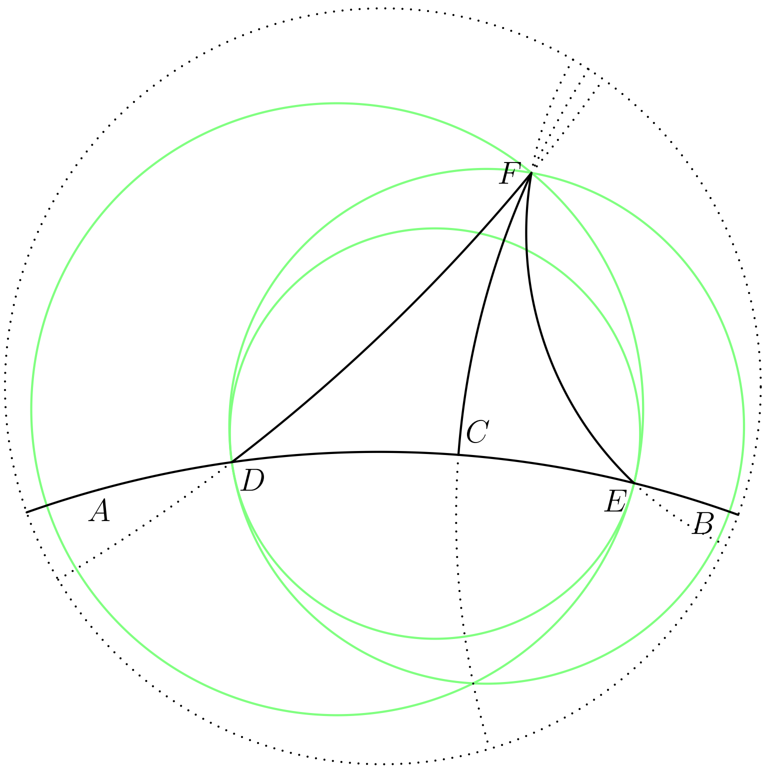
To prove this, note that the sides AD, DF and AF of the geodesic triangle ADF are equal in hyperbolic length to the sides AE, EF and AF respectively of the sides of the geodesic triangle AEF. Applying the SSS Congruence Rule (Proposition 8), we conclude that those two geodesic triangles are congruent to one another, and therefore the angles DAF and EAF of those geodesic triangles at the vertex A are equal to one another. The angle BAC between the geodesic segments AB and AC is thus bisected by the geodesic segment AF. ∎

Hyperbolic Proposition 10 (Construction) *To bisect a geodesic segment in the hyperbolic plane.*



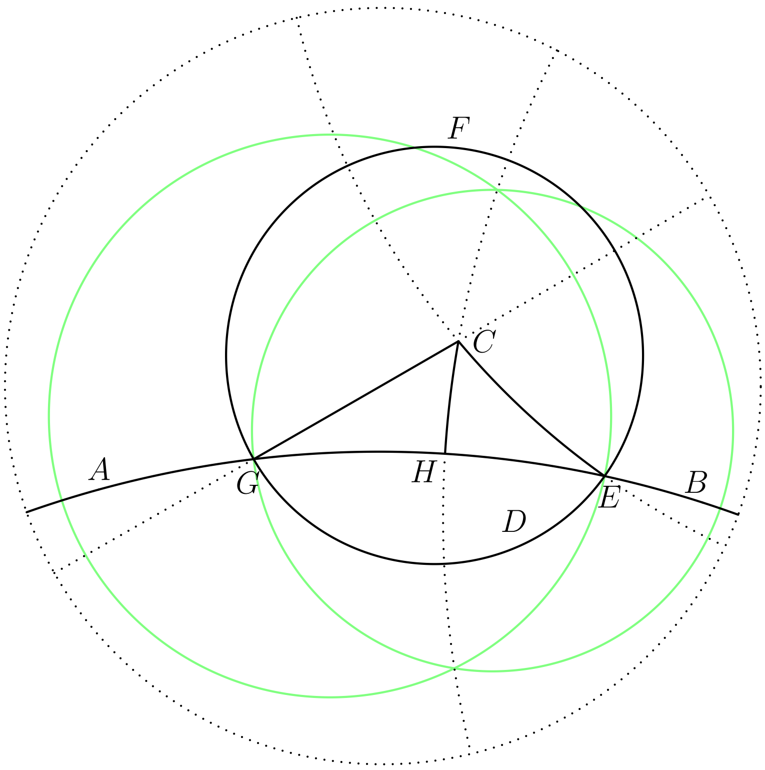
Construction Let A and B be points in the hyperbolic plane. It is required to bisect the geodesic segment AB by locating a point D on that geodesic segment for which the geodesic segments AD and DB are equal to one another in hyperbolic length. To achieve this, construct an equilateral geodesic triangle ABC on the geodesic segment AB (Proposition 1), and bisect the angle ACB between the sides CA and CB of this geodesic triangle by a geodesic ray which intersects the geodesic segment AB at the point D (Proposition 9). The sides AC and CD of the geodesic triangle ACD are respectively equal in hyperbolic length to the sides BC and CD of the geodesic triangle BCD, and the included angles ACD and BCD are equal. It follows on applying the SAS Congruence Rule (Proposition 4) that the geodesic triangles ACD and BCD are congruent. Consequently the sides AD and BD of those geodesic triangles are equal in hyperbolic length, and thus the geodesic segment AB has been bisected at the point D. ■

Hyperbolic Proposition 11 (Construction) *At a point C on a geodesic in the hyperbolic plane that passes through distinct points A and B, to draw a geodesic segment CF that intersects the geodesic AB at right angles at the point C.*



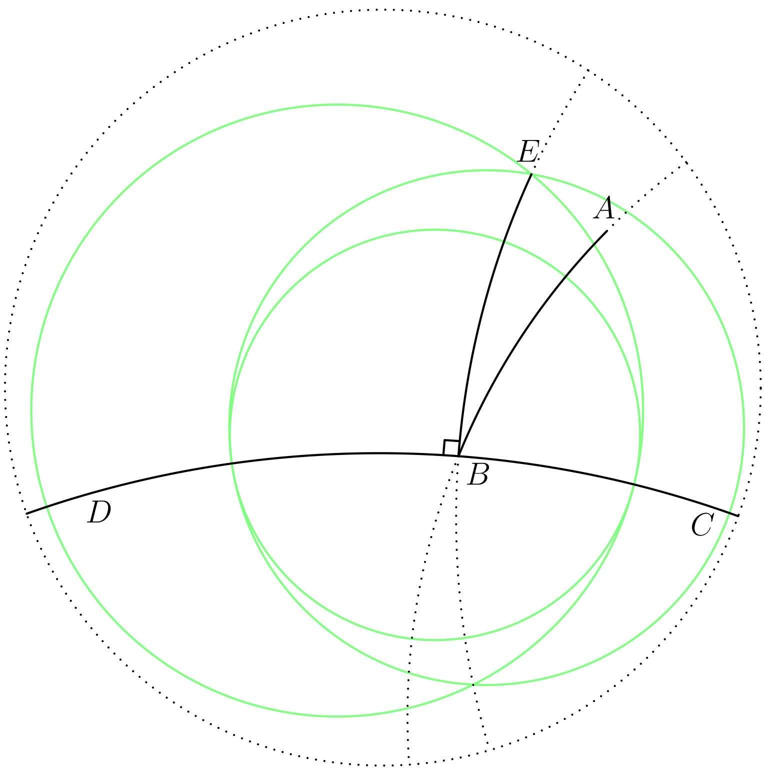
Construction Take points D and E on the geodesic AB, lying on either side of the chosen point C so that the geodesic segments DC and CE are equal in hyperbolic length. (This can be achieved by letting D and E be the points at which a circle with hyperbolic centre C intersects the geodesic AB.) Then construct an equilateral geodesic triangle DFE on the geodesic segment DE (Proposition 1), and join the points C and F by a geodesic segment. Now the sides DF, FC and DC of the geodesic triangle DFC are respectively equal in hyperbolic length to the sides EF, FC and EC of the geodesic triangle EFC. It follows on applying the SSS Congruence Rule (Proposition 8) that the angles DCF and ECF are equal to one another. Consequently the geodesic segment CF meets the geodesic AB at right angles at the chosen point C of the geodesic C, and thus the requirements of the construction have been achieved. ∎

Hyperbolic Proposition 12 (Construction) *To draw a geodesic segment from a given point C of the hyperbolic plane to a point H lying on a given geodesic AB in the hyperbolic plane that does not pass though the given point C, where the geodesic segment CH intersects the given geodesic AB at right angles at the point H.*



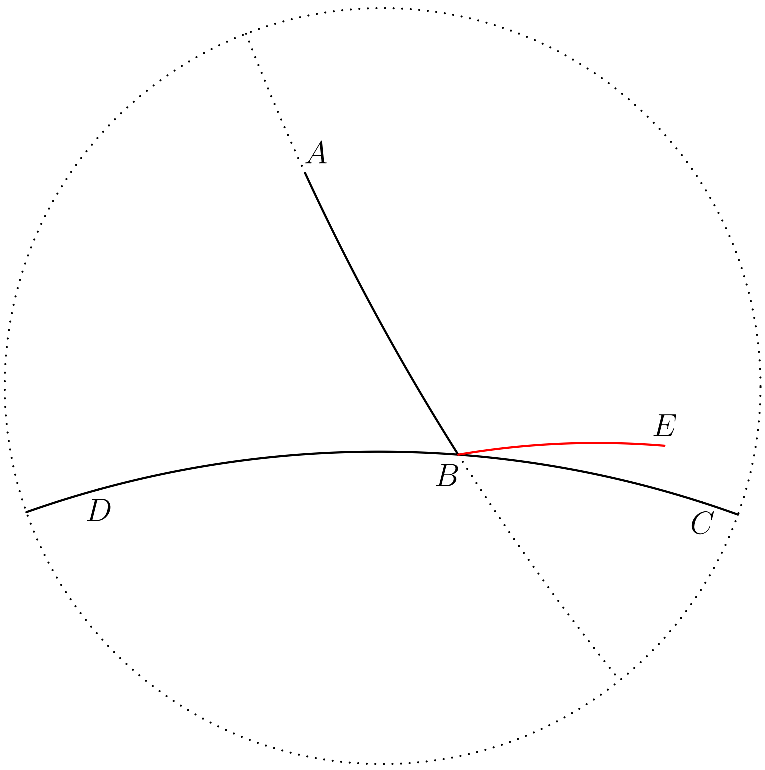
Construction Take a point D in the hyperbolic plane so that the points C and D lie on opposite sides of the geodesic AB. Then the circle in the hyperbolic plane with hyperbolic centre C intersects the geodesic AB at two points G and E. Let the geodesic segment GE be bisected at the point H (Proposition 10), and join the points G and E and H to the point C by geodesic segments. Now the sides GC, CH are GH of the geodesic triangle GCH are respectively equal in hyperbolic length to the sides EC, CH and HE of the geodesic triangle ECH. Applying the SSS Congruence Rule (Proposition 8), it follows that the geodesic triangles GCH and ECH are congruent, and consequently the angles GHC and EHC are equal to one another. Thus the geodesic segment CH meets the geodesic AB at right angles at the point H, as required. ∎

Hyperbolic Proposition 13 (Supplementary Angles) *If a geodesic segment BA be taken intersecting a geodesic DC at a point B between C and D, then the sum of the angle ABC with its supplementary angle ABD is equal to two right angles.*



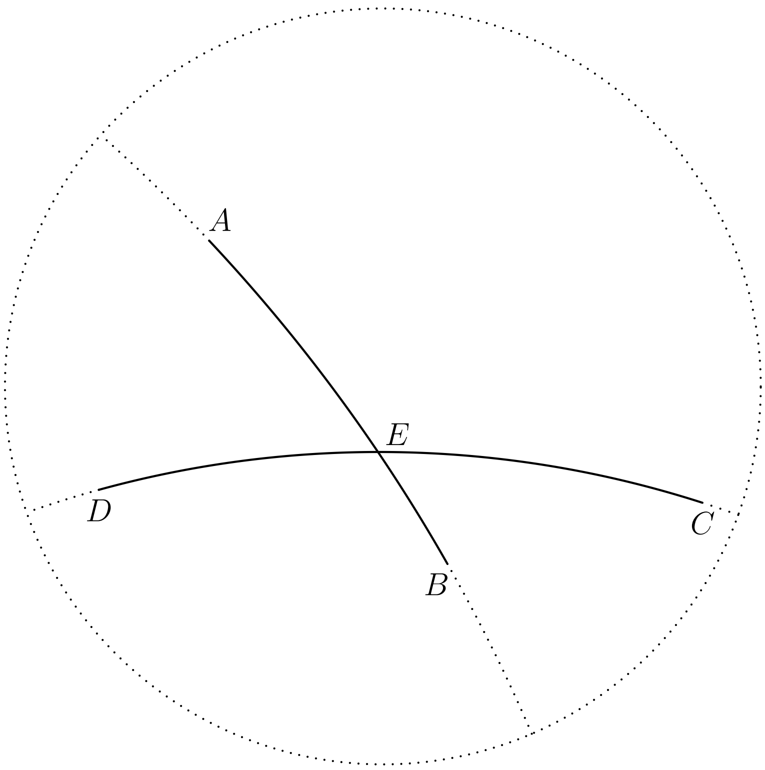
Proof Let the geodesic segment BE be taken with an endpoint at the point B so as to intersect the geodesic CD at right angles at the point B, ensuring that the points A and E lie on the same side of the geodesic CD. Suppose that the point A lies in the interior of the angle CBE. Then the sum of the two angles DBA and ABC is equal to the sum of the three angles DBE, EBA and ABC, and is thus equal to sum of the two right angles DBE and EBC. A similar argument applies when the point A lies in the interior of the angle DBE, and the result is immediate when the point A lies on the geodesic ray AE. Thus the required result can be established in all relevant cases. ∎

Hyperbolic Proposition 14 (Adjacent angles summing to two right angles) *If geodesic segments DB and BC in the hyperbolic plane make angles at the point B with a geodesic segment AB that sum to two right angles, where the points C and D lie on opposite sides of AB, then some geodesic passes though the three points D, B and C.*



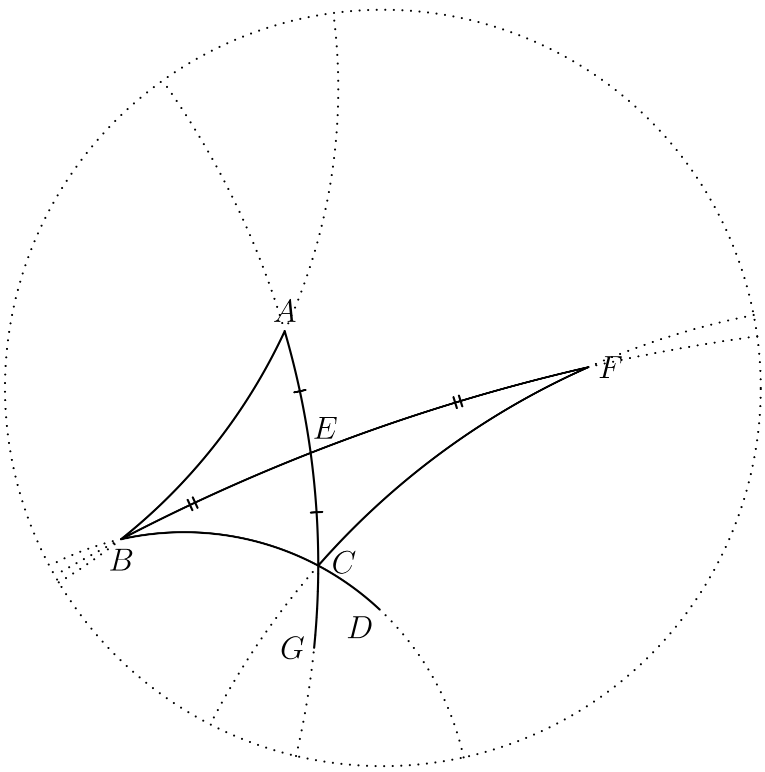
Proof Suppose that the geodesic segments DB and BC were not both parts of a single geodesic in the hyperbolic plane passing through the points C and D. Then the geodesic segment DB could be produced to a point E that does not lie on the geodesic passing through the points B and C. Suppose that the point E were located on the same side of the geodesic segment BC as the point A. Then the angle ABE would be less than the angle ABC. But the angles DBA and ABE must sum to two right angles. Consequently the angles DBA and ABC would sum to more than two right angles, contradicting the conditions of the proposition. A similar argument shows that the points A and E cannot lie on opposite sides of the geodesic BC. Consequently the geodesic segments DB and BC must be parts of a single geodesic that passes though the points D, B and C, as required. ∎

Hyperbolic Proposition 15 (Vertically-opposite angles) *If geodesics AB and CD intersect at some point E then the vertically opposite angles AED and BEC are equal to one another, as are the vertically opposite angles CEA and DEB.*



Proof If the angle CAE is added to either of the angles AED or BEC, then the sum of the relevant angles is equal to two right angles. But where the same angle is subtracted from equal angle sums, the remaining angles or angle sums are equal. Consequently the angles EAD and BEC are equal to one another. Similarly the angles CEA and DEB are equal to one another, as required. ∎

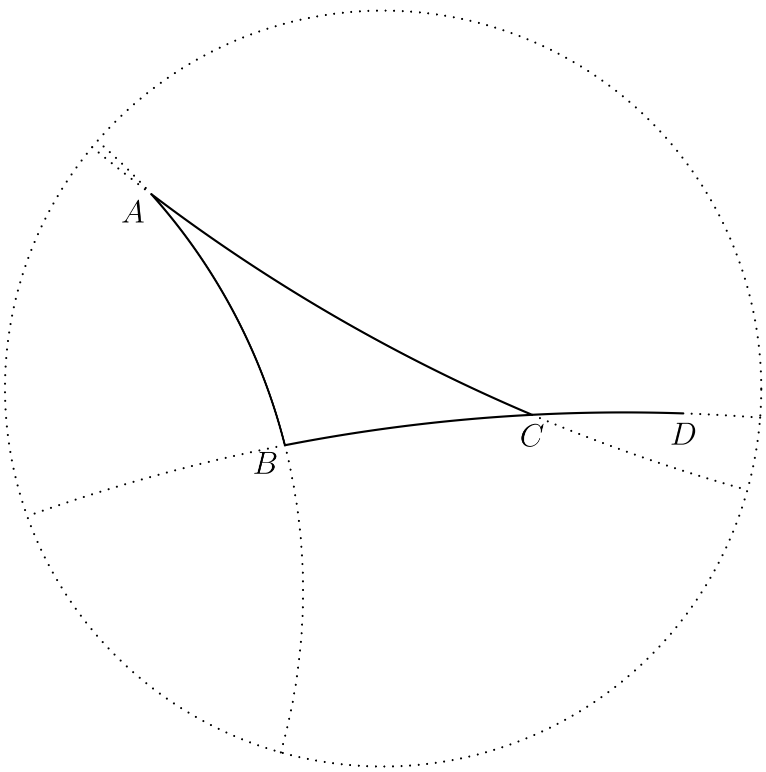
Hyperbolic Proposition 16 (Exterior angle greater than interior and opposite angles *Let a side BC of a geodesic triangle ABC in the hyperbolic plane be produced past B to a point C. Then the external angle ACD of the geodesic triangle at C is greater than the internal and opposite angles of the geodesic triangle ABC at the vertices A and B.*



Proof Bisect the side AC of the geodesic triangle ABC at E, and produce the geodesic segment BE past E to a point F so that the segments BE and EF of the geodesic BF are equal in hyperbolic length. Then the geodesic segments EA and EB are respectively equal in hyperbolic length to the geodesic segments EC and EF, and moreover the included angles AEB and CEF are vertically-opposite angles, and are therefore equal to one another (Proposition 15). Applying the SAS Congruence Rule (Proposition 4), we see that the geodesic triangles EAB and ECF are congruent, and therefore the angle ECF is equal to the angle EAB. But the points A and F all lie on the same side of the geodesic in the hyperbolic plane that passes through the points B, C and D. Consequently the angle ECF is less than the angle ECD. It follows that the internal angle CAB of the geodesic triangle ABC at A is less than the external angle ACD of the geodesic triangle ABC at

C. Similarly the angle ABC of the given geodesic triangle at the point B is less than the external angle BCG of that geodesic triangle. But the external angles ACD and BCG of the geodesic triangle ABC at the vertex C are equal to one another, because they are vertically-opposite angles (Proposition 15). Thus the internal angles of the geodesic triangle ABC are the vertices A and B are both less than the external angles of that geodesic triangle at the vertex C. The result follows. ∎

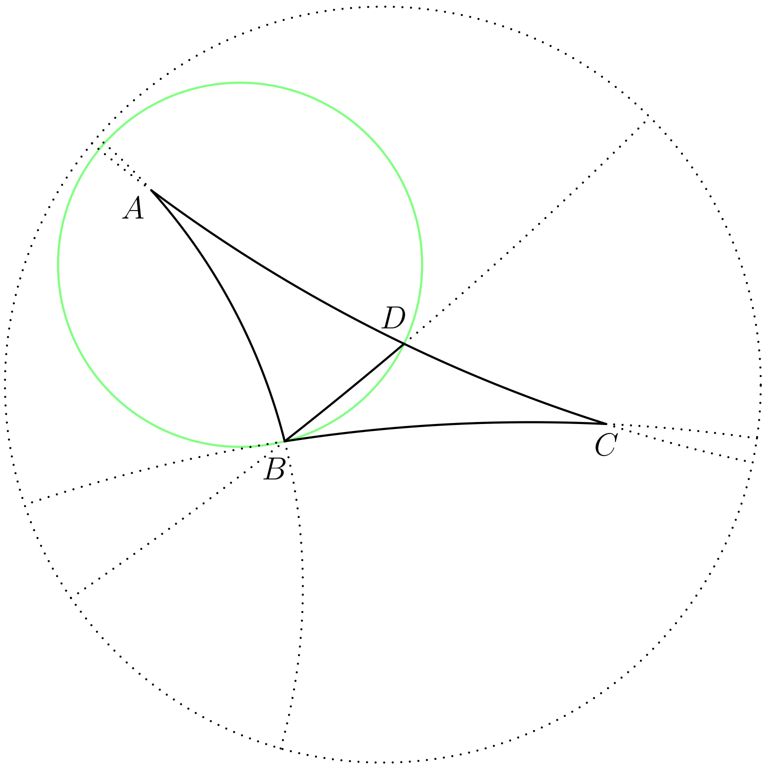
Hyperbolic Proposition 17 *In any geodesic triangle in the hyperbolic plane, two angles taken together in any manner are less than two right angles.*



Proof Let ABC be a geodesic triangle in the hyperbolic plane. We must show that two angles of the geodesic triangle ABC taken together are less than two right angles.

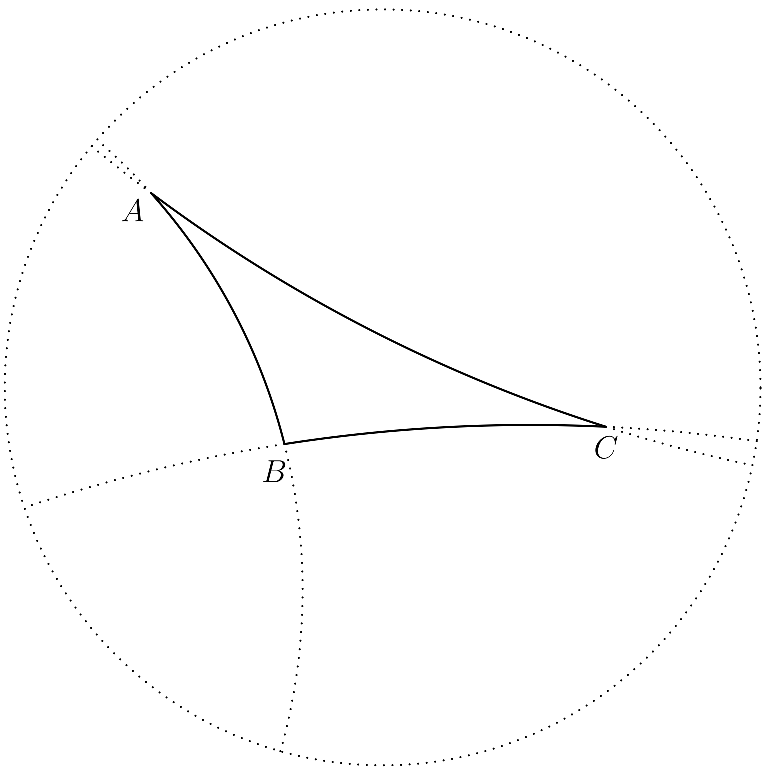
To show this let the geodesic BC be produced beyond C to D, ensuring that BD is a geodesic. Then the interior angle ABC of the geodesic triangle at the vertex B is less than the exterior angle ACD of that geodesic triangle at C. It follows, on adding the angle ACB to each of ABC and ACD, that the sum of the angles ABC and ACB is less than the sum of the angles ACD and ACB, and is therefore less than two right angles (Proposition 13). Consequently the sum of the interior angles ABC and ACB of the geodesic triangle ABC at vertices B and C is less than two right angles. Similarly the sum of the interior angles of the geodesic triangle at vertices A and B is less than two right angles, as is the sum of the interior angles of that geodesic triangle at A and C. This completes the proof. ∎

Hyperbolic Proposition 18 *In a geodesic triangle ABC in the hyperbolic plane, if the side AC is greater in hyperbolic length than the side AB, then the angle ABC that is subtended by the greater side is greater than the angle ACB that is subtended by the lesser side.*



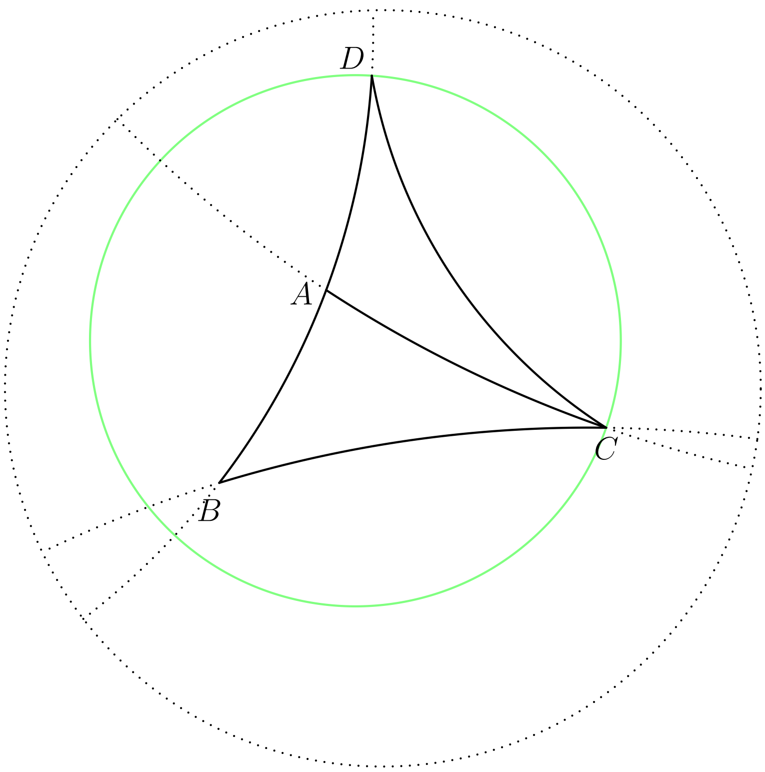
Proof Let a geodesic segment AD be cut off from the greater side AC so that AD is equal in hyperbolic length to the lesser side AB (Proposition 3), and let the points B and D be joined by a geodesic segment. Then ABD is an isoceles geodesic triangle, and therefore the angles ABD and ADB are equal to one another (Proposition 5). Now ADB is an external angle of the geodesic triangle BCD. It follows that the angle ADB is greater than the internal angle BCD of that geodesic triangle at the vertex C (Proposition 16). Morever the angles BCD and BCA are identical. Also the angle ABD is less than the angle ABC. Consequently the angle ACB, being less than ADB, and thus less than ABD, must be less than ABC, as required. ∎

Hyperbolic Proposition 19 *In a geodesic triangle ABC in the hyperbolic plane, if the angle ABC is greater than the angle ACB, then the side AC that subtends the greater angle is greater in hyperbolic length than the side AB that subtends the lesser angle.*



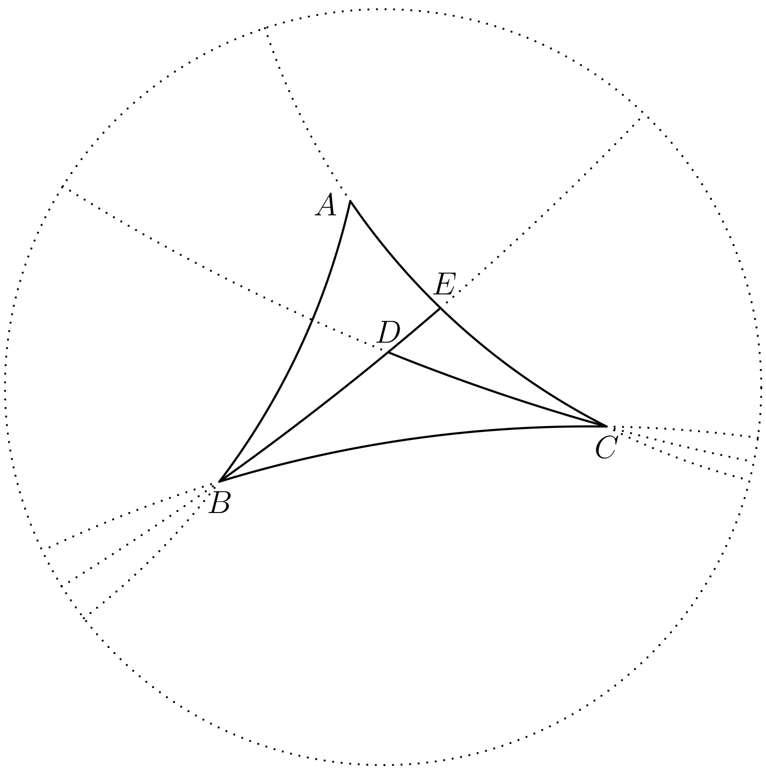
Proof Suppose that, in the geodesic triangle ABC, the angle ABC is greater than the side ACB. If the side AC were equal in hyperbolic length to the side AB then the angle ABC would be equal to the angle ACB (Proposition 5). But it is not. If the side AC were less than the side AB in hyperbolic length then the angle ABC would be less than ACB (Proposition 18). But it is not. Therefore the side AC must be greater in hyperbolic length than the side AB, as claimed. ∎

Hyperbolic Proposition 20 *In a geodesic triangle in the hyperbolic plane, two sides taken together in any manner are greater in hyperbolic length than the remaining one.*



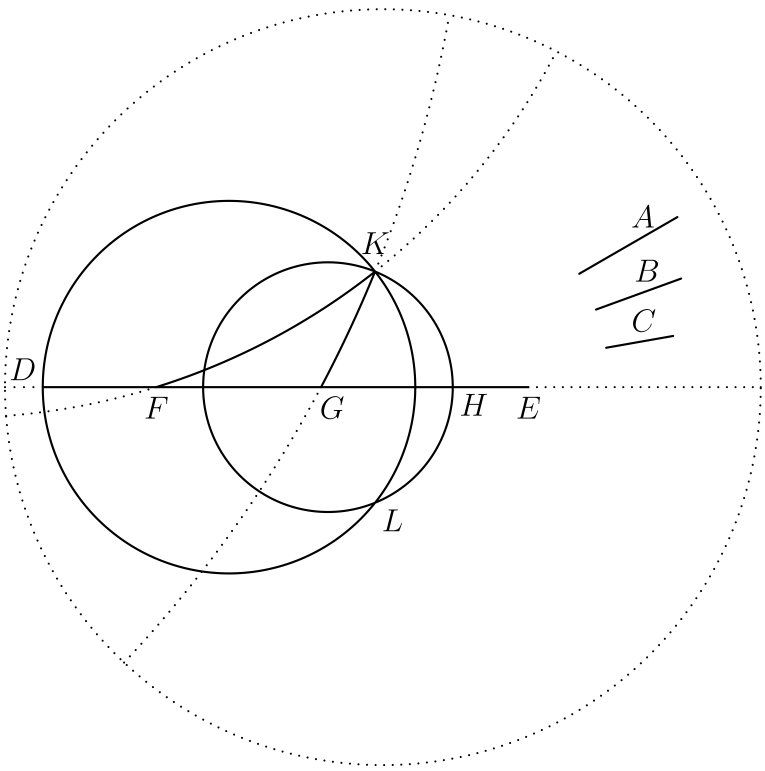
Proof Let ABC be a geodesic triangle in the hyperbolic plane. Produce the side BA of the geodesic triangle beyond A to D so as to ensure that the part AD of the geodesic BD is equal in hyperbolic length to the side AC of the geodesic triangle (Proposition 3). Then ACD is an isosceles geodesic triangle in which the sides AC and AD are equal in hyperbolic length. It then follows that the angles ACD and ADC are equal (Proposition 5). Consequently the angle BCD is greater than the angle BDC, and therefore the geodesic segment BD is greater in hyperbolic length than the geodesic segment BC. But the sides AB and AC of the geodesic triangle ABC taken together are equal to BD. Consequently the sides AB and AC taken together are greater in hyperbolic length than the side AB. Similarly any other two sides of the geodesic triangle taken together are greater than the remaining side. ∎

Hyperbolic Proposition 21 *If ABC is a geodesic triangle in the hyperbolic plane, and if D is a point in the interior of the geodesic triangle, then the sum of the sides BD and DC of the geodesic triangle DBC in hyperbolic length less than the sum of the sides BA and AC of the geodesic triangle ABC, and the angle BDC is greater than the angle BAC.*



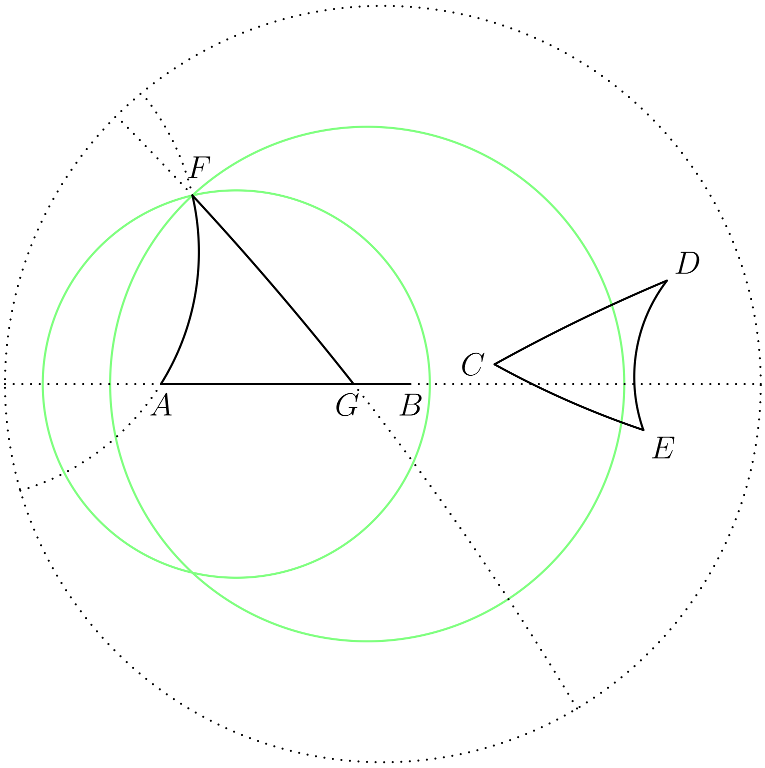
Proof The side DC of the geodesic triangle DEC is in hyperbolic length less than the sum of the sides DE and EC. Conseqently the sum of BD and DC is in hyperbolic length less than the sum of BE and EC, which in turn, and for similar reasons, is in hyperbolic length less than the sum of BA and AC. Also the angle BDC, being an exterior angle of the geodesic triangle EDC, is greater than the interior angle DEC of that geodesic triangle. But DEC, being an exterior angle of the geodesic triangle BAE, is greater than the interior angle BAE of that geodesic triangle. Consequently the angle BDC is greater than the angle BAC, as required. ∎

Hyperbolic Proposition 22 (Construction) *Given geodesic segments A, B and C in the hyperbolic plane, where the sum of any two of these geodesic segments is greater in hyperbolic length than the remaining one, to construct a geodesic triangle FGK whose sides KF, FG, GK are respectively equal in hyperbolic length to A, B and C.*



Construction On a geodesic ray DE in the hyperbolic plane starting at the point D, mark off segments DF, FG, GH, equal in hyperbolic length to the geodesic segments A, B and C respectively. Then draw two circles in the hyperbolic plane where the first circle has hyperbolic centre located at the point F and passes through the point D and th second circle has hyperbolic centre located at the point G and passes through the point H. The condition that A be less than the sum of B and C ensures that second circle is not contained in the first circle. The condition that C be less than the sum of A and B ensures that the second circle does not contain the the first circle. Then condition that B be less than the sum of A and C ensures that the two circles are not separated. Accordingly the two circles intersect. Let K be the point of intersection. Then the sides KF, FG and GK of the geodesic triangle KFG are respectively equal in hyperbolic length to the geodesic segments A, B and C, as required. ∎

Hyperbolic Proposition 23 (Construction) *On a given geodesic AB in the hyperbolic plane, and at a point A on it, to construct a geodesic segment AF starting at the point A which makes an an angle with AB equal to a given angle.*



Proof Let the given angle be that between geodesic segments CD and CE at a point C of the hyperbolic plane. Join points D and E taken on those geodesic segments by a geodesic segment DE. Then take the point G on the geodesic ray AB so as to ensure that AG and CE are equal in hyperbolic length (Proposition 3). Then construct a geodesic triangle AFG on AG so that CD and DE are equal in hyperbolic length to CD and DE respectively (Proposition 22). It then follows, on applying the SSS Congruence Rule (Proposition 8) that the angles of the geodesic triangle AFG at vertices A, F and G are respectively equal to the angles of the geodesic triangle CDE at C, D and E respectively. Thus the geodesic segments AF and AB makes an angle with one another at A equal the given angle, which is the angle between the geodesic segments CD and CE. The required construction has therefore been achieved. ∎

Hyperbolic Proposition 24 *If two geodesic triangles ABC and DEF in the hyperbolic plane have the two sides AB and AC respectively equal in hyperbolic length to the two sides DE and DF, and if the angle CAB is greater than the angle FDE, then then the side BC of the geodesic triangle ABC is in hyperbolic length greater than the side EF of the geodesic triangle DEF.*

Proof A geodesic segment DG can be constructed so that the angles CAB and GDE are equal and the geodesic segments AC and DG are equal in hyperbolic length (Proposition 23 and Proposition 3). It then follows, on applying the SAS Congruence Rule (Proposition 4) that the geodesic segments BC and EG are equal in hyperbolic length.

Now the point F lies in the interior of the angle EDG, because the angle FDE is less than the angle CAB and thus less than GDE. There are three cases to be considered:

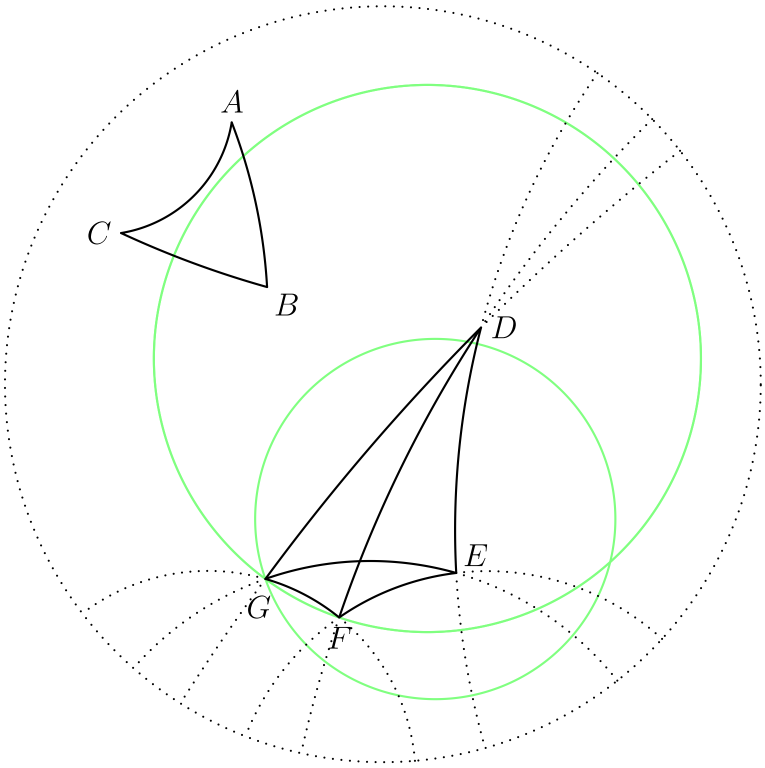
(i) this is the case when the points D and F lie on opposite sides of the geodesic that passes through the points E and G;

(i) this is the case when the point F lies on the geodesic that passes through the points E and G;

(iii) this is the case when the points D and F lie on the same side of the geodesic that passes through the points E and G.

We first prove the result in case (i). In this case the points D and F lie on opposite sides of the geodesic that passes through the points E and G, and, because the point F lies in the interior of the angle EDG, the geodesic segment DF must cross the geodesic segment EG.

The geodesic segments DG and DF are equal in hyperbolic length, because both are equal in hyperbolic length to AC. Consequently DGF is a isosceles geodesic triangle, and therefore the angles DGF and DFG opposite the equal sides are equal to one another (Proposition 5). The angle EGF is less than DGF, because the points D and F lie on opposite sides of the geodesic through G and E. Also F lies in the interior of the angle GDE and consequently the points G and E lie on opposite sides of the geodesic through D and F. It follows that The angle EFG is greater than DFG. Consequently the angle EFG is greater than the angle EGF, and therefore the side EG of the geodesic triangle EGF is in hyperbolic length greater than the side EF of that geodesic triangle. But EG and BC are equal in hyperbolic length. Consequently BC is greater in hyperbolic length than EF, as required in this case.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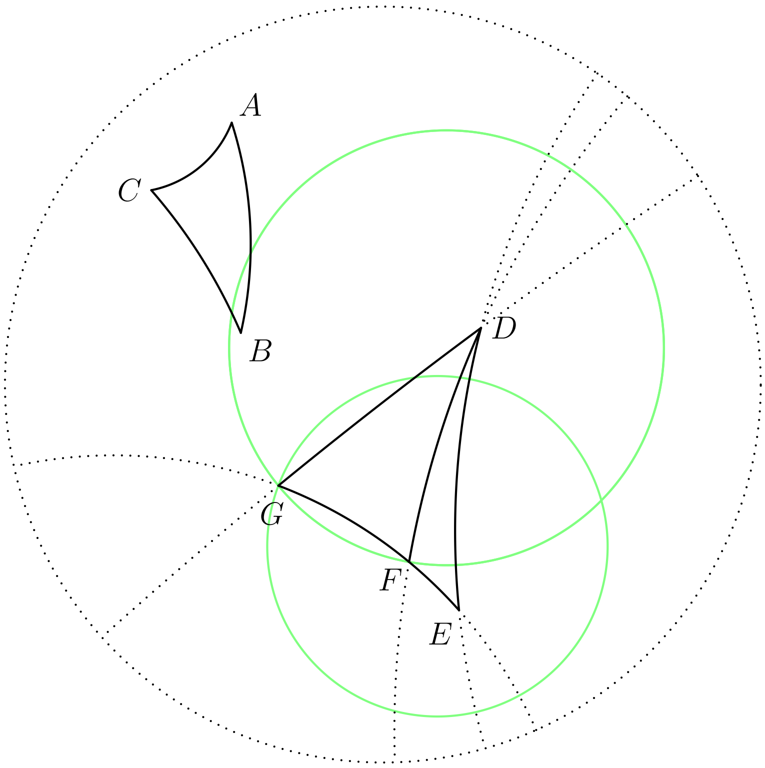
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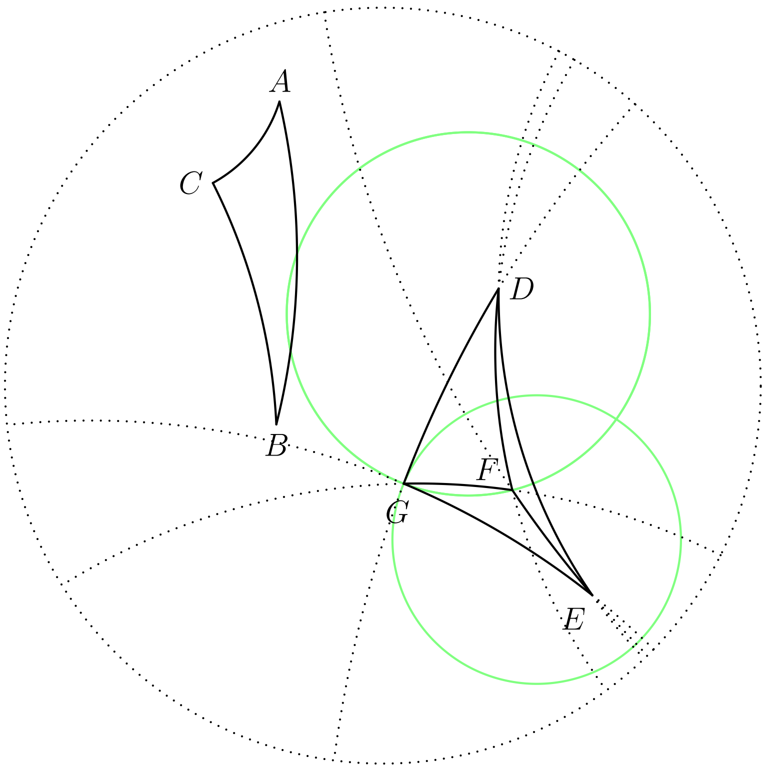
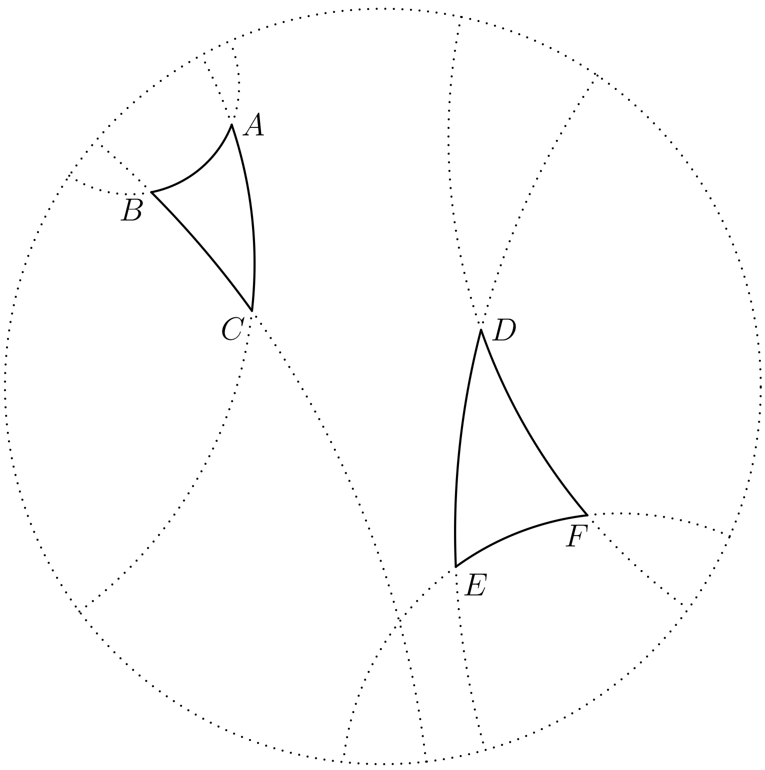
In case (ii) the point F lies in on the geodesic segment EG between E and G, and moreover the geodesic segments BC and EG are equal in hyperbolic length. Consequently the geodesic segment BC is greater than the geodesic segment EF in this case also.



In case (iii), the final case to consider, the point F lies in the interior of the geodesic triangle DGE. Consequently EF and FD are together less than EG and GD in hyperbolic length (Proposition 21). But FD and GD are equal in hyperbolic length, because both are equal to CA. Consequently EG is in hyperbolic length greater than EF. Now BC and EG are equal in hyperbolic length. It follows that BC is in hyperbolic length greater than EF in this case also. This completes the proof. ∎

Hyperbolic Proposition 25 *If two geodesic triangles ABC and DEF in the hyperbolic plane have the two sides AB and AC respectively equal in hyperbolic length to the two sides DE and DF, and if the side CB is in hyperbolic length greater than the side FE, then then the angle BAC of the geodesic triangle ABC is greater than the angle EDF of the geodesic triangle DEF.*

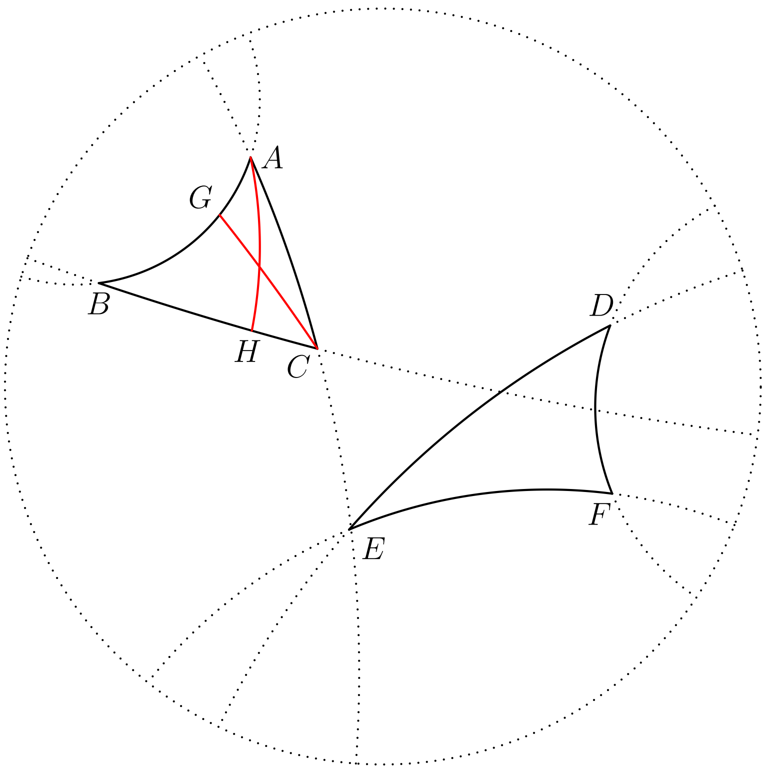


Proof If the angle BAC were less than the angle EDF, then the side BC would be in hyperbolic length less than the side EF (Proposition 24). But it is not. If the angle BAC were equal to the angle EDF then it would follow from the SAS Congruence Rule (Proposition 4) that the geodesic triangles ABC and DEF would be congruent, and therefore BC would be equal in hyperbolic length to EF. But it is not. Consquently the angle BAC must be greater than the angle EFG, as required. ∎

Hyperbolic Proposition 26 (ASA and SAA Congruence Rules) *Let ABC and DEF be geodesic triangles in the hyperbolic plane.*

> ***(ASA Congruence Rule)**. If angles ABC and DEF are equal to one another, angles ACB and DFE are equal to one another, and sides BC and EF equal to one another in hyperbolic length then the geodesic triangles ABC and DEF are congruent to one another.*
>
> ***(SAA Congruence Rule)**. If angles ABC and DEF are equal to one another, angles ACB and DFE are equal to one another, and sides AB and DE equal to one another in hyperbolic length then the geodesic triangles ABC and DEF are congruent to one another.*

Accordingly, when the hypotheses of either congruence rule are satisfied, the sides and angles of the geodesic triangle ABC are equal to the corresponding sides and angles of the geodesic triangle DEF.



Proof First suppose that the conditions of the ASA Congruence Rule are satisfied. If the sides AB and DE of the respective geodesic triangles were unequal then one would exceed the other in hyperbolic length. Suppose that AB were greater than DE in hyperbolic length. Then a point G could be found on BA so as to make the geodesic segment BG equal in length to the geodesic segment DE. Then the sides GB and BC of the geodesic triangle GBC would be equal in hyperbolic length to the respective sides of the geodesic triangle DEF, and the included angle GBC would be equal to the included angle DEF. The SAS Congruence Rule (Proposition 4) would then ensure the congruence of the geodesic triangles GBC and DEF. The angles GCB and DFE would therefore be equal to one another.

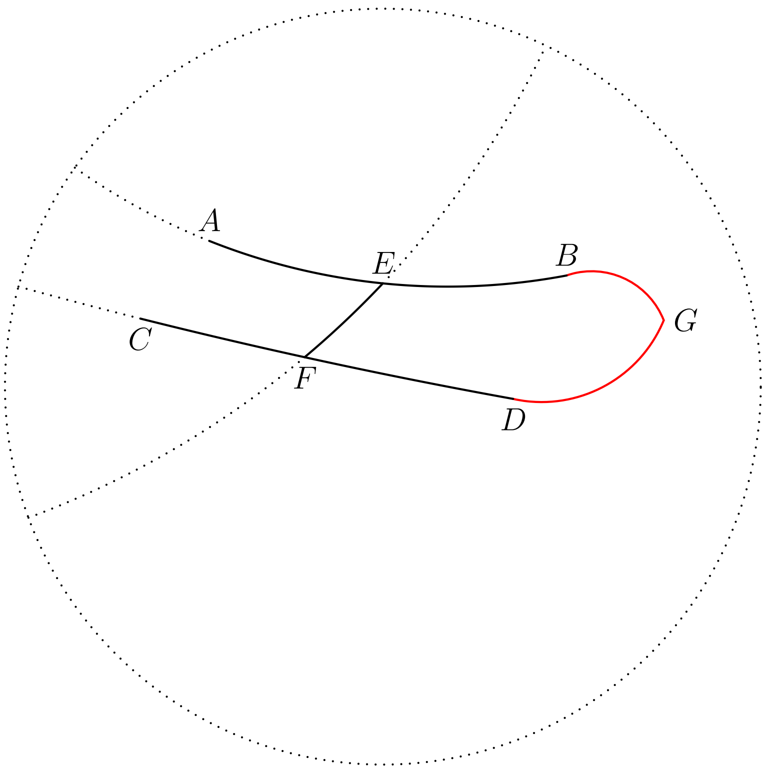
But the angle DFE is equal to the angle ACB, and therefore could not be equal to the angle GCB. We conclude therefore that the side AB cannot exceed the side DE in hyperbolic length. Nor can the side DE exceed the side AB in hyperbolic length. The sides AB and DE of the geodesic triangles ABC and DEF are therefore equal in hyperbolic length. An application of the ASA Congruence Rule (Proposition 4) now shows that the geodesic triangles ABC and DEF are congruent.

Now suppose that the conditions of the SAA Congruence Rule are satisfied. If the sides BC and EF of the respective geodesic triangles were unequal then one would exceed the other in hyperbolic length. Suppose that BC were greater than EF in hyperbolic length. Then a point H could be found on BC so as to make the geodesic segment BH equal in length to the geodesic segment EF. Then the sides BC and BH of the geodesic triangle ABH would be equal in hyperbolic length to the respective sides of the geodesic triangle DEF, and the included angle ABH would be equal to the included angle DEF. The SAS Congruence Rule (Proposition 4) would then ensure the congruence of the geodesic triangles ABH and DEF. The angles AHB and DFE would therefore be equal to one another.

But the angle DFE is equal to the angle ACB, and the exterior angle AHB of the geodesic triangle ACH is greater than the internal and opposite angle ACB. Therefore the angle AHB could not be equal to the angle DFE. We conclude therefore that the side BC cannot exceed the side EF in hyperbolic length. Nor can the side EF exceed the side BC in hyperbolic length. The sides BC and EF of the geodesic triangles ABC and DEF are therefore equal in hyperbolic length. An application of the ASA Congruence Rule (Proposition 4) now shows that the geodesic triangles ABC and DEF are congruent.

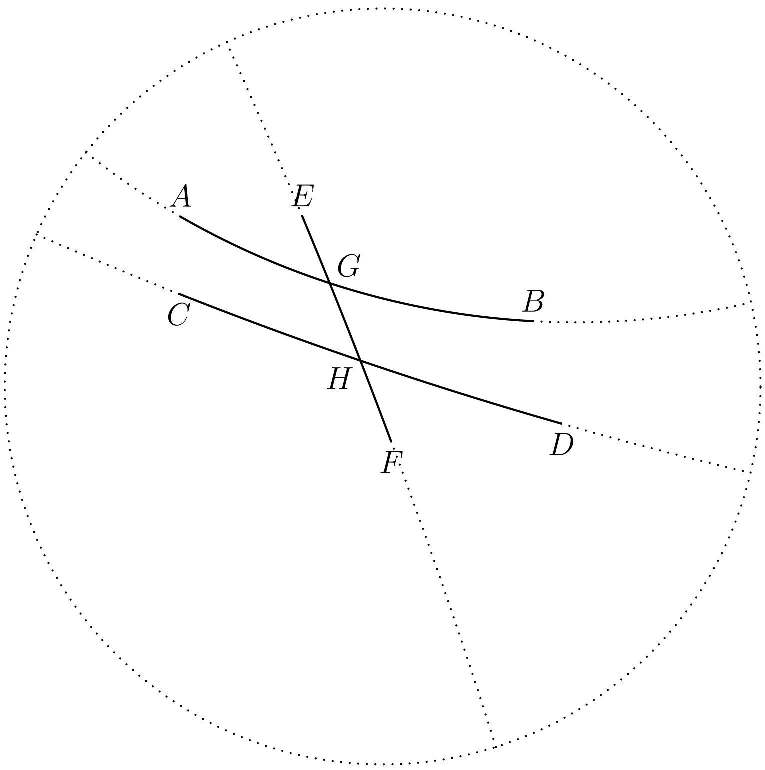
Thus under the hypotheses of the ASA Congruence Rule, or of the SAA Congruence Rule, the geodesic triangles ABC and DEF are congruent, and the sides and angles of the geodesic triangle ABC are respectively equal to the sides and angles of the geodesic triangle DEF, as required. ∎

Hyperbolic Proposition 27 *If a geodesic EF in the hyperbolic plane falling on two geodesics AB and CD make the alternative angles AEF and EFD equal to one another then the complete geodesic passing through the points A and B does not intersect the complete geodesic passing through the points C and D.*



Proof Suppose that the geodesics AB and CD could be produced beyond B and D respectively so as to intersect at a point G of the hyperbolic plane. Then, in the geodesic triangle GEF, the exterior angle AEF would be greater than the interior and opposite angle EFD (Proposition 16), contrary to hypothesis. Therefore the complete geodesic passing through the points A and B cannot intersect the complete geodesic passing through the points C and D at any point of the hyperbolic plane that lies on the same side of the geodesic EF as the point B and D. A similar argument shows that the complete geodesic passing through the points A and B cannot intersect the complete geodesic passing through the points C and D at any point of the hyperbolic plane that lies on the opposite side of the geodesic EF as the point B and D. The result follows. ∎

Hyperbolic Proposition 28 *If a geodesic EF in the hyperbolic plane falling on two geodesics AB and CD make the exterior angle EGB equal to the interior and opposite angle GHD, or make the sum of the interior angles BGH, GHD equal to two right angles, then the complete geodesic passing through the points A and B does not intersect the complete geodesic passing through the points C and D.*



Proof Suppose EGB and GHD are equal. Now the angles EGB and AGH are equal (Proposition 15). It follows that the alternate angles AGH and GHD are equal, and therefore the complete geodesic passing through the points A and B does not intersect the complete geodesic passing through the points C and D (Proposition 27). Similarly if the angles BGH and GHD sum to two right angles then the alternating angles AGH and GGD are equal, because the angles AGH and BGH also sum to two right angles, and consequently the complete geodesic passing through the points A and B does not intersect the complete geodesic passing through the points C and D. ∎