The First Twenty-Eight Propositions of Euclid (Euclidean and Non-Euclidean Geometry)

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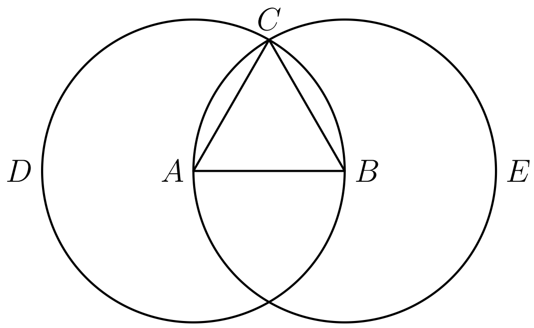
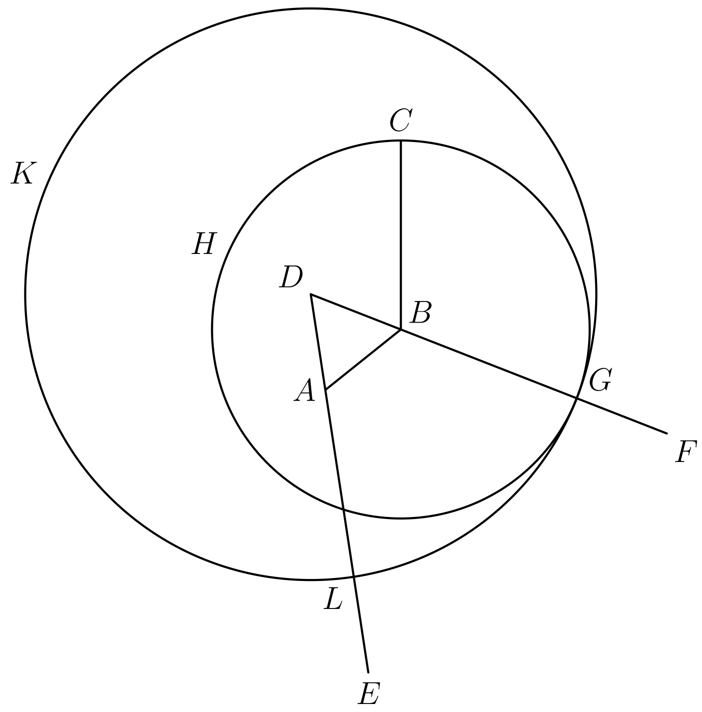
Contents

Euclidean Proposition 1 (Construction) *On a straight line segment AB in the Euclidean plane, to construct an equilateral triangle ABC.*



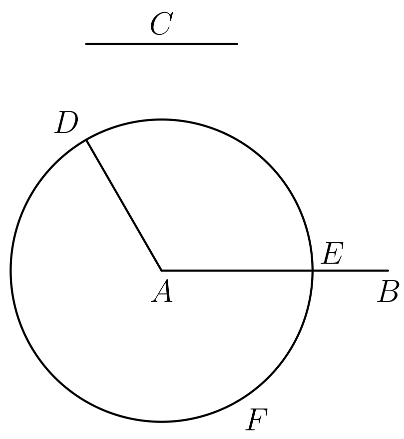
Construction To construct the equilateral triangle, first draw a circle in the Euclidean plane passing though the point B with centre at the point A. Also draw a circle in the Euclidean plane passing through the point A with centre at the point B. Now the second circle passes though the centre of the first circle, but also passes through points that lie outside the first circle. Consequently there are points at which those two circles intersect. Let C be a point at which the two circles intersect, and construct the triangle with vertices at A, B and C. Then the straight line segments AB and AC are equal in length because the points B and C lie on a circle whose centre is located at the point A. Similarly the straight line segments AB and BC are equal in length, because the points A and C lie on a circle whose centre is located at the point B. Consequently the triangle ABC is the required equilateral triangle. ∎

Euclidean Proposition 2 (Construction) *To place at a point A of the Euclidean plane a straight line segment equal in length to a given straight line segment BC.*



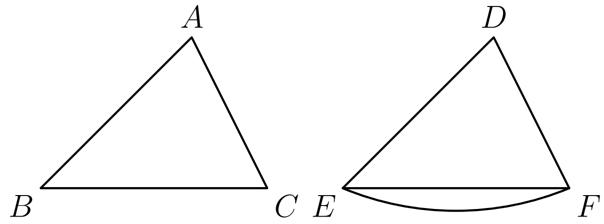
Construction First construct an equilateral triangle ABD on the straight line segment AB (Proposition 1). Then draw a circle in the Euclidean plane passing though the point C whose centre is located at the point B. Let the straight line segment DB be produced beyond B to intersect that circle just constructed at the point G. Then draw a circle in the Euclidean plane passing through the point G whose centre is located at the point D. Then let the straight line segment DA be produced beyond A to meet this new circle at the point L. Then the straight line segments DG and DL are equal with respect to length. Also the parts DB and DA of those straight line segments are equal with respect to length. Consequently the remainders BG and AL are equal with respect to length. But the straight line segment BC is also equal in length to the straight line segment BG, because the points C and G are located on a circle in the Euclidean plane whose centre is located at the point B. Consequently the straight line segments BC, BG and AL are all equal to one another with respect to length, and thus AL is the required straight line segment. ∎

Euclidean Proposition 3 (Construction) *Given two unequal straight line segments AB and C in the Euclidean plane, to cut off from the longer straight line segment AB a straight line segment equal in length to the shorter straight line segment C.*



Construction First construct a straight line segment AD, with one endpoint at the point A, so that AD is equal to C with respect to length. Then draw a circle through the point D whose centre is located at the point A, and let that circle intersect the straight line segment AB at the point E. Then AE is the required straight line segment cut off from the segment AB. ∎

Euclidean Proposition 4 (SAS Congruence Rule) *If two triangles ABC and DEF in the Euclidean plane have the two sides AB, AC equal in length to the two sides DE, DF respectively, and have the contained angles BAC and EDF equal to one another, then the triangles ABC and DEF are congruent, and thus the sides BC and EF are equal to one another in length, and the angles of the triangle ABC at B and C are equal to the angles of the triangle DEF at E and F respectively.*

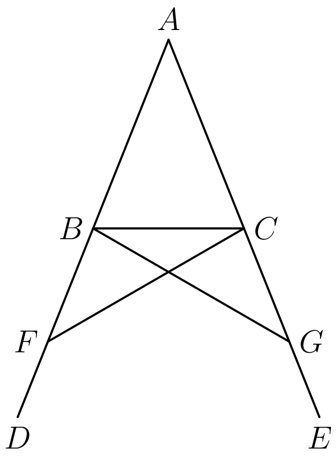


Proof The sides AB and DE of the respective triangles ABC and DEF are equal in length. Consequently there exists a Euclidean motion φ of the Euclidean plane, preserving both the Euclidean distance between pairs of points and the angles between straight line segments at the points at which they intersect, where this Euclidean motion φ maps the points A and B of the Euclidean plane onto the points D and E of that plane, and also maps any point of the Euclidean plane that lies on the same side of the straight line AB as the point C to some point that lies on the same side of the straight line DE as the point F. The equality of the angles BAC and EDF then ensures that points on the straight ray starting from the point A and passing through the point C are mapped to points of the straight ray starting from the point D and passing through the point F. But the straight line segments AC and DF are also equal in length. It follows that the Euclidean motion φ must map the point C to the point F. Thus

$$\varphi(A) = D, \quad \varphi(B) = E \quad \text{and} \quad \varphi(C) = F.$$

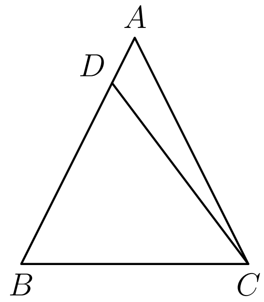
Now the distance-preserving property of the Euclidean motion φ ensures that BA, AC and BC are equal to ED, DF and EF respectively with regard to length. Also the angle-preserving property of the Euclidean motion φ ensures that the angles BAC, ABC and BCA are equal to angles EDF, DEF and EFD respectively. The result follows. ∎

Euclidean Proposition 5 (Isosceles Geodesic Triangles) *Let ABC be an isosceles triangle in the Euclidean plane, and let the equal sides AB and AC be produced to points D and E. Then the angles CBD and BCE under the base BC are equal to one another, as are the angles ABC and ACB of the isosceles triangle ABC at the endpoints B and C of the base.*



Proof Let points F and G be constructed on the straight lines AD and AE so that B lies between A and F, C lies between A and G and BF is equal to CG wih respect to length (Proposition 3). Now the straight line segments AB and AC are equal in length. Consequently the straight line segments AF and AC are equal in length to the finite straight lines AG and AB respectively. It therefore follows from the SAS Congruence Rule (Proposition 4) that the triangles AFC and AGB are congruent, and consequently the sides FC and GB are equal to one another in length. Moreover the angles BFC and CGB, being identical to the angles AFC and AGB respectively, are equal to one another. Consequently the triangles BFC and CGB are congruent. It then follows that the angles CBD and BCE, being identical to the angles CBF and BCG respectively, are equal to one another. Thus the angles under the base of the isosceles triangle ABC are equal to one another. Now the congruence of the triangles BFC and CGB also ensures that the angles BCF and CBG are equal to one another. We previously showed that the triangles AFC and AGB are congruent, from which it follows that the angles ACF and ABG are equal to one another. Subtracting the equal angles BCF and CBG from the equal angles ACF and ABG, we conclude that the angles ABC and ACB are equal to one another. Thus the angles of the isosceles triangle ABC at the endpoints B and C of the base are equal to one another, as required. ∎

Euclidean Proposition 6 (Converse of Proposition 5) *If in a triangle ABC in the Euclidean plane two angles B and C be equal to one another, the sides AC and AB which subtend the equal angles will also be equal to one another.*



Proof Suppose that the two sides AC and AB were not equal to one another with respect to length. Then one would be longer than the other. Suppose therefore that the side AB were longer than the side AC. We would then be able to cut off from AB a straight line segment BD equal in length to the side AC of the triangle. Applying the SAS Congruence Rule (Proposition 4), we would then conclude that the triangles ABC and DCB would be congruent, and therefore the angles ABC and ACD would be equal to one another. But the angles ABC and ACB are equal to one another. Thus, under the assumption that the straight line segment BD is equal to AC, we would have to conclude that the angles ACB and DCB would be equal to one another. But this is impossible since DCB would also be a proper part of the angle ACB. Consequently neither of the sides AB and AC can be longer than the other, and therefore these two sides are equal in length, as claimed. ∎

Euclidean Proposition 7 *Given a straight line segment AB in the Euclidean plane, there cannot be constructed two distinct triangles ABC and ABD on the same side of the straight line AB with the properties that the sides AC and AD are equal to one another in length and the sides BC and BD are also equal to one another in length.*

Proof Let C be a point of the Euclidean plane that does not lie on the complete straight line that passes through the points A and B, where A and B are distinct. Now it is not possible to find any point D distinct from the point$C$ but lying on the straight ray starting at the point A and passing through the point C so as to make the straight line segments AC and AD equal to one another in length and also make the straight line segments BC and BD equal in length. Similarly it is not possible to find any point D distinct from the point$C$ but lying on the straight ray starting at the point A and passing through the point C so as to make the straight line segments AC and AD equal to one another in length and also make the straight line segments BC and BD equal in length. We may therefore restrict our attention to cases in which the point D does not lie on the straight rays starting at the points A and B that pass through the point C.

Now the removal of these two straight rays divides the side of the Euclidean plane to which the point C belongs into four regions. A complete investigation of all relevant cases therefore needs to consider the following four cases:—

Case (i): this is the case where the point D lies inside the angle BAC but outside the angle ABC;

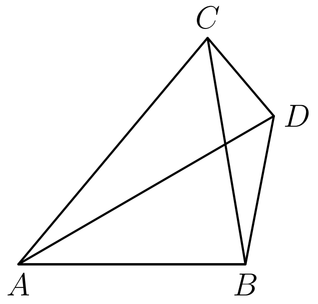
Case (ii): this is the case where the point D lies inside the angle BAC and also inside the angle ABC;

Case (iii): this is the case where the point D lies outside the angle BAC but inside the angle ABC;

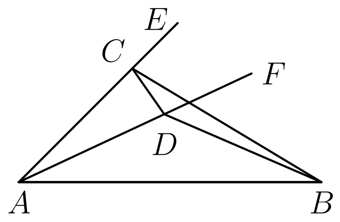
Case (iv): this is the case where the point D lies outside the angle BAC and also outside the angle ABC.

Now it should be noted that the point D lies outside the angle BAC if and only if the point C lies inside the angle BAD, and the point D lies inside the angle BAC if and only if the point C lies outside the angle BAD. Consequently the four cases to consider are the following:

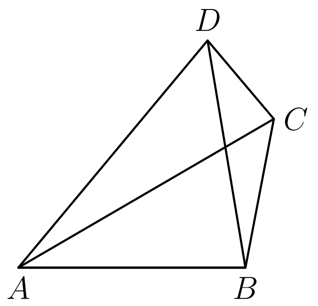
Case (i): this is the case where the point D lies inside the angle BAC but outside the angle ABC;



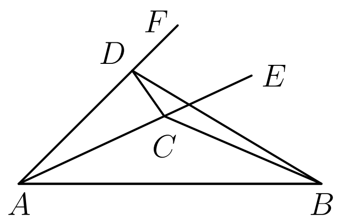
Case (i)



Case (ii)



Case (iii)



Case (iv)

Case (ii): this is the case where the point D lies inside the angle BAC and also inside the angle ABC;

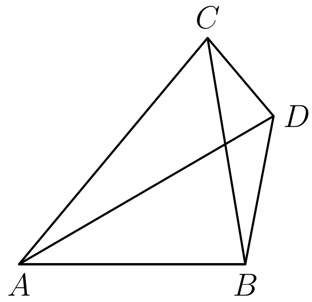
Case (iii): this is the case where the point C lies inside the angle BAD but outside the angle ABD;

Case (iv): this is the case where the point C lies inside the angle BAD and also inside the angle ABD.

A comparison of these characterizations of the cases shows that the result in cases (iii) and (iv) follows from the corresponding result in cases (i) and (ii) on interchanging the roles of the points C and D. Thus to prove the proposition in full generality, it only remains to prove the result in cases (i) and (ii).

Accordingly we first prove the result in case (i). Accordingly suppose that the straight line segments AC and AD are equal in length and that the point D lies in the angle BAC but outside the triangle ABC. This is the configuration depicted in the following figure:

Join the points C and D by a straight line segment. Then ACD is an isosceles triangle with equal sides AC and AD, and consequently the angles ACD and ADC at the base of the triangle are equal. Now the points A

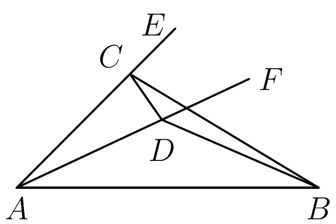


and D lie on the opposite sides of the straight line that passes through the points B and C, and consequently the point B lies in the interior of the angle ACD. The angle BCD between the straight line segments CB and CD at the point C is therefore less than the angle ACD between the straight line segments CA and CD at that point.

Also, considering angles at the point D, it can be seen that the points A and C lie on the same side of the straight line passing through D and B, and the points A and B lie on the same side of the straight line passing through D and C. Consequently the point A lies in the interior of the angle BDC between the straight line segments DB and DC, and therefore the angle ADC is less than BCD.

We have now demonstrated the following: the angle BCD is less than the angle ACD; the angle ACD is equal to the angle ADC; the angle ADC is less than the angle BDC. It follows that the angle BCD is less than the angle BDC. Consequently the triangle BCD is not an isosceles triangle with equal sides BC and BD, for it it were, we would have arrived at a result contradicting Proposition 5. We have thus shown that, in case (i), if the straight line segments AC and AD are equal to one another in length, then the straight line segments BC and BD must by unequal in length.

We now turn our attention to case (ii). In this case the configuration is as depicted in the following diagram:



In this case the point D lies in the interior of the triangle ABC.

We produce the straight line segements AC and AD to points E and F of the Euclidean plane, so that the points E and F are joined to the point A

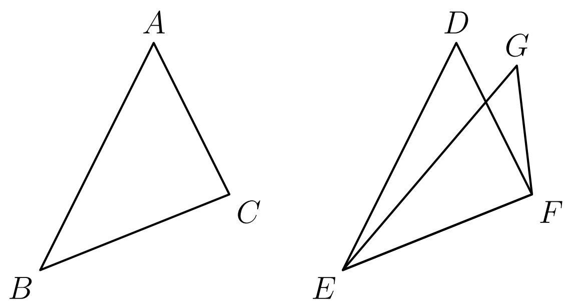
by straight lines passing through the points C and D respectively.

Suppose that the straight line segments AC and AD are equal in length. It then follows from Proposition 5 that the angles ECD and FDC are equal, because they are the angles under the base of an isosceles triangle. Now, in this case, the points D and E of the Euclidean plane lie on opposite sides of the straight line passing through the points B and C and therefore the angle ECD between the finite straight lines CE and CD is greater than the angle ECB between the straight line segments CE and CB. Also the point F lies in the interior of the angle CDB between the straight line segments AD and DB, and therefore the angle CDF between the straight line segments DC and DF is less than the angle CDB between the straight line segments DC and DB.

We have now demonstrated the following: the angle BCD is less than the angle ECD; the angle ECD is equal to the angle FDC; the angle FDC is less than the angle BDC. It follows that the angle BCD is less than the angle BDC. Consequently the triangle BCD is not an isosceles triangle with equal sides BC and BD, for it it were, we would have arrived at a result contradicting Proposition 5. We have thus shown that, in case (ii), if the straight line segments AC and AD are equal to one another in length, then the straight line segments BC and BD must by unequal in length.

As explained previously, the required result in cases (iii) and (iv) follows from the results proved in cases (i) and (ii) on interchanging the roles of the points C and D. Consequently the result of Proposition 7 has been proved in full generality, as required. ∎

Euclidean Proposition 8 (SSS Congruence Rule) *If, in triangles ABC and DEF in the Euclidean plane, the sides AB, BC, CA of the triangle ABC are respectively equal to the sides DE, EF and FD in length, then the triangles are congruent, and consequently the angles between the sides of of the triangle ABC at the vertices A, B and C are respectively equal to the angles between the sides of the triangle DEF at the vertices D, E and F.*

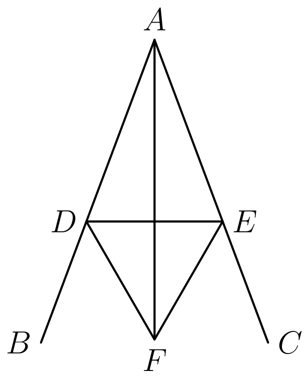


Proof The straight line segments BC and EF are equal in length, and therefore there exists a Euclidean motion φ of the Euclidean plane that maps the points B and C onto the points E and F respectively, where this Euclidean motion φ preserves both the lengths of straight line segments and that angles between straight line segments at their points of intersection. Moreover the Euclidean motion φ may be chosen so that it maps the point A onto a point G of the Euclidean plane that lies on the same side of the straight line through the points E and F as the point D. The sides GE, EF and FG of the triangle GEF are respectively equal to the sides AB, BC and CA of the triangle ABC, because

$$\varphi(A) = G, \quad \varphi(B) = E \quad \text{and} \quad \varphi(C) = F.$$

But the sides AB, BC and CA of the triangle ABC are respectively equal in length to the sides DE, EF, FD of the triangle DEF. Consequently the sides GE, EF and FG of the triangle GEF are respectively equal in length to the sides DE, DF and FA of the tringle DEF. It now follows immediately from Proposition 7 that the points D and G coincide. Consequently the triangles ABC and DEF are congruent, as required. ∎

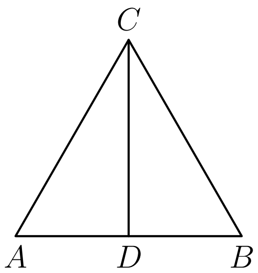
Euclidean Proposition 9 (Construction) *To bisect the angle between two straight lines in the Euclidean plane at a point at which they intersect.*



Construction We seek to bisect the angle between two straight line segments AB and AC that intersect at a point A of the Euclidean plane. Take points D and E on the straight line segments AB and AC respectively, chosen so that AD and AE are equal in length. (The possibility of finding such points D and E is guaranteed by Proposition 3, and can be achieved in practice by choosing a point D and E to be the points where a circle of sufficiently small radius in the Euclidean plane with centre A intersects the straight line segments AB and AC.) Then construct the equilateral triangle DEF on the straight line segment DE so that the points A and F of the Euclidean plane lie on opposite sides of the straight line that passes through the points D and F. Then join the points A and F be a straight line segment. It can be shown that this straight line segment bisects the angle BAC.

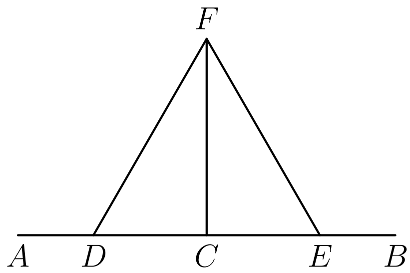
To prove this, note that the sides AD, DF and AF of the triangle ADF are equal in length to the sides AE, EF and AF respectively of the sides of the triangle AEF. Applying the SSS Congruence Rule (Proposition 8), we conclude that those two triangles are congruent to one another, and therefore the angles DAF and EAF of those triangles at the vertex A are equal to one another. The angle BAC between the straight line segments AB and AC is thus bisected by the straight line segment AF. ∎

Euclidean Proposition 10 (Construction) *To bisect a straight line segment in the Euclidean plane.*



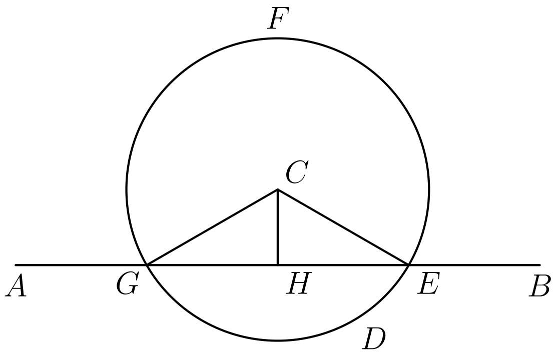
Construction Let A and B be points in the Euclidean plane. It is required to bisect the straight line segment AB by locating a point D on that straight line segment for which the straight line segments AD and DB are equal to one another in length. To achieve this, construct an equilateral triangle ABC on the straight line segment AB (Proposition 1), and bisect the angle ACB between the sides CA and CB of this triangle by a straight ray which intersects the straight line segment AB at the point D (Proposition 9). The sides AC and CD of the triangle ACD are respectively equal in length to the sides BC and CD of the triangle BCD, and the included angles ACD and BCD are equal. It follows on applying the SAS Congruence Rule (Proposition 4) that the triangles ACD and BCD are congruent. Consequently the sides AD and BD of those triangles are equal in length, and thus the straight line segment AB has been bisected at the point D. ∎

Euclidean Proposition 11 (Construction) *At a point C on a straight line in the Euclidean plane that passes through distinct points A and B, to draw a straight line segment CF that intersects the straight line AB at right angles at the point C.*



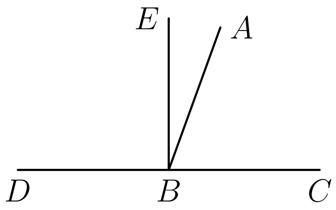
Construction Take points D and E on the straight line AB, lying on either side of the chosen point C so that the straight line segments DC and CE are equal in length. (This can be achieved by letting D and E be the points at which a circle with centre C intersects the straight line AB.) Then construct an equilateral triangle DFE on the straight line segment DE (Proposition 1), and join the points C and F by a straight line segment. Now the sides DF, FC and DC of the triangle DFC are respectively equal in length to the sides EF, FC and EC of the triangle EFC. It follows on applying the SSS Congruence Rule (Proposition 8) that the angles DCF and ECF are equal to one another. Consequently the straight line segment CF meets the straight line AB at right angles at the chosen point C of the straight line C, and thus the requirements of the construction have been achieved. ∎

Euclidean Proposition 12 (Construction) *To draw a straight line segment from a given point C of the Euclidean plane to a point H lying on a given straight line AB in the Euclidean plane that does not pass though the given point C, where the straight line segment CH intersects the given straight line AB at right angles at the point H.*



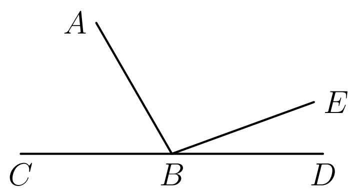
Construction Take a point D in the Euclidean plane so that the points C and D lie on opposite sides of the straight line AB. Then the circle in the Euclidean plane with centre C intersects the straight line AB at two points G and E. Let the straight line segment GE be bisected at the point H (Proposition 10), and join the points G and E and H to the point C by straight line segments. Now the sides GC, CH are GH of the triangle GCH are respectively equal in length to the sides EC, CH and HE of the triangle ECH. Applying the SSS Congruence Rule (Proposition 8), it follows that the triangles GCH and ECH are congruent, and consequently the angles GHC and EHC are equal to one another. Thus the straight line segment CH meets the straight line AB at right angles at the point H, as required. ■

Euclidean Proposition 13 (Supplementary Angles) *If a straight line segment BA be taken intersecting a straight line DC at a point B between C and D, then the sum of the angle ABC with its supplementary angle ABD is equal to two right angles.*



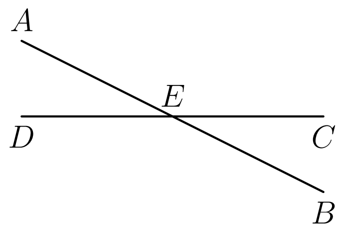
Proof Let the straight line segment BE be taken with an endpoint at the point B so as to intersect the straight line CD at right angles at the point B, ensuring that the points A and E lie on the same side of the straight line CD. Suppose that the point A lies in the interior of the angle CBE. Then the sum of the two angles DBA and ABC is equal to the sum of the three angles DBE, EBA and ABC, and is thus equal to sum of the two right angles DBE and EBC. A similar argument applies when the point A lies in the interior of the angle DBE, and the result is immediate when the point A lies on the straight ray AE. Thus the required result can be established in all relevant cases. ∎

Euclidean Proposition 14 (Adjacent angles summing to two right angles) *If straight line segments DB and BC in the Euclidean plane make angles at the point B with a straight line segment AB that sum to two right angles, where the points C and D lie on opposite sides of AB, then some straight line passes though the three points D, B and C.*



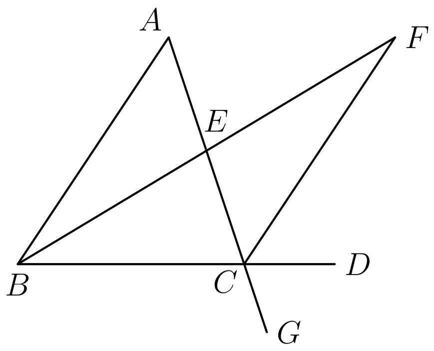
Proof Suppose that the straight line segments DB and BC were not both parts of a single straight line in the Euclidean plane passing through the points C and D. Then the straight line segment DB could be produced to a point E that does not lie on the straight line passing through the points B and C. Suppose that the point E were located on the same side of the straight line segment BC as the point A. Then the angle ABE would be less than the angle ABC. But the angles DBA and ABE must sum to two right angles. Consequently the angles DBA and ABC would sum to more than two right angles, contradicting the conditions of the proposition. A similar argument shows that the points A and E cannot lie on opposite sides of the straight line BC. Consequently the straight line segments DB and BC must be parts of a single straight line that passes though the points D, B and C, as required. ∎

Euclidean Proposition 15 (Vertically-opposite angles) *If straight lines AB and CD intersect at some point E then the vertically opposite angles AED and BEC are equal to one another, as are the vertically opposite angles CEA and DEB.*



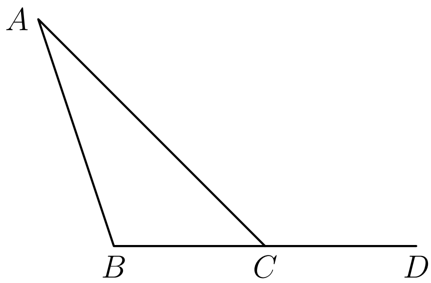
Proof If the angle CAE is added to either of the angles AED or BEC, then the sum of the relevant angles is equal to two right angles. But where the same angle is subtracted from equal angle sums, the remaining angles or angle sums are equal. Consequently the angles EAD and BEC are equal to one another. Similarly the angles CEA and DEB are equal to one another, as required. ∎

Euclidean Proposition 16 (Exterior angle greater than interior and opposite angles) *Let a side BC of a triangle ABC in the Euclidean plane be produced past B to a point C. Then the external angle ACD of the triangle at C is greater than the internal and opposite angles of the triangle ABC at the vertices A and B.*



Proof Bisect the side AC of the triangle ABC at E, and produce the straight line segment BE past E to a point F so that the segments BE and EF of the straight line BF are equal in length. Then the straight line segments EA and EB are respectively equal in length to the finite straight lines EC and EF, and moreover the included angles AEB and CEF are vertically-opposite angles, and are therefore equal to one another (Proposition 15). Applying the SAS Congruence Rule (Proposition 4), we see that the triangles EAB and ECF are congruent, and therefore the angle ECF is equal to the angle EAB. But the points A and F all lie on the same side of the straight line in the Euclidean plane that passes through the points B, C and D. Consequently the angle ECF is less than the angle ECD. It follows that the internal angle CAB of the triangle ABC at A is less than the external angle ACD of the triangle ABC at C. Similarly the angle ABC of the given triangle at the point B is less than the external angle BCG of that triangle. But the external angles ACD and BCG of the triangle ABC at the vertex C are equal to one another, because they are vertically-opposite angles (Proposition 15). Thus the internal angles of the triangle ABC are the vertices A and B are both less than the external angles of that triangle at the vertex C. The result follows. ∎

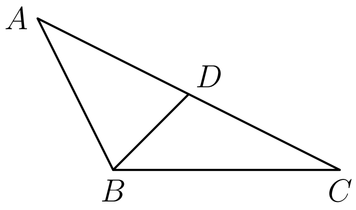
Euclidean Proposition 17 *In any triangle in the Euclidean plane, two angles taken together in any manner are less than two right angles.*



Proof Let ABC be a triangle in the Euclidean plane. We must show that two angles of the triangle ABC taken together are less than two right angles.

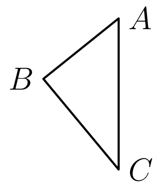
To show this let the straight line BC be produced beyond C to D, ensuring that BD is a straight line. Then the interior angle ABC of the triangle at the vertex B is less than the exterior angle ACD of that triangle at C. It follows, on adding the angle ACB to each of ABC and ACD, that the sum of the angles ABC and ACB is less than the sum of the angles ACD and ACB, and is therefore less than two right angles (Proposition 13). Consequently the sum of the interior angles ABC and ACB of the triangle ABC at vertices B and C is less than two right angles. Similarly the sum of the interior angles of the triangle at vertices A and B is less than two right angles, as is the sum of the interior angles of that triangle at A and C. This completes the proof. ∎

Euclidean Proposition 18 *In a triangle ABC in the Euclidean plane, if the side AC is greater in length than the side AB, then the angle ABC that is subtended by the greater side is greater than the angle ACB that is subtended by the lesser side.*



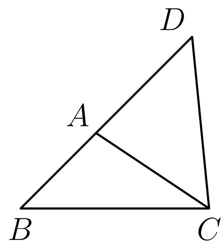
Proof Let a straight line segment AD be cut off from the greater side AC so that AD is equal in length to the lesser side AB (Proposition 3), and let the points B and D be joined by a straight line segment. Then ABD is an isoceles triangle, and therefore the angles ABD and ADB are equal to one another (Proposition 5). Now ADB is an external angle of the triangle BCD. It follows that the angle ADB is greater than the internal angle BCD of that triangle at the vertex C (Proposition 16). Morever the angles BCD and BCA are identical. Also the angle ABD is less than the angle ABC. Consequently the angle ACB, being less than ADB, and thus less than ABD, must be less than ABC, as required. ∎

Euclidean Proposition 19 *In a triangle ABC in the Euclidean plane, if the angle ABC is greater than the angle ACB, then the side AC that subtends the greater angle is greater in length than the side AB that subtends the lesser angle.*



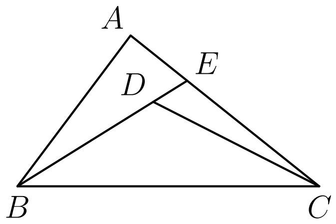
Proof Suppose that, in the triangle ABC, the angle ABC is greater than the side ACB. If the side AC were equal in length to the side AB then the angle ABC would be equal to the angle ACB (Proposition 5). But it is not. If the side AC were less than the side AB in length then the angle ABC would be less than ACB (Proposition 18). But it is not. Therefore the side AC must be greater in length than the side AB, as claimed. ∎

Euclidean Proposition 20 *In a triangle in the Euclidean plane, two sides taken together in any manner are greater in length than the remaining one.*



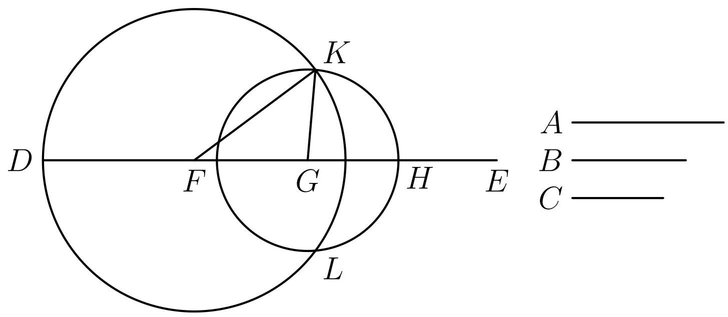
Proof Let ABC be a triangle in the Euclidean plane. Produce the side BA of the triangle beyond A to D so as to ensure that the part AD of the straight line BD is equal in length to the side AC of the triangle (Proposition 3). Then ACD is an isosceles triangle in which the sides AC and AD are equal in length. It then follows that the angles ACD and ADC are equal (Proposition 5). Consequently the angle BCD is greater than the angle BDC, and therefore the straight line segment BD is greater in length than the straight line segment BC. But the sides AB and AC of the triangle ABC taken together are equal to BD. Consequently the sides AB and AC taken together are greater in length than the side AB. Similarly any other two sides of the triangle taken together are greater than the remaining side. ∎

Euclidean Proposition 21 *If ABC is a triangle in the Euclidean plane, and if D is a point in the interior of the triangle, then the sum of the sides BD and DC of the triangle DBC in length less than the sum of the sides BA and AC of the triangle ABC, and the angle BDC is greater than the angle BAC.*



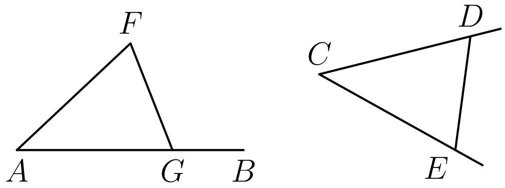
Proof The side DC of the triangle DEC is in length less than the sum of the sides DE and EC. Conseqently the sum of BD and DC is in length less than the sum of BE and EC, which in turn, and for similar reasons, is in length less than the sum of BA and AC. Also the angle BDC, being an exterior angle of the triangle EDC, is greater than the interior angle DEC of that triangle. But DEC, being an exterior angle of the triangle BAE, is greater than the interior angle BAE of that triangle. Consequently the angle BDC is greater than the angle BAC, as required. ∎

Euclidean Proposition 22 (Construction) *Given straight line segments A, B and C in the Euclidean plane, where the sum of any two of these straight line segments is greater in length than the remaining one, to construct a triangle FGK whose sides KF, FG, GK are respectively equal in length to A, B and C.*



Construction On a straight ray DE in the Euclidean plane starting at the point D, mark off segments DF, FG, GH, equal in length to the straight line segments A, B and C respectively. Then draw two circles in the Euclidean plane where the first circle has centre located at the point F and passes through the point D and the second circle has centre located at the point G and passes through the point H. The condition that A be less than the sum of B and C ensures that second circle is not contained in the first circle. The condition that C be less than the sum of A and B ensures that the second circle does not contain the the first circle. Then condition that B be less than the sum of A and C ensures that the two circles are not separated. Accordingly the two circles intersect. Let K be the point of intersection. Then the sides KF, FG and GK of the triangle KFG are respectively equal in length to the straight line segments A, B and C, as required. ∎

Euclidean Proposition 23 (Construction) *On a given straight line AB in the Euclidean plane, and at a point A on it, to construct a straight line segment AF starting at the point A which makes an an angle with AB equal to a given angle.*



Proof Let the given angle be that between straight line segments CD and CE at a point C of the Euclidean plane. Join points D and E taken on those straight line segments by a straight line segment DE. Then take the point G on the straight ray AB so as to ensure that AG and CE are equal in length (Proposition 3). Then construct a triangle AFG on AG so that CD and DE are equal in length to CD and DE respectively (Proposition 22). It then follows, on applying the SSS Congruence Rule (Proposition 8) that the angles of the triangle AFG at vertices A, F and G are respectively equal to the angles of the triangle CDE at C, D and E respectively. Thus the straight line segments AF and AB makes an angle with one another at A equal the given angle, which is the angle between the straight line segments CD and CE. The required construction has therefore been achieved. ∎

Euclidean Proposition 24 *If two triangles ABC and DEF in the Euclidean plane have the two sides AB and AC respectively equal in length to the two sides DE and DF, and if the angle CAB is greater than the angle FDE, then then the side BC of the triangle ABC is in length greater than the side EF of the triangle DEF.*

Proof A straight line segment DG can be constructed so that the angles CAB and GDE are equal and the straight line segments AC and DG are equal in length (Proposition 23 and Proposition 3). It then follows, on applying the SAS Congruence Rule (Proposition 4) that the straight line segments BC and EG are equal in length.

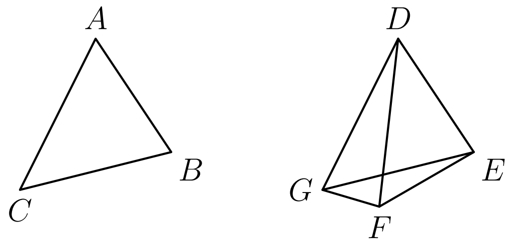
Now the point F lies in the interior of the angle EDG, because the angle FDE is less than the angle CAB and thus less than GDE. There are three cases to be considered:

(i) this is the case when the points D and F lie on opposite sides of the straight line that passes through the points E and G;

(i) this is the case when the point F lies on the straight line that passes through the points E and G;

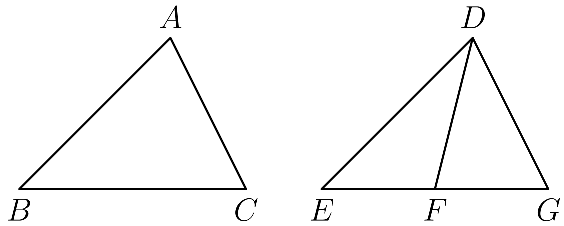
(iii) this is the case when the points D and F lie on the same side of the straight line that passes through the points E and G.

We first prove the result in case (i). In this case the points D and F lie on opposite sides of the straight line that passes through the points E and G, and, because the point F lies in the interior of the angle EDG, the straight line segment DF must cross the straight line segment EG.

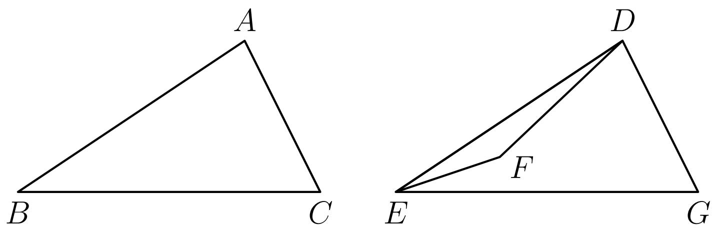


The straight line segments DG and DF are equal in length, because both are equal in length to AC. Consequently DGF is a isosceles triangle, and therefore the angles DGF and DFG opposite the equal sides are equal to one another (Proposition 5). The angle EGF is less than DGF, because the points D and F lie on opposite sides of the straight line through G and E. Also F lies in the interior of the angle GDE and consequently the points G

and E lie on opposite sides of the straight line through D and F. It follows that The angle EFG is greater than DFG. Consequently the angle EFG is greater than the angle EGF, and therefore the side EG of the triangle EGF is in length greater than the side EF of that triangle. But EG and BC are equal in length. Consequently BC is greater in length than EF, as required in this case.

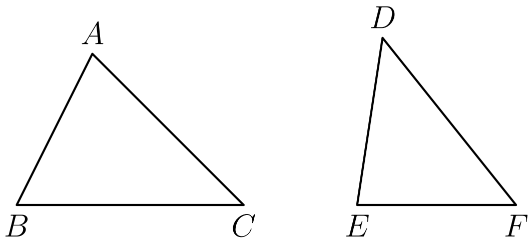


In case (ii) the point F lies in on the straight line segment EG between E and G, and moreover the straight line segments BC and EG are equal in length. Consequently the straight line segment BC is greater than the straight line segment EF in this case also.



In case (iii), the final case to consider, the point F lies in the interior of the triangle DGE. Consequently EF and FD are together less than EG and GD in length (Proposition 21). But FD and GD are equal in length, because both are equal to CA. Consequently EG is in length greater than EF. Now BC and EG are equal in length. It follows that BC is in length greater than EF in this case also. This completes the proof. ∎

Euclidean Proposition 25 *If two triangles ABC and DEF in the Euclidean plane have the two sides AB and AC respectively equal in length to the two sides DE and DF, and if the side CB is in length greater than the side FE, then then the angle BAC of the triangle ABC is greater than the angle EDF of the triangle DEF.*

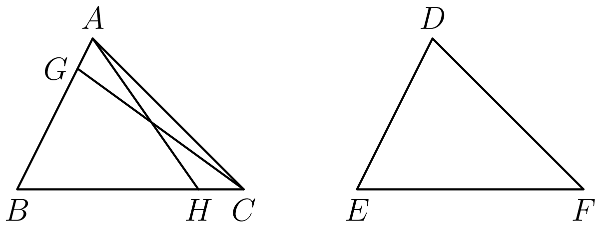


Proof If the angle BAC were less than the angle EDF, then the side BC would be in length less than the side EF (Proposition 24). But it is not. If the angle BAC were equal to the angle EDF then it would follow from the SAS Congruence Rule (Proposition 4) that the triangles ABC and DEF would be congruent, and therefore BC would be equal in length to EF. But it is not. Consquently the angle BAC must be greater than the angle EFG, as required. ∎

Euclidean Proposition 26 (ASA and SAA Congruence Rules) *Let ABC and DEF be triangles in the Euclidean plane.*

> ***(ASA Congruence Rule)**. If angles ABC and DEF are equal to one another, angles ACB and DFE are equal to one another, and sides BC and EF equal to one another in length then the triangles ABC and DEF are congruent to one another.*
> ***(SAA Congruence Rule)**. If angles ABC and DEF are equal to one another, angles ACB and DFE are equal to one another, and sides AB and DE equal to one another in length then the triangles ABC and DEF are congruent to one another.*

Accordingly, when the hypotheses of either congruence rule are satisfied, the sides and angles of the triangle ABC are equal to the corresponding sides and angles of the triangle DEF.



Proof First suppose that the conditions of the ASA Congruence Rule are satisfied. If the sides AB and DE of the respective triangles were unequal then one would exceed the other in length. Suppose that AB were greater than DE in length. Then a point G could be found on BA so as to make the straight line segment BG equal in length to the straight line segment DE. Then the sides GB and BC of the triangle GBC would be equal in length to the respective sides of the triangle DEF, and the included angle GBC would be equal to the included angle DEF. The SAS Congruence Rule (Proposition 4) would then ensure the congruence of the triangles GBC and DEF. The angles GCB and DFE would therefore be equal to one another.

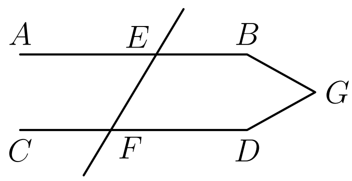
But the angle DFE is equal to the angle ACB, and therefore could not be equal to the angle GCB. We conclude therefore that the side AB cannot exceed the side DE in length. Nor can the side DE exceed the side AB in length. The sides AB and DE of the triangles ABC and DEF are therefore equal in length. An application of the ASA Congruence Rule (Proposition 4) now shows that the triangles ABC and DEF are congruent.

Now suppose that the conditions of the SAA Congruence Rule are satisfied. If the sides BC and EF of the respective triangles were unequal then one would exceed the other in length. Suppose that BC were greater than EF in length. Then a point H could be found on BC so as to make the straight line segment BH equal in length to the straight line segment EF. Then the sides BC and BH of the triangle ABH would be equal in length to the respective sides of the triangle DEF, and the included angle ABH would be equal to the included angle DEF. The SAS Congruence Rule (Proposition 4) would then ensure the congruence of the triangles ABH and DEF. The angles AHB and DFE would therefore be equal to one another.

But the angle DFE is equal to the angle ACB, and the exterior angle AHB of the triangle ACH is greater than the internal and opposite angle ACB. Therefore the angle AHB could not be equal to the angle DFE. We conclude therefore that the side BC cannot exceed the side EF in length. Nor can the side EF exceed the side BC in length. The sides BC and EF of the triangles ABC and DEF are therefore equal in length. An application of the ASA Congruence Rule (Proposition 4) now shows that the triangles ABC and DEF are congruent.

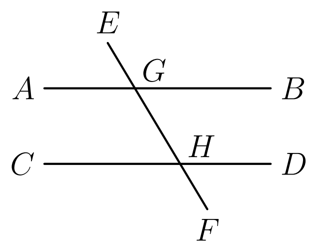
Thus under the hypotheses of the ASA Congruence Rule, or of the SAA Congruence Rule, the triangles ABC and DEF are congruent, and the sides and angles of the triangle ABC are respectively equal to the sides and angles of the triangle DEF, as required. ∎

Euclidean Proposition 27 *If a straight line EF in the Euclidean plane falling on two straight lines AB and CD make the alternative angles AEF and EFD equal to one another then the complete straight line passing through the points A and B does not intersect the complete straight line passing through the points C and D.*



Proof Suppose that the straight lines AB and CD could be produced beyond B and D respectively so as to intersect at a point G of the Euclidean plane. Then, in the triangle GEF, the exterior angle AEF would be greater than the interior and opposite angle EFD (Proposition 16), contrary to hypothesis. Therefore the complete straight line passing through the points A and B cannot intersect the complete straight line passing through the points C and D at any point of the Euclidean plane that lies on the same side of the straight line EF as the point B and D. A similar argument shows that the complete straight line passing through the points A and B cannot intersect the complete straight line passing through the points C and D at any point of the Euclidean plane that lies on the opposite side of the straight line EF as the point B and D. The result follows. ∎

Euclidean Proposition 28 *If a straight line EF in the Euclidean plane falling on two straight lines AB and CD make the exterior angle EGB equal to the interior and opposite angle GHD, or make the sum of the interior angles BGH, GHD equal to two right angles, then the complete straight line passing through the points A and B does not intersect the complete straight line passing through the points C and D.*



Proof Suppose EGB and GHD are equal. Now the angles EGB and AGH are equal (Proposition 15). It follows that the alternate angles AGH and GHD are equal, and therefore the complete straight line passing through the points A and B does not intersect the complete straight line passing through the points C and D (Proposition 27). Similarly if the angles BGH and GHD sum to two right angles then the alternating angles AGH and GGD are equal, because the angles AGH and BGH also sum to two right angles, and consequently the complete straight line passing through the points A and B does not intersect the complete straight line passing through the points C and D. ∎